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# SUBORDINATING RESULTS OF CLASSES OF MULTIVALENT MEROMORPHIC FUNCTIONS

M. A. MOWAFY, A. O. MOSTAFA, S. M. MADIAN AND A. I. ELMAHDY

Abstract. In this paper, we defined a class of meromorphic functions which are analytic and multivalent in punctured unit disk. After that by using the operator which are defined by ( see Mostafa [8] ) we defined a new operator using meromorphic functions which are analytic and p-valent in punctured unit disk. Also in the present paper we defined a new class of meromorphic functions by using this new operator. Furthermer we use the concepts of differential subordination and Hadamard product or (convolution) in our proving theorems. Plus more we use the definition of hypergeometric function in our proof. After that we derive several inclusion relationships for this class. In order to prove our main results we need the following Lemmas which presented in our paper. Also we investigate some properties of certain classes of multivalent meromorphic functions by making use of the method of differential subordination, which are defined by means of a certain operator.

## 1. Introduction

For any integer $\epsilon > -\varsigma$, let $\sum_{\varsigma,\epsilon}$ be the class of meromorphic functions:

$$F(\varkappa) = \varkappa^{-\varsigma} + \sum_{k=\epsilon}^{\infty} a_k \varkappa^k, \ \varsigma, \epsilon \in \mathbb{N} = \{1, 2, ...\}, \tag{1.1}$$

which are analytic and $\varsigma$-valent in $\mathbb{U}^* = \{\varkappa : \varkappa \in \mathbb{C}, 0 < |\varkappa| < 1\} = \mathbb{U}\backslash\{0\}$.
For $\alpha \geq 0$ and $\beta > -1$, Mostafa [8] defined the operator

$$\Omega^{\alpha}_{\varsigma,\beta,\mu} F(\varkappa) = \frac{1}{\varkappa^{\varsigma}} + \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \sum_{k=1-\varsigma}^{\infty} \frac{\Gamma(k+\varsigma+\alpha+\beta)(\mu)_{k+\varsigma}}{\Gamma(k+\varsigma+\beta)(k+\varsigma)!} a_k \varkappa^k,$$

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where

$$(d)_s = \frac{\Gamma(d+s)}{\Gamma(d)} = \begin{cases} 1 & (s=0; d \in \mathbb{C}^* = \mathbb{C}/\{0\}) \\ d(d+1)\ldots\ldots(d+s-1) & (s \in \mathbb{N}_0; d \in \mathbb{C}) \end{cases}.$$

For $F$ as $(1.1)$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mu > 0$, using $\Omega^{\alpha}_{\varsigma,\beta,\mu} F$ and $\lambda \geq 0$ we define the operator $\Upsilon^{\alpha,n}_{\beta,\lambda,\mu} : \sum_{\varsigma,\epsilon} \to \sum_{\varsigma,\epsilon}$ as

$$\begin{aligned} \Upsilon^{0,0}_{\beta,\lambda,1} F(\varkappa) &= F(\varkappa), \\ \Upsilon^{\alpha,1}_{\beta,\lambda,\mu} F(\varkappa) &= (1+\lambda)\,\Omega^{\alpha}_{\varsigma,\beta,\mu} F(\varkappa) + \frac{\lambda}{\varsigma}\varkappa\left(\Omega^{\alpha}_{\varsigma,\beta,\mu} F(\varkappa)\right)', \\ \Upsilon^{\alpha,2}_{\beta,\lambda,\mu} F(\varkappa) &= (1+\lambda)\,\Upsilon^{\alpha,1}_{\beta,\lambda,\mu} F(\varkappa) + \frac{\lambda}{\varsigma}\varkappa\left(\Upsilon^{\alpha,1}_{\beta,\lambda,\mu} F(\varkappa)\right)', \\ \Upsilon^{\alpha,n}_{\beta,\lambda,\mu} F(\varkappa) &= (1+\lambda)\,\Upsilon^{\alpha,n-1}_{\beta,\lambda,\mu} F(\varkappa) + \frac{\lambda}{\varsigma}\varkappa\left(\Upsilon^{\alpha,n-1}_{\beta,\lambda,\mu} F(\varkappa)\right)', \\ &= \frac{1}{\varkappa^{\varsigma}} + \sum_{k=\epsilon}^{\infty} \Lambda_k(\alpha,\beta,\lambda,\mu,\varsigma)\, a_k \varkappa^k, \end{aligned} \tag{1.2}$$

where

$$\Lambda_k(\alpha,\beta,\lambda,\mu,\varsigma) = \frac{\Gamma(\beta)\left[1+\lambda\left(1+\frac{k}{\varsigma}\right)\right]^n \Gamma(k+\varsigma+\alpha+\beta)\,(\mu)_{k+\varsigma}}{\Gamma(\alpha+\beta)\,\Gamma(k+\varsigma+\beta)\,(k+\varsigma)!}, \tag{1.3}$$

which satisfies

$$\varkappa\left(\Upsilon^{\alpha,n}_{\beta,\lambda,\mu} F(\varkappa)\right)' = \mu \Upsilon^{\alpha,n}_{\beta,\lambda,\mu+1} F(\varkappa) - (\mu+\varsigma)\Upsilon^{\alpha,n}_{\beta,\lambda,\mu} F(\varkappa), \tag{1.4}$$

$$\varkappa\left(\Upsilon^{\alpha,n}_{\beta,\lambda,\mu} F(\varkappa)\right)' = (\alpha+\beta)\,\Upsilon^{\alpha+1,n}_{\beta,\lambda,\mu} F(\varkappa) - (\alpha+\beta+\varsigma)\,\Upsilon^{\alpha,n}_{\beta,\lambda,\mu} F(\varkappa), \tag{1.5}$$

and

$$\varkappa\left(\Upsilon^{\alpha,n}_{\beta,\lambda,\mu} F(\varkappa)\right)' = \frac{\varsigma}{\lambda}\Upsilon^{\alpha,n+1}_{\beta,\lambda,\mu} F(\varkappa) - \frac{\varsigma(\lambda+1)}{\lambda}\Upsilon^{\alpha,n}_{\beta,\lambda,\mu} F(\varkappa). \tag{1.6}$$

