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A NEW GENERAL CONFORMABLE FRACTIONAL DERIVATIVE AND SOME APPLICATIONS

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Abstract. This paper introduces a new local fractional derivative, called the $\mathcal{M}$-conformable derivative. It is defined by the following formula:

$$\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\vartheta(t) = \lim_{\epsilon \to 0} \frac{\vartheta\left(t + \epsilon q\left(t\right)^{1-\rho} \mathcal{M}\left(\vartheta\left(t\right)\right)\right) - \vartheta(t)}{\epsilon},$$

where $\mathcal{M}(\cdot)$ and $q(\cdot)$ are two functions that satisfy some conditions and $0 < \rho < 1$.

First, we investigate fundamental properties of the $\mathcal{M}$-conformable derivative and derive analogues of Rolle's Theorem, the Mean Value Theorem and L'Hôpital's Rule within this framework. Furthermore, we define the $\tilde{\mathcal{M}}$-conformable integral via the fundamental theorem of calculus and establish an integration by parts formula. Finally, we present examples of $\mathcal{M}$-conformable fractional differential equations to illustrate the applicability of the proposed derivative.

1. Introduction

Fractional calculus, which is considered a generalization of traditional calculus, has seen remarkable advancements in recent decades. It has become one of the fastest-growing research areas, thanks to the compelling results derived from applying fractional operators to model real-world problems ([14], [18]). A key feature of this field is the variety of fractional operators available, enabling researchers to select the most suitable one for the problem under study. Among the most well-known fractional derivatives are Riemann–Liouville, Caputo, Hadamard, Caputo-Hadamard, Riesz and Grünwald-Letnikov ([20], [21]).

Recently, several articles have emerged addressing new types of fractional operators known as local fractional derivatives. The authors in [17] define a new

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well-behaved simple fractional derivative called "the conformable fractional derivative" depending just on the basic limit definition of the derivative

$$T_{\rho}\vartheta\left(t\right)=\lim_{\epsilon\to 0}\frac{\vartheta\left(t+\epsilon t^{1-\rho}\right)-\vartheta\left(t\right)}{\epsilon},$$

where ϑ a real function and $\rho\in\left]0,1\right[$. Unfortunately, his definition does not accommodate zero or negative values. Katugampola introduced a new derivative in [16], defined by

$$D^{\rho}\vartheta\left(t\right)=\lim_{\epsilon\to 0}\frac{\vartheta\left(te^{\epsilon t^{-\rho}}\right)-\vartheta\left(t\right)}{\epsilon}.$$

Atangana and Goufo [3], introduced the beta operator which has been applied to problems involving the asymptotic method. The beta operator is defined by the following formula

$${}_{0}^{A}D_{t}^{\beta}\left(\vartheta\left(t\right)\right)=\lim_{\epsilon\to 0}\frac{\vartheta\left(t+\epsilon\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right)-\vartheta\left(t\right)}{\epsilon},\ \ \beta\in\left]0,1\right[.$$

In 2016, Almeida et al. [2] generalized the beta operator and the conformable fractional derivative by introducing a new type of fractional derivative with a kernel, defined as follows

$$\vartheta^{(\rho)}\left(t\right)=\lim_{\epsilon\to 0}\frac{\vartheta\left(t+\epsilon q\left(t\right)^{1-\rho}\right)-\vartheta\left(t\right)}{\epsilon},$$

where the kernel $q:[a,b]\to\mathbb{R}$ is a continuous nonnegative map. Camrud [6], defined the conformable ratio derivative for a positive function $\vartheta(t)$ with $\vartheta'\left(t\right)\geq 0$ as follows

$$k_{\rho}(\vartheta\left(t\right))=\lim_{\epsilon\to 0}\vartheta\left(t\right)^{1-\rho}\left(\frac{\vartheta(t+\epsilon)-\vartheta\left(t\right)}{\epsilon}\right)^{\rho},$$

for all $t>0$ and $\rho\in[0,1]$. Guebbai and Ghiat in [9], gave another new definition of the fractional derivative by

$$\vartheta^{(\rho)}\left(t\right)=\lim_{\epsilon\to 0}\left(\frac{\vartheta\left(t+\epsilon\left(\vartheta\left(t\right)\right)^{(1-\rho)/\rho}\right)-\vartheta\left(t\right)}{\epsilon}\right)^{\rho},$$