**Definition 1.1.** *[1], [6] and [7] If $F$, $g$ are analytic in $\mathbb{U}$, then $F$ is subordinate to $g$, written $F \prec g$ if there exists a Schwarz function $w(\varkappa)$ analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(\varkappa)| < 1$ for all $\varkappa \in \mathbb{U}$, such that*

$$F(\varkappa) = g(w(\varkappa)).$$

**Definition 1.2.** *For $F \in \sum_{\varsigma,\epsilon}$, $(-1 \leq B < A \leq 1)$ and $F \in \sum^{\alpha,n}_{\beta,\lambda}(\mu; A, B)$ if it satisfies:*

$$-\frac{\varkappa^{\varsigma+1}\left(\Upsilon^{\alpha,n}_{\beta,\lambda,\mu} F(\varkappa)\right)'}{\varsigma} \prec \frac{1+A\varkappa}{1+B\varkappa}. \tag{1.7}$$

*Let $\sum^{\alpha,n}_{\beta,\lambda}\left(\mu; 1-\frac{2\eta}{\varsigma}, -1\right) = \sum^{\alpha,n}_{\beta,\lambda}(\mu,\eta)$, $0 \leq \eta < \varsigma$, where $\sum^{\alpha,n}_{\beta,\lambda}(\mu,\eta)$ denotes the class of functions in $\sum_{\varsigma,\epsilon}$ satisfying*

$$Re\left\{-\varkappa^{\varsigma+1}\left(\Upsilon^{\alpha,n}_{\beta,\lambda,\mu} F(\varkappa)\right)'\right\} > \eta. \tag{1.8}$$

Many authors obtained subordination results for classes of meromorphic functions ex: [2], [3] and [3].

In the present paper, we derive several inclusion relationships for the function class $\digamma \in \sum_{\beta,\lambda}^{\alpha,n}(\mu; A, B)$. To prove our main results we need the following Lemmas.

**Lemma 1.1.** *[4] Let $\hbar$ be a convex (univalent) function with $\hbar(0) = 1$. Also let*

$$\phi(\varkappa) = 1 + c_{\varsigma+\epsilon}\varkappa^{\varsigma+\epsilon} + c_{\varsigma+\epsilon+1}\varkappa^{\varsigma+\epsilon+1}..., \tag{1.9}$$

*$\gamma \in \mathbb{C}/\{0\}$ be analytic in $\mathbb{U}$. If*

$$\phi(\varkappa) + \frac{\varkappa\phi'(\varkappa)}{\gamma} \prec \hbar(\varkappa) \quad (Re(\gamma) \geq 0, \varkappa \in \mathbb{U}), \tag{1.10}$$

*then*

$$\phi(\varkappa) \prec \psi(\varkappa) = \frac{\gamma\varkappa^{-\frac{\gamma}{\varsigma+\epsilon}}}{\varsigma+\epsilon}\int_0^{\varkappa} t^{\frac{\gamma}{\varsigma+\epsilon}-1}\hbar(t)\,dt \tag{1.11}$$

*and $\psi$ is the best dominant. We denote by $P(\gamma)$ the class of functions $\varphi$, given by*

$$\varphi(\varkappa) = 1 + c_1\varkappa + c_2\varkappa^2 + ..., \; Re\{\varphi(\varkappa)\} > \gamma, \; 0 \leq \gamma < 1. \tag{1.12}$$

**Lemma 1.2.** *[10] Let $\varphi$ as (1.12) be in the class $P(\gamma)$. Then*

$$Re\{\varphi(\varkappa)\} \geq 2\gamma - 1 + \frac{2(1-\gamma)}{1+|\varkappa|} \; (\, 0 \leq \gamma < 1).$$

**Lemma 1.3.** *[15] Let Let $\mu$ be a positive measure on $[0,1]$. Let $g(\varkappa, t)$ be a complex valued function defined on $\mathbb{U} \times [0,1]$ such that $g(., t)$ is analytic in $\mathbb{U}$ for each $t \in [0,1]$, and $g(\varkappa, .)$ is $\mu-$ integrable on $[0,1]$, for all $\varkappa \in \mathbb{U}$. In addition suppose that $Re\{g(\varkappa, t)\} > 0$, $g(-r, t)$ is real and*

$$Re\left\{\frac{1}{g(\varkappa, t)}\right\} \geq \frac{1}{g(-r, t)} \; (|\varkappa| \leq r < 1; \; t \in [0,1]).$$

*If*

$$g(\varkappa) = \int_0^1 g(\varkappa, t)d\mu(t),$$

*then*

$$Re\left\{\frac{1}{g(\varkappa)}\right\} \geq \frac{1}{g(-r)} \; (|\varkappa| \leq r < 1).$$

Each of the identities (asserted by Lemma 1.4) is fairly well known (cf., e.g., [14], Ch.14)

**Lemma 1.4.** *[14] For real or complex numbers $a, b$ and $c$ $(c \neq 0, -1, -2, ...)$,*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-t\varkappa)^{-a}\,dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} \; {}_2\digamma_1(a, b; c; \varkappa) \; Re(c) > Re(b) > 0, \tag{1.13}$$

*where*

$${}_2\digamma_1(a, b; c; \varkappa) = (1-\varkappa)_2^{-a}\digamma_1\left(a, c-b; c; \frac{\varkappa}{\varkappa-1}\right) \tag{1.14}$$

*and*

$${}_2\digamma_1(a, b; c; \varkappa) = {}_2\digamma_1(b, a; c; \varkappa). \tag{1.15}$$

**Lemma 1.5.** *[13] Let $\varnothing$ be analytic in $\mathbb{U}$ with*

$$\varnothing(0) = 1 \text{ and } Re\{\varnothing(\varkappa)\} > \frac{1}{2}.$$

*Then, for any $F$ analytic in $\mathbb{U}$, $(\varnothing * F)(\mathbb{U})$ is contained in the convex hull of $F(\mathbb{U})$ where $*$ denotes convolution.*

## 2. Main results

The best dominant of the differential subordination solution will be found in the following theorems. Also for the function class $F \in \sum_{\beta,\lambda}^{\alpha,n}(\mu; A, B)$ ,we obtain a variety of inclusion relationships.

*Theorem* 2.1. For $F$ as (1.1) satisfies:

$$-\frac{(1-\delta)\varkappa^{\varsigma+1}\left(\Upsilon_{\beta,\lambda,\mu}^{\alpha,n}F(\varkappa)\right)' + \delta^{\varsigma+1}\left(\Upsilon_{\beta,\lambda,\mu+1}^{\alpha,n}F(\varkappa)\right)'}{\varsigma} \prec \frac{1+A\varkappa}{1+B\varkappa}, \tag{2.1}$$

then

$$-\frac{\varkappa^{\varsigma+1}\left(\Upsilon_{\beta,\lambda,\mu}^{\alpha,n}F(\varkappa)\right)'}{\varsigma} \prec Q^*(\varkappa) \prec \frac{1+A\varkappa}{1+B\varkappa}, \tag{2.2}$$

where

$$Q^*(\varkappa) = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1+B\varkappa)^{-1}\ {}_2F_1\left(1,1;\frac{\mu}{\delta(\varsigma+\epsilon)}+1;\frac{B\varkappa}{1+B\varkappa}\right) & B \neq 0 \\ 1 + \frac{\mu}{\delta(\varsigma+\epsilon)+\mu}A\varkappa & B = 0 \end{cases},$$

is the best dominant of (2.2). Furthermore

$$Re\left\{-\frac{\varkappa^{\varsigma+1}\left(\Upsilon_{\beta,\lambda,\mu}^{\alpha,n}F(\varkappa)\right)'}{\varsigma}\right\} > \rho \ \ (0 \leq \rho < 1), \tag{2.3}$$

where

$$\rho = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1-B)^{-1}\ {}_2F_1\left(1,1;\frac{\mu}{\delta(\varsigma+\epsilon)}+1;\frac{B}{B-1}\right) & B \neq 0 \\ 1 - \frac{\mu}{\delta(\varsigma+\epsilon)+\mu}A & B = 0 \end{cases}. \tag{2.4}$$

The result is the best possible.

*Proof.* Let

$$\phi(\varkappa) = -\frac{\varkappa^{\varsigma+1}\left(\Upsilon_{\beta,\lambda,\mu}^{\alpha,n}F(\varkappa)\right)'}{\varsigma}. \tag{2.5}$$

Then $\phi$ as (1.9). Applying (1.4) in (2.5) and differentiating, we get

$$-\frac{(1-\delta)\varkappa^{\varsigma+1}\left(\Upsilon_{\beta,\lambda,\mu}^{\alpha,n}F(\varkappa)\right)' + \delta\varkappa^{\varsigma+1}\left(\Upsilon_{\beta,\lambda,\mu+1}^{\alpha,n}F(\varkappa)\right)'}{\varsigma} = \phi(\varkappa) + \frac{\delta\varkappa\phi'(\varkappa)}{\mu} \prec \frac{1+A\varkappa}{1+B\varkappa}.$$

Using Lemma 1.1, for $\gamma = \frac{\mu}{\delta}$, we have

$$-\frac{\varkappa^{\varsigma+1}\left(\Upsilon^{\alpha,n}_{\beta,\lambda,\mu}F(\varkappa)\right)'}{\varsigma} \prec Q^*(\varkappa) = \frac{\mu\varkappa^{-\frac{\mu}{\delta(\varsigma+\epsilon)}}}{\delta(\varsigma+\epsilon)}\int_0^{\varkappa} t^{\frac{\mu}{\delta(\varsigma+\epsilon)}-1}\left(\frac{1+At}{1+Bt}\right)dt$$
$$= \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1+B\varkappa)^{-1}\ {}_2F_1\left(1,1;\frac{\mu}{\delta(\varsigma+\epsilon)}+1;\frac{B\varkappa}{1+B\varkappa}\right) & B \neq 0 \\ 1+\frac{\mu}{\delta(\varsigma+\epsilon)+\mu}A\varkappa & B = 0 \end{cases}.$$

This proves (2.2). Next we shall show that

$$\inf_{|\varkappa|<1}\left\{Re\left(Q^*(\varkappa)\right)\right\} = Q^*(-1). \tag{2.6}$$

For $|\varkappa| \leq r < 1$, we have

$$Re\left(\frac{1+A\varkappa}{1+B\varkappa}\right) \geq \frac{1-Ar}{1-Br}.$$

Setting

$$g(\varkappa, s) = \frac{1+A\varkappa s}{1+B\varkappa s} \text{ and } d\mu(s) = \frac{\mu s^{\frac{\mu}{\delta(\varsigma+\epsilon)}-1}}{\delta(\varsigma+\epsilon)}ds \ \ (0 \leq s \leq 1),$$

which is a positive measure on $[0,1]$, we get

$$Re\left[Q^*(\varkappa)\right] \geq \int_0^1 \frac{1-Asr}{1-Bsr}d\mu(s) = Q^*(-1) \ \ (|\varkappa| \leq r < 1).$$

Letting $r \to 1^-$, we obtain (2.3).