where $\vartheta(t)$ is an increasing and positive function for all $t>0$ and $\rho\in\left]0,1\right]$. In 2020, Nápoles Valdés et al. [19], introduced a definition of a non-conformable fractional derivative, denoted by $N_{F}^{\rho}\vartheta\left(\cdot\right)$, which is defined as follows

$$N_{F}^{\rho}\vartheta\left(t\right)=\lim_{\epsilon\to 0}\frac{\vartheta\left(t+\epsilon F\left(t,\rho\right)\right)-\vartheta\left(t\right)}{\epsilon},$$

where $F(\cdot,\cdot)$ is an absolutely continuous function depending on $t>0$ and $\rho\in\left]0,1\right]$. In paper [15], the authors introduce a new definition of the local conformable fractional derivative, as follows

$$D^{\rho}\vartheta\left(t\right)=\lim_{\epsilon\to 0}\frac{\vartheta\left(t+\epsilon e^{(\rho-1)t}\right)-\vartheta\left(t\right)}{\epsilon},$$

for all $t>0$ and $\rho\in\left]0,1\right[$.

In addition to the previous definitions, recent research has proposed various generalizations of conformable derivatives, enriching their theoretical foundations and expanding their range of applications. For instance, Fleitas et al. [8] introduced a generalized conformable derivative of order $\alpha > 0$, where α is not restricted to integer values. This definition addresses certain limitations of classical local derivatives, whether conformable or not, and allows for the computation of fractional derivatives of functions defined on any open subset of the real line. Vivas-Cortés et al. [22] proposed a multi-index approach to generalized derivatives, further expanding the framework for applications in fractional calculus. In [7], the authors introduced the omega derivative, which presents a new concept that generalizes the classical derivative. Additionally, in the paper [10], Guzmán et al. proposed another generalized derivative that extends the class of continuous and differentiable functions an important objective in the development of new differential operators.

Beyond the definition of fractional derivatives, fractional integral inequalities particularly within the framework of conformable fractional calculus have gained significant attention due to their applicability in refining classical inequalities and developing new analytical techniques. Recent studies have introduced novel approaches and refinements in this area. For example, Hezenci et al. [13] provided remarks on inequalities with parameters via conformable fractional integrals, offering new perspectives on integral inequalities. In paper [11], Haider et al. studied Hermite-Hadamard inequalities for subadditive functions using conformable fractional integrals. Several versions of these inequalities were established, contributing to the improvement and extension of previous results for convex and subadditive functions. In [1], the authors extended Euler-Maclaurin type inequalities by introducing refined versions via conformable fractional integrals. The authors in [5] proposed a new approach to examine certain Newton-type inequalities for differentiable convex functions using conformable fractional operators. Additionally, Hezenci and Budak [12] introduced new Bullen-type inequalities for twice-differentiable functions, leveraging the properties of conformable fractional integrals to derive more generalized and precise formulations.

The purpose of this paper is to introduce a new generalization of the conformable fractional derivative ($\mathcal{M}$-conformable fractional derivative), extending the well-known definition of the derivative of a function at a given point "t". Additionally, the paper aims to generalize some results previously obtained in the papers [2], [6], [9], [16], [15] and [17].

The paper is organized as follows. In Section 2, we present a new general definition of local conformable fractional derivative ($\mathcal{M}$-conformable fractional derivative). In Section 3, we introduce the generalized conformable fractional integral ($\tilde{\mathcal{M}}$-conformable fractional integral) with formula for integration by parts. In Section 4, we prove some important theorems about $\mathcal{M}$-conformable fractional derivative, including Rolle's Theorem, Mean Value Theorem and the L'Hôspital's rule. In Section 5, we give some applications to $\mathcal{M}$-conformable fractional differential equations.

2. $\mathcal{M}$-Conformable Fractional Derivative

In this section, we present the main definition of the paper and establish a relationship between this new concept and ordinary differentiation. Using this formula, most of the fundamental properties of the fractional derivative can be derived directly.

Definition 2.1. *Let $q : [a, b] \longrightarrow \mathbb{R}$, and $\mathcal{M} : \mathbb{R} \longrightarrow [c, d]$ be two continuous maps such that nonnegative $q(t) \neq 0$; whenever $t > a$ and $\mathcal{M} \neq 0$. Given a function $\vartheta : [a, b] \longrightarrow \mathbb{R}$ and $\rho \in]0, 1[$, the $\mathcal{M}$-conformable derivative of ϑ of order ρ is defined by*

$$\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\vartheta(t) = \lim_{\epsilon \to 0} \frac{\vartheta\left(t + \epsilon q(t)^{1-\rho} \mathcal{M}(\vartheta(t))\right) - \vartheta(t)}{\epsilon},$$

for $t \in]a, b[$ and $\rho \in]0, 1[$. If ϑ is ρ-differentiable at $t = a$ and $\lim_{t \to a^+} \mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\vartheta(t)$ exists, then

$$\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\vartheta(a) = \lim_{t \to a^+} \mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\vartheta(t).$$

Remark 1. *When $\mathcal{M}(\cdot) := 1$ and $q(\cdot) := 1$, we get the classical derivation $\mathfrak{D}^{\rho}_{1,1}\vartheta(t) = \vartheta'(t)$.*

When $\mathcal{M}(\cdot) := 1$, the operator $\mathfrak{D}^{\rho}_{q(\cdot),1}\vartheta(t)$ coincides with the definition of the derivative of a function as presented in [2].

When $\mathcal{M}(\cdot) := 1$ and $q(t) := t$, the operator $\mathfrak{D}^{\rho}_{t,1}\vartheta(t)$ coincides with the definition of the derivative of a function as presented in [17].

Theorem 2.1. *If a function ϑ is differentiable at a point $t \in]a, b[$, then it is also ρ-differentiable at that point for any $\rho \in]0, 1[$. Furthermore, in this case we have the following*

$$\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\vartheta(t) = q(t)^{1-\rho} \vartheta'(t)\mathcal{M}(\vartheta(t)). \tag{1}$$

Proof. By definition, we have

$$\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\vartheta(t) = \lim_{\epsilon \to 0} \frac{\vartheta\left(t + \epsilon q(t)^{1-\rho} \mathcal{M}(\vartheta(t))\right) - \vartheta(t)}{\epsilon}.$$

So, we conclude that

$$\begin{aligned} \mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\vartheta(t) &= q(t)^{1-\rho} \mathcal{M}(\vartheta(t)) \lim_{\epsilon \to 0} \frac{\vartheta\left(t + \epsilon q(t)^{1-\rho} \mathcal{M}(\vartheta(t))\right) - \vartheta(t)}{\epsilon q(t)^{1-\rho} \mathcal{M}(\vartheta(t))} \\ &= q(t)^{1-\rho} \mathcal{M}(\vartheta(t)) \vartheta'(t). \end{aligned}$$

□

Lemma 2.1. *If ϑ is ρ-differentiable at $t \in]a, b[$ for some ρ, then ϑ is locally bounded in that region.*

Proof. Suppose ϑ is ρ-differentiable at t. Then, there exist a number $\delta > 0$ such that

$$\left|\vartheta(t + \epsilon q(t)^{1-\rho} \mathcal{M}(\vartheta(t))) - \vartheta(t) - \epsilon \mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\vartheta(t)\right| \leq |\epsilon|, \quad \text{for} \quad |\epsilon| < \delta,$$

this implies that

$$\left|\vartheta\left(t + \epsilon q(t)^{1-\rho} \mathcal{M}(\vartheta(t))\right)\right| \leq |\epsilon| + \left|\vartheta(t) + \epsilon \mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\vartheta(t)\right|, \quad \text{for} \quad |\epsilon| < \delta,$$

this means that

$$\left|\vartheta\left(t + \epsilon q(t)^{1-\rho} \mathcal{M}(\vartheta(t))\right)\right| \leq |\epsilon| + |\vartheta(t)| + |\epsilon| \left|\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\vartheta(t)\right|, \text{ for } |\epsilon| < \delta,$$

then, we get

$$\left|\vartheta\left(t + \epsilon q(t)^{1-\rho} \mathcal{M}(\vartheta(t))\right)\right| \leq |\vartheta(t)| + |\epsilon| \left(1 + \left|\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\vartheta(t)\right|\right), \text{ for } |\epsilon| < \delta.$$

So, there exists positive numbers m and δ, such that

$$\left|\vartheta\left(t+\epsilon q\left(t\right)^{1-\rho}\mathcal{M}\left(\vartheta\left(t\right)\right)\right)\right| \leq m, \text{ whenever } |\epsilon|<\delta.$$