Finally, the estimate (2.3) is the best possible as $Q^*(\varkappa)$ is the best dominant of (2.2). □

Putting $\delta = 1$, $A = 1 - \frac{2\eta}{\varsigma}$ and $B = -1$ in Theorem 2.1, we have the following inclusion property

**Corollary 2.1.**

$$\sum_{\beta,\lambda}^{\alpha,n}(\mu+1,\eta) \subset \sum_{\beta,\lambda}^{\alpha,n}(\mu,\omega(\varsigma,\epsilon,\mu,\eta)) \subset \sum_{\beta,\lambda}^{\alpha,n}(\mu,\eta),$$

*where*

$$\omega(\varsigma,\epsilon,\mu,\eta) = \eta + (\varsigma-\eta)\left[{}_2F_1\left(1,1;\frac{\mu}{(\varsigma+\epsilon)}+1;\frac{1}{2}\right)-1\right].$$

The result is the best possible.

Taking $\delta = 1$ and $\epsilon = 1-\varsigma$ in Theorem 2.1, we obtain the following inclusion property

**Corollary 2.2.**

$$\sum_{\beta,\lambda}^{\alpha,n}(\mu+1,A,B) \subset \sum_{\beta,\lambda}^{\alpha,n}\left(\mu,1-\frac{2\eta}{\varsigma},-1\right) \subset \sum_{\beta,\lambda}^{\alpha,n}(\mu,A,B),$$

*where*

$$\eta = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1 + B\varkappa)^{-1} \;{}_2F_1\left(1, 1; \mu + 1; \frac{B}{B-1}\right) & B \neq 0 \\ 1 - \frac{\mu A}{1+\mu} & B = 0 \end{cases}.$$

The result is the best possible.

*Theorem* 2.2. Let $F \in \sum_{\beta,\lambda}^{\alpha,n}(\mu,\gamma)$, then

$$Re\left[-(1-\delta)\varkappa^{\varsigma+1}\left(\Upsilon_{\beta,\lambda,\mu}^{\alpha,n}F(\varkappa)\right)' - \delta\varkappa^{\varsigma+1}\left(\Upsilon_{\beta,\lambda,\mu+1}^{\alpha,n}F(\varkappa)\right)'\right] > \gamma \;\; (|\varkappa| < R), \tag{2.7}$$

where

$$R = \left\{\frac{\sqrt{\delta^2(\varsigma+\epsilon)^2 + \mu^2} - \delta(\varsigma+\epsilon)}{\mu}\right\}^{\frac{1}{(\varsigma+\epsilon)}}. \tag{2.8}$$

The result is the best possible.

*Proof.* Since $F \in \sum_{\beta,\lambda}^{\alpha,n}(\mu,\gamma)$, we write

$$-\varkappa^{\varsigma+1}\left(\Upsilon_{\beta,\lambda,\mu}^{\alpha,n}F(\varkappa)\right)' = \gamma + (\varsigma - \gamma)\,u(\varkappa), \tag{2.9}$$

where $u$ as (1.9) and $Re[u(\varkappa)] > 0$. Using of (1.4) in (2.9) and differentiating the resulting equation, we have

$$-\frac{\varkappa^{\varsigma+1}\left[(1-\delta)\left(\Upsilon_{\beta,\lambda,\mu}^{\alpha,n}F(\varkappa)\right)' + \delta\left(\Upsilon_{\beta,\lambda,\mu+1}^{\alpha,n}F(\varkappa)\right)'\right] + \gamma}{\varsigma - \gamma} = u(\varkappa) + \frac{\delta\varkappa u'(\varkappa)}{\mu}. \tag{2.10}$$

Applying the following estimate [5]:

$$\frac{|\varkappa u'(\varkappa)|}{Re[u(\varkappa)]} \leq \frac{2(\varsigma+\epsilon)\,r^{\varsigma+\epsilon}}{1 - r^{2(\varsigma+\epsilon)}} \;\; (|\varkappa| \leq r < 1),$$

in (2.10), we get

$$\begin{aligned} & Re\left\{-\frac{\varkappa^{\varsigma+1}\left[(1-\delta)\left(\Upsilon_{\beta,\lambda,\mu}^{\alpha,n}F(\varkappa)\right)' + \delta\left(\Upsilon_{\beta,\lambda,\mu+1}^{\alpha,n}F(\varkappa)\right)'\right] + \gamma}{\varsigma - \gamma}\right\} \\ \geq\;\; & Re[u(\varkappa)]\left[1 - \frac{2\delta(\varsigma+\epsilon)\,r^{\varsigma+\epsilon}}{\mu\left[1 - r^{2(\varsigma+\epsilon)}\right]}\right] > 0, \end{aligned} \tag{2.11}$$

if $r < R$, $R$ as $(2.8)$. In order to show that the bound $R$ is the best possible, we consider $F \in \sum_{\varsigma,\epsilon}$ defined by

$$-\varkappa^{\varsigma+1}\left(\Upsilon_{\beta,\lambda,\mu}^{\alpha,n}F(\varkappa)\right)' = \gamma + (\varsigma - \gamma)\frac{1 + \varkappa^{\varsigma+\epsilon}}{1 - \varkappa^{\varsigma+\epsilon}} \;\; (0 \leq \gamma < \varsigma).$$

Noting that

$$-\frac{\varkappa^{\varsigma+1}\left[(1-\delta)\left(\Upsilon_{\beta,\lambda,\mu}^{\alpha,n}F(\varkappa)\right)'+\delta\left(\Upsilon_{\beta,\lambda,\mu+1}^{\alpha,n}F(\varkappa)\right)'\right]+\gamma}{\varsigma-\gamma}$$
$$=\frac{\mu\left[1-\varkappa^{2(\varsigma+\epsilon)}\right]+2\delta(\varsigma+\epsilon)\varkappa^{\varsigma+\epsilon}}{\mu\left(1-\varkappa^{\varsigma+\epsilon}\right)^2}=0,$$

for $\varkappa = R\exp\left(\frac{i\pi}{\varsigma+\epsilon}\right)$. This completes the proof. □

Putting $\delta = 1$ in Theorem 2.2, we have

**Corollary 2.3.** *If* $F \in \sum_{\beta,\lambda}^{\alpha,n}(\mu,\gamma)$, *then* $F \in \sum_{\beta,\lambda}^{\alpha,n}(\mu+1,\gamma)$ *for* $|\varkappa| < R^*$, *where*

$$R^* = \left\{\frac{\sqrt{(\varsigma+\epsilon)^2+\mu^2}-(\varsigma+\epsilon)}{\mu}\right\}^{\frac{1}{(\varsigma+\epsilon)}}.$$

The result is the best possible.

*Theorem* 2.3. If $F \in \sum_{\varsigma,\epsilon}$ satisfies:

$$\varkappa^{\varsigma}\left[(1-\delta)\left(\Upsilon_{\beta,\lambda,\mu}^{\alpha,n}F(\varkappa)\right)'+\delta\left(\Upsilon_{\beta,\lambda,\mu+1}^{\alpha,n}F(\varkappa)\right)'\right]\prec\frac{1+A\varkappa}{1+B\varkappa},$$

then

$$\varkappa^{\varsigma}\Upsilon_{\beta,\lambda,\mu}^{\alpha,n}F(\varkappa)\prec Q^*(\varkappa)\prec\frac{1+A\varkappa}{1+B\varkappa}$$

and

$$Re\left[\varkappa^{\varsigma}\Upsilon_{\beta,\lambda,\mu}^{\alpha,n}F(\varkappa)\right]>\rho,$$

where $Q^*(\varkappa)$ and $\rho$ as in Theorem 2.1. The result is the best possible.

*Proof.* Using the same lines as in the proof of Theorem 2.1, by taking $\phi(\varkappa) = \varkappa^{\varsigma}\Upsilon_{\beta,\lambda,\mu}^{\alpha,n}F(\varkappa)$ in (2.5).

For $F \in \sum_{\varsigma,\epsilon}$ and $F_{c,\varsigma} : \sum_{\varsigma,\epsilon} \to \sum_{\varsigma,\epsilon}$, such that

$$\begin{aligned} F_{c,\varsigma}F(\varkappa) &= \frac{c}{\varkappa^{c+\varsigma}}\int_0^{\varkappa} t^{c+\varsigma-1}F(t)dt \\ &= \varkappa^{-\varsigma}+\sum_{k=\epsilon}^{\infty}\frac{c}{c+\varsigma+k}a_k\varkappa^k \ \ (c>0) \end{aligned} \tag{2.12}$$

and satisfies

$$\varkappa\left(\Upsilon_{\beta,\lambda,\mu}^{\alpha,n}F_{c,\varsigma}F(\varkappa)\right)'=c\Upsilon_{\beta,\lambda,\mu}^{\alpha,n}F(\varkappa)-(c+\varsigma)\Upsilon_{\beta,\lambda,\mu}^{\alpha,n}F_{c,\varsigma}F(\varkappa). \tag{2.13}$$

□

*Theorem* 2.4. Let $F \in \sum_{\beta,\lambda}^{\alpha,n}(\mu;A,B)$ and $F_{c,\varsigma}F(\varkappa)$ as (2.12) then

$$-\frac{\varkappa^{\varsigma+1}\left(\Upsilon_{\beta,\lambda,\mu}^{\alpha,n}F_{c,\varsigma}F(\varkappa)\right)'}{\varsigma}\prec\Theta(\varkappa)\prec\frac{1+A\varkappa}{1+B\varkappa}, \tag{2.14}$$

where

$$\Theta(\varkappa)=\begin{cases}\frac{A}{B}+\left(1-\frac{A}{B}\right)(1+B\varkappa)_2^{-1}F_1\left(1,1;\frac{c}{(\varsigma+\epsilon)}+1;\frac{B\varkappa}{1+B\varkappa}\right) & B\neq 0\\ 1+\frac{c}{(\varsigma+\epsilon)+c}A\varkappa & B=0\end{cases},$$

is the best dominant of (2.14). Furthermore

$$Re\left\{-\frac{\varkappa^{\varsigma+1}\left(\Upsilon_{\beta,\lambda,\mu}^{\alpha,n}F_{c,\varsigma}F(\varkappa)\right)'}{\varsigma}\right\}>\upsilon, \tag{2.15}$$

where

$$\upsilon=\begin{cases}\frac{A}{B}+\left(1-\frac{A}{B}\right)(1-B)^{-1}\ {}_2F_1\left(1,1;\frac{c}{(\varsigma+\epsilon)}+1;\frac{B}{B-1}\right) & B\neq 0\\ 1-\frac{c}{(\varsigma+\epsilon)+c}A & B=0\end{cases}.$$

The result is the best possible.