We take δ small enough so that $t+\epsilon q\left(t\right)^{1-\rho}\mathcal{M}\left(\vartheta\left(t\right)\right)\in\,]a,b[$. This yields that ϑ is locally bounded at t. □

Theorem 2.2. *If a function $\vartheta:[a,b]\to\mathbb{R}$ is ρ-differentiable at $t>a$, for $\rho\in\,]0,1[$, then ϑ is continuous at t.*

Proof. We know that

$$\vartheta\left(t+\epsilon q\left(t\right)^{1-\rho}\mathcal{M}\left(\vartheta\left(t\right)\right)\right)-\vartheta(t)=\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\vartheta(t)\cdot\epsilon,$$

then, we have

$$\lim_{\epsilon\to0}\left[\vartheta\left(t+\epsilon q\left(t\right)^{1-\rho}\mathcal{M}\left(\vartheta\left(t\right)\right)\right)-\vartheta(t)\right]=\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\vartheta(t)\cdot 0,$$

this implies that

$$\lim_{\epsilon\to0}\left[\vartheta\left(t+\epsilon q\left(t\right)^{1-\rho}\mathcal{M}\left(\vartheta\left(t\right)\right)\right)-\vartheta(t)\right]=0.$$

Now, setting $h:=\epsilon q\left(t\right)^{1-\rho}\mathcal{M}\left(\vartheta\left(t\right)\right)$ and using Lemma 2.1, we find

$$\lim_{h\to0}\left[\vartheta\left(t+h\right)-\vartheta\left(t\right)\right]=0.$$

This concludes the proof. □

Corollary 2.1. *A function ϑ that is ρ-differentiable on $]a,b[$ is also differentiable on that interval.*

Proof. Using the classical derivative definition, we obtain

$$\begin{aligned}\vartheta'\left(t\right) &= \lim_{\epsilon\to0}\frac{\left[\vartheta\left(t+\epsilon q\left(t\right)^{1-\rho}\mathcal{M}\left(\vartheta(t)\right)\right)-\vartheta(t)\right]}{\epsilon q\left(t\right)^{1-\rho}\mathcal{M}\left(\vartheta\left(t\right)\right)}\\ &= \lim_{\epsilon\to0}\frac{\left[\vartheta\left(t+\epsilon q\left(t\right)^{1-\rho}\mathcal{M}\left(\vartheta(t)\right)\right)-\vartheta(t)\right]}{\epsilon q\left(t\right)^{1-\rho}\mathcal{M}\left(\vartheta\left(t\right)\right)}\\ &= \frac{q\left(t\right)^{\rho-1}}{\mathcal{M}\left(\vartheta\left(t\right)\right)}\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\vartheta(t).\end{aligned}$$

So, we get the result done. □

Theorem 2.3. *Let ϑ be defined in $[a,b]$. For any $\rho\in\,]0,1[$, the function ϑ is ρ-differentiable if and only if it is differentiable.*

Proof. It is concluded from Theorem 2.1 and Corollary above. □

Next, we consider the possibility of $\rho\in\,]n,n+1[$ for some $n\in\mathbb{N}$. We have the following definition.

Definition 2.2. *Let ϑ be n-times differentiable at $t\in\,]a,b[$. For any $\rho\in\,]n,n+1[$ we naturally extend the concept of the $\mathcal{M}$-conformable derivative and define it using the following limit*

$$\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\vartheta(t):=\lim_{\epsilon\to0}\frac{\vartheta^{(n)}\left(t+\epsilon q\left(t\right)^{n+1-\rho}\mathcal{M}\left(\vartheta^{(n)}\left(t\right)\right)\right)-\vartheta^{(n)}(t)}{\epsilon}.$$

Remark 2. *If $\vartheta^{(n+1)}$ exists, we have*

$$\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\vartheta(t) = q\left(t\right)^{n+1-\rho}\vartheta^{(n+1)}(t)\mathcal{M}\left(\vartheta^{(n)}\left(t\right)\right),$$

for any $\rho \in \left]n, n+1\right[$.

We now present some properties of the $\mathcal{M}$-conformable derivative.

Theorem 2.4. *Let $\rho \in \left]0,1\right[$ and $\vartheta, \eth$ be two functions that are ρ-differentiable at a point $t > 0$. Then*

1. $\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}[\mu\vartheta + \nu\eth] = \left(\mu\frac{\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\vartheta}{\mathcal{M}(\vartheta)} + \nu\frac{\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\eth}{\mathcal{M}(\eth)}\right)\mathcal{M}\left(\mu\vartheta + \nu\eth\right)$*, for all* $\mu, \nu \in \mathbb{R}$.
2. $\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}(C) = 0$.
3. $\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\left(\vartheta.\eth\right) = \left(\eth\frac{\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\vartheta}{\mathcal{M}(\vartheta)} + \vartheta\frac{\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\eth}{\mathcal{M}(\eth)}\right)\mathcal{M}\left(\vartheta\eth\right)$.
4. $\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\left(\frac{\vartheta}{\eth}\right) = \left(\frac{\eth\,\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\vartheta}{\eth^2\mathcal{M}(\vartheta)} - \frac{\vartheta\,\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\eth}{\eth^2\mathcal{M}(\eth)}\right)\mathcal{M}\left(\frac{\vartheta}{\eth}\right)$*, for all* $\eth$ *is a non-zero function.*
5. $\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}(\vartheta\circ\eth) = q\left(\cdot\right)^{1-\rho}\eth'\left(\vartheta'\circ\eth\right)\mathcal{M}\left(\vartheta\circ\eth\right)$.

Proof. 1. From definition, we have

$$\begin{aligned}\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\left(\mu\vartheta + \nu\eth\right) &= q(\cdot)^{1-\rho}\left(\mu\vartheta' + \nu\eth'\right)\mathcal{M}\left(\mu\vartheta + \nu\eth\right) \\ &= \left(\mu\frac{\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\vartheta}{\mathcal{M}\left(\vartheta\right)} + \nu\frac{\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\eth}{\mathcal{M}\left(\eth\right)}\right)\left(\mathcal{M}\left(\mu\vartheta + \nu\eth\right)\right).\end{aligned}$$

2. Is evident from equality (1).

3. Product rule

$$\begin{aligned}\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\left(\vartheta\eth\right) &= q(\cdot)^{1-\rho}\left(\vartheta'\eth + \vartheta\eth'\right)\mathcal{M}\left(\vartheta\eth\right) \\ &= \left(\eth\frac{\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\vartheta}{\mathcal{M}\left(\vartheta\right)} + \vartheta\frac{\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\eth}{\mathcal{M}\left(\eth\right)}\right)\mathcal{M}\left(\vartheta\eth\right).\end{aligned}$$

4. Quotient rule

$$\begin{aligned}\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\left(\frac{\vartheta}{\eth}\right) &= q(\cdot)^{1-\rho}\left(\frac{\vartheta'\eth - \vartheta\ \eth'}{\eth^2}\right)\mathcal{M}\left(\frac{\vartheta}{\eth}\right) \\ &= \left(\frac{\eth\ \mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}(\vartheta)}{\eth^2\mathcal{M}\left(\vartheta\right)} - \frac{\vartheta\ \mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}(\eth)}{\eth^2\mathcal{M}\left(\eth\right)}\right)\mathcal{M}\left(\frac{\vartheta}{\eth}\right).\end{aligned}$$

5. Chain rule

$$\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}(\vartheta\circ\eth) = q\left(\cdot\right)^{1-\rho}\left(\vartheta\circ\eth\right)'\mathcal{M}\left(\vartheta\circ\eth\right).$$

□

The $\mathcal{M}$-conformable fractional derivative of certain functions.

Proposition 1. *1.* $\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\left(t^n\right) = nt^{n-1}q\left(t\right)^{1-\rho}\mathcal{M}\left(t^n\right)$*, for all* $n \in \mathbb{R}$.

2. $\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\left(e^t\right) = q\left(t\right)^{1-\rho}e^t\mathcal{M}\left(e^t\right)$.
3. $\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\left(\cos\left(t\right)\right) = -q\left(t\right)^{1-\rho}\sin\left(t\right)\mathcal{M}\left(\cos\left(t\right)\right)$.
4. $\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\left(\sin\left(t\right)\right) = q\left(t\right)^{1-\rho}\cos\left(t\right)\mathcal{M}\left(\sin\left(t\right)\right)$.
5. $\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\left(\log\left(t\right)\right) = q\left(t\right)^{1-\rho}t^{-1}\mathcal{M}\left(\log\left(t\right)\right)$.