*Proof.* Let

$$\phi(\varkappa)=-\frac{\varkappa^{\varsigma+1}\left(\Upsilon_{\beta,\lambda,\mu}^{\alpha,n}F_{c,\varsigma}F(\varkappa)\right)'}{\varsigma},$$

then $\phi$ as (1.9). Using (2.13) in (2.14) and differentiating, we have

$$-\frac{\varkappa^{\varsigma+1}\left(\Upsilon_{\beta,\lambda,\mu}^{\alpha,n}F(\varkappa)\right)'}{\varsigma}=\phi(\varkappa)+\frac{\varkappa\phi'(\varkappa)}{c}\prec\frac{1+A\varkappa}{1+B\varkappa}.$$

Now the remaining part of Theorem 2.4 follows by employing the technique used in proving Theorem 2.1. □

*Theorem* 2.5. For $F_{c,\varsigma}F(\varkappa)$ as (2.12), satisfy

$$\varkappa^{\varsigma}\left[(1-\delta)\left(\Upsilon_{\beta,\lambda,\mu}^{\alpha,n}F_{c,\varsigma}F(\varkappa)\right)+\delta\left(\Upsilon_{\beta,\lambda,\mu}^{\alpha,n}F(\varkappa)\right)\right]\prec\frac{1+A\varkappa}{1+B\varkappa}, \tag{2.16}$$

then

$$Re\left[\varkappa^{\varsigma}\Upsilon_{\beta,\lambda,\mu}^{\alpha,n}F_{c,\varsigma}F(\varkappa)\right]>\tau, \tag{2.17}$$

where

$$\tau=\begin{cases}\frac{A}{B}+\left(1-\frac{A}{B}\right)(1-B)^{-1}\ {}_2F_1\left(1,1;\frac{c}{\delta(\varsigma+\epsilon)}+1;\frac{B}{B-1}\right) & B\neq 0\\ 1-\frac{c}{\delta(\varsigma+\epsilon)+c}A & B=0\end{cases}. \tag{2.18}$$

*Proof.* Let

$$\phi(\varkappa)=\varkappa^{\varsigma}\Upsilon_{\beta,\lambda,\mu}^{\alpha,n}F_{c,\varsigma}F(\varkappa), \tag{2.19}$$

then $\phi$ as (1.9). Differentiating (2.19) and using (2.13) and (2.16), we have

$$\phi(\varkappa)+\frac{\varkappa\phi'(\varkappa)}{c}\prec\frac{1+A\varkappa}{1+B\varkappa}.$$

Now the remaining part of Theorem 2.5, follows by employing the technique used in proving Theorem 2.1. □

Putting $n = 0$, $\alpha = 0$, $\mu = 1$, $B = -1$, $A = 1 - 2\eta$ $(0 \leq \eta < 1)$ and $\delta = 1$ in Theorem 2.5, we obtain

**Corollary 2.4.** *For $F \in \sum_{\varsigma,\epsilon}$ satisfies:*

$$Re\left[\varkappa^{\varsigma}F(\varkappa)\right] > \eta,$$

*then*

$$Re\left[\frac{c}{\varkappa^{c}}\int_{0}^{\varkappa} t^{c+\varsigma-1}F(t)dt\right] > \eta + (1-\eta)\left[{}_{2}F_{1}\left(1,1;\frac{c}{(\varsigma+\epsilon)}+1;\frac{1}{2}\right)-1\right].$$

The result is the best possible.

*Theorem* 2.6. Let $F \in \sum_{\varsigma,\epsilon}$ satisfying

$$-\frac{\varkappa^{\varsigma+1}\left[(1-\delta)\left(\Upsilon^{\alpha,n}_{\beta,\lambda,\mu}F_{c,\varsigma}F(\varkappa)\right)' + \delta\left(\Upsilon^{\alpha,n}_{\beta,\lambda,\mu}F(\varkappa)\right)'\right]}{\varsigma} \prec \frac{1+A\varkappa}{1+B\varkappa},$$

then

$$Re\left[-\frac{\varkappa^{\varsigma+1}\left(\Upsilon^{\alpha,n}_{\beta,\lambda,\mu}F_{c,\varsigma}F(\varkappa)\right)'}{\varsigma}\right] > \tau,$$

where $F_{c,\varsigma}F(\varkappa)$ as (2.12) and $\tau$ as ( 2.18) . The result is the best possible.

*Proof.* The proof follows by taking the same lines as in Theorem 2.5. □

Considering the fact that:

$$\varkappa^{\varsigma+1}\left(\Upsilon^{\alpha,n}_{\beta,\lambda,\mu}F_{c,\varsigma}F(\varkappa)\right)' = \frac{c}{\varkappa^{c}}\int_{0}^{\varkappa} t^{c+\varsigma}\left[\Upsilon^{\alpha,n}_{\beta,\lambda,\mu}F(t)\right]' dt,$$

then taking $n = 0$, $\alpha = 0$, $\mu = 1$, $B = -1$, $A = 1 - \frac{2\eta}{\varsigma}$ $(0 \leq \eta < \varsigma)$ and $\delta = 1$ in Theorem 2.6, we have

**Corollary 2.5.** *For $F \in \sum_{\varsigma,\epsilon}$ satisfies:*

$$Re\left[-\varkappa^{\varsigma+1}F'(\varkappa)\right] > \eta,$$

*then*

$$Re\left[-\frac{c}{\varkappa^{c}}\int_{0}^{\varkappa} t^{c+\varsigma}F'(t)dt\right] > \eta + (\varsigma-\eta)\left[{}_{2}F_{1}\left(1,1;\frac{c}{(\varsigma+\epsilon)}+1;\frac{1}{2}\right)-1\right].$$

The result is the best possible.

*Theorem* 2.7. Let $F \in \sum_{\varsigma,\epsilon}$ and $g \in \sum_{\varsigma,\epsilon}$, such that

$$Re\left[\varkappa^{\varsigma}\left(\Upsilon^{\alpha,n}_{\beta,\lambda,\mu}g(\varkappa)\right)\right] > 0.$$

If

$$\left|\frac{\Upsilon^{\alpha,n}_{\beta,\lambda,\mu}F(\varkappa)}{\Upsilon^{\alpha,n}_{\beta,\lambda,\mu}g(\varkappa)} - 1\right| < 1,$$

then

$$Re\left[-\frac{\varkappa\left(\Upsilon^{\alpha,n}_{\beta,\lambda,\mu}F\left(\varkappa\right)\right)'}{\Upsilon^{\alpha,n}_{\beta,\lambda,\mu}F\left(\varkappa\right)}\right]>0,$$

where

$$R_0=\frac{\sqrt{9\left(\varsigma+\epsilon\right)^2+4\varsigma\left(2\varsigma+\epsilon\right)}-3\left(\varsigma+\epsilon\right)}{2\left(2\varsigma+\epsilon\right)}. \tag{2.20}$$

*Proof.* Let

$$\varphi\left(\varkappa\right)=\frac{\Upsilon^{\alpha,n}_{\beta,\lambda,\mu}F\left(\varkappa\right)}{\Upsilon^{\alpha,n}_{\beta,\lambda,\mu}g\left(\varkappa\right)}-1=e_{\varsigma+\epsilon}\varkappa^{\varsigma+\epsilon}+e_{\varsigma+\epsilon+1}\varkappa^{\varsigma+\epsilon+1}..., \tag{2.21}$$

we note that $\varphi(0)=0$ and $|\varphi(\varkappa)|\leq|\varkappa|^{\varsigma+\epsilon}$. Then by applying the familiar Schwarz Lemma [9], we have $\varphi(\varkappa)=\varkappa^{\varsigma+\epsilon}\Psi(\varkappa)$, where $\Psi$ is analytic in $\mathbb{U}$ and $|\Psi(\varkappa)|\leq 1$. Therefore (2.21) , leads to

$$\Upsilon^{\alpha,n}_{\beta,\lambda,\mu}F\left(\varkappa\right)=\Upsilon^{\alpha,n}_{\beta,\lambda,\mu}g\left(\varkappa\right)\left[1+\varkappa^{\varsigma+\epsilon}\Psi\left(\varkappa\right)\right]. \tag{2.22}$$

Differentiating (2.22) logarithmically , we obtain

$$\frac{\varkappa\left[\Upsilon^{\alpha,n}_{\beta,\lambda,\mu}F\left(\varkappa\right)\right]'}{\Upsilon^{\alpha,n}_{\beta,\lambda,\mu}F\left(\varkappa\right)}=\frac{\varkappa\left[\Upsilon^{\alpha,n}_{\beta,\lambda,\mu}g\left(\varkappa\right)\right]'}{\Upsilon^{\alpha,n}_{\beta,\lambda,\mu}g\left(\varkappa\right)}+\frac{\varkappa^{\varsigma+\epsilon}\left[\left(\varsigma+\epsilon\right)\Psi\left(\varkappa\right)+\varkappa\Psi'\left(\varkappa\right)\right]}{1+\varkappa^{\varsigma+\epsilon}\Psi\left(\varkappa\right)}. \tag{2.23}$$

Letting $\chi\left(\varkappa\right)=\varkappa^{\varsigma}\Upsilon^{\alpha,n}_{\beta,\lambda,\mu}g\left(\varkappa\right)$, we see that $\chi$ as (1.9) is analytic in $\mathbb{U}$, $Re\left[\chi\left(\varkappa\right)\right]>0$ and

$$\frac{\varkappa\left[\Upsilon^{\alpha,n}_{\beta,\lambda,\mu}g\left(\varkappa\right)\right]'}{\Upsilon^{\alpha,n}_{\beta,\lambda,\mu}g\left(\varkappa\right)}=\frac{\varkappa\chi'\left(\varkappa\right)}{\chi\left(\varkappa\right)}-\varsigma,$$

using (2.23) we have

$$Re\left\{\frac{\varkappa\left[\Upsilon^{\alpha,n}_{\beta,\lambda,\mu}F\left(\varkappa\right)\right]'}{\Upsilon^{\alpha,n}_{\beta,\lambda,\mu}F\left(\varkappa\right)}\right\}\geq\varsigma-\left|\frac{\varkappa\chi'\left(\varkappa\right)}{\chi\left(\varkappa\right)}\right|-\left|\frac{\varkappa^{\varsigma+\epsilon}\left[\left(\varsigma+\epsilon\right)\Psi\left(\varkappa\right)+\varkappa\Psi'\left(\varkappa\right)\right]}{1+\varkappa^{\varsigma+\epsilon}\Psi\left(\varkappa\right)}\right|. \tag{2.24}$$

Using the following known estimates [11], (see also [5])

$$\left|\frac{\varkappa\chi'\left(\varkappa\right)}{\chi\left(\varkappa\right)}\right|\leq\frac{2\left(\varsigma+\epsilon\right)r^{\varsigma+\epsilon-1}}{1-r^{2(\varsigma+\epsilon)}}\text{ and }\left|\frac{\varkappa^{\varsigma+\epsilon}\left[\left(\varsigma+\epsilon\right)\Psi\left(\varkappa\right)+\varkappa\Psi'\left(\varkappa\right)\right]}{1+\varkappa^{\varsigma+\epsilon}\Psi\left(\varkappa\right)}\right|\leq\frac{\varsigma+\epsilon}{1-r^{(\varsigma+\epsilon)}}\quad\left(|\varkappa|<r<1\right),$$

in (2.24) , we have

$$Re\left\{\frac{\varkappa\left[\Upsilon^{\alpha,n}_{\beta,\lambda,\mu}F\left(\varkappa\right)\right]'}{\Upsilon^{\alpha,n}_{\beta,\lambda,\mu}F\left(\varkappa\right)}\right\}\geq\frac{\varsigma-3\left(\varsigma+\epsilon\right)r^{(\varsigma+\epsilon)}-\left(2\varsigma+\epsilon\right)r^{2(\varsigma+\epsilon)}}{1-r^{2(\varsigma+\epsilon)}}=0,$$

provided that $r<R_0$, $R_0$ as (2.20) . □

*Theorem* 2.8. For $F\in\sum_{\varsigma,\epsilon}$ and satisfies:

$$\left(1-\delta\right)\varkappa^{\varsigma}\left(\Upsilon^{\alpha,n}_{\beta,\lambda,\mu}F\left(\varkappa\right)\right)+\delta\varkappa^{\varsigma}\left(\Upsilon^{\alpha,n}_{\beta,\lambda,\mu+1}F\left(\varkappa\right)\right)\prec\frac{1+A\varkappa}{1+B\varkappa},$$

then

$$Re\left\{\left[\varkappa^{\varsigma}\Upsilon_{\beta,\lambda,\mu}^{\alpha,n}F(\varkappa)\right]^{\frac{1}{q}}\right\}>\xi^{\frac{1}{q}}\quad(q\in\mathbb{N}),$$

where $\xi$ as (2.4). The result is best possible.

*Proof.* Let

$$\phi(\varkappa)=\varkappa^{\varsigma}\Upsilon_{\beta,\lambda,\mu}^{\alpha,n}F(\varkappa), \tag{2.25}$$

then $\phi$ as (1.9). Using (1.4) and differentiating the resulting, we have

$$(1-\delta)\varkappa^{\varsigma}\left(\Upsilon_{\beta,\lambda,\mu}^{\alpha,n}F(\varkappa)\right)+\delta\varkappa^{\varsigma}\left(\Upsilon_{\beta,\lambda,\mu+1}^{\alpha,n}F(\varkappa)\right)=\phi(\varkappa)+\frac{\delta\varkappa\phi'(\varkappa)}{\mu}\prec\frac{1+A\varkappa}{1+B\varkappa}. \tag{2.26}$$

□

Following the lines of the proof of Theorem 2.1, mutatis mutandis and using:

$$Re\left(w^{\frac{1}{q}}\right)>\left[Re(w)\right]^{\frac{1}{q}}\quad(Re(w)>0),$$

we have the result asserted by Theorem 2.8.

*Theorem* 2.9. For $F\in\sum_{\beta,\lambda}^{\alpha,n}(\mu;A,B)$ and $g\in\sum_{\varsigma,\epsilon}$ satisfy:

$$Re\left[\varkappa^{\varsigma}g(\varkappa)\right]>\frac{1}{2},$$

then

$$(F*g)(\varkappa)\in\sum_{\beta,\lambda}^{\alpha,n}(\mu;A,B).$$

*Proof.* We have

$$-\frac{\varkappa^{\varsigma+1}\left[\Upsilon_{\beta,\lambda,\mu}^{\alpha,n}(F*g)(\varkappa)\right]'}{\varsigma}=-\frac{\varkappa^{\varsigma+1}\left[\Upsilon_{\beta,\lambda,\mu}^{\alpha,n}F(\varkappa)\right]'}{\varsigma}*\varkappa^{\varsigma}g(\varkappa).$$

Since

$$Re\left[\varkappa^{\varsigma}g(\varkappa)\right]>\frac{1}{2},$$

and $\frac{1+A\varkappa}{1+B\varkappa}$ is convex (univalent) in $\mathbb{U}$, it follows from (1.7) and Lemma 1.5, that $(F*g)(\varkappa)\in\sum_{\beta,\lambda}^{\alpha,n}(\mu;A,B)$. □

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M. A. Mowafy

Mathematics department, Faculty of Science, Mansoura University, Mansoura Egypt.

*Email address*: mohamed1976224@gmail.com

A. O. Mostafa

Mathematics department, Faculty of Science, Mansoura University, Mansoura Egypt.

*Email address*: adelaeg254@yahoo.com

S. M. Madian

Basic Science Dept., Higher Institute for, Engineering and Technology, New Damietta, Egypt.

*Email address*: samar_math@yahoo.com

A. I. Elmahdy

Mathematics department, Faculty of Science, Mansoura University, Mansoura Egypt.

*Email address*: atefmahdy05@mans.edu.eg