3. $\tilde{\mathcal{M}}$-Conformable Fractional Integral

In fractional calculus the fractional integral which serves as the inverse of the fractional derivative is just as crucial as the fractional derivative itself. Now, we introduce the fractional integral as the inverse operator for the $\mathcal{M}$-conformable derivative. Throughout this section we assume all functions to be continuous.

In the following, let $\tilde{\mathcal{M}}$ denote the primitive function of $\mathcal{M}$ and $\tilde{\mathcal{M}}^{-1}$ the inverse function of $\tilde{\mathcal{M}}$.

Definition 3.3 ($\tilde{\mathcal{M}}$-Conformable Fractional Integral)**.** *Let $t \in [a,b]$ and ϑ be a function defined on $]a,t]$. We assume that the function $\tilde{\mathcal{M}}$ is bijective. Then, the $\tilde{\mathcal{M}}$-conformable fractional integral of ϑ is defined by*

$$ {}_{a}^{t}\mathcal{I}^{\rho}_{q(\cdot),\tilde{\mathcal{M}}}\vartheta(t) = \tilde{\mathcal{M}}^{-1}\left(\int_a^t \frac{\vartheta(x)}{q(x)^{1-\rho}}dx\right), \tag{2} $$

if the Riemann improper integral exists and $\left(\int_a^t \frac{\vartheta(x)}{q(x)^{1-\rho}}dx\right) \in Dom\left(\tilde{\mathcal{M}}^{-1}\right)$.

Remark 3. *If $q(x) := (t-x)$ we get ${}_{a}^{t}\mathcal{I}^{\rho}_{q(\cdot),\tilde{\mathcal{M}}}\vartheta(t) = \tilde{\mathcal{M}}^{-1}\left(\Gamma(\rho)\, {}^{RL}I_a^{\rho}\vartheta(t)\right)$, with ${}^{RL}I_a^{\rho}\vartheta(t)$ is Riemann-Liouville fractional integral.*

It is interesting to observe that the $\mathcal{M}$-conformable fractional derivative and the $\mathcal{M}$-conformable fractional integral are inverse of each other as given in the next result.

Theorem 3.5 (Inverse property)**.** *Let $\rho \in]0,1[$ and ϑ be a continuous function such that ${}_{a}^{t}\mathcal{I}^{\rho}_{q(\cdot),\mathcal{M}}\vartheta(t)$ exists. Then*

$$ \mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\left({}_{a}^{t}\mathcal{I}^{\rho}_{q(\cdot),\tilde{\mathcal{M}}}\vartheta(t)\right) = \vartheta(t), \quad \text{for } t \geq a, $$

and

$$ {}_{a}^{t}\mathcal{I}^{\rho}_{q(\cdot),\tilde{\mathcal{M}}}(\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\vartheta(t)) = \tilde{\mathcal{M}}^{-1}\left(\tilde{\mathcal{M}}(\vartheta(t)) - \tilde{\mathcal{M}}(\vartheta(a))\right), \quad \text{for } t \geq a. \tag{3} $$

Proof. Let $P(t)$ is a differentiable function over $[a,b]$. Since ϑ is given to be continuous so ${}_{a}^{t}\mathcal{I}^{\rho}_{q(\cdot),\tilde{\mathcal{M}}}\vartheta(t)$ is ρ-differentiable.

If we put $P(t) := {}_{a}^{t}\mathcal{I}^{\rho}_{q(\cdot),\tilde{\mathcal{M}}}\vartheta(t)$, then we have

$$ \begin{aligned} \mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\left({}_{a}^{t}\mathcal{I}^{\rho}_{q(\cdot),\tilde{\mathcal{M}}}\vartheta(t)\right) &= \mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}P(t) \\ &= q(t)^{1-\rho}P'(t)\mathcal{M}(P). \end{aligned} $$

We know that a particular solution of the differential equation

$$ q(t)^{1-\rho}P'(t)\mathcal{M}(P) = \vartheta(t), $$

is given as

$$ P(t) = \tilde{\mathcal{M}}^{-1}\left(\int_a^t \frac{\vartheta(x)}{q(x)^{1-\rho}}dx\right). $$

For the second part, we set

$$ P(t) := \mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\vartheta(t) = q(t)^{1-\rho}\vartheta'(t)\mathcal{M}(\vartheta). $$

By integrating both sides, we notice that

$$\begin{aligned} {}_{a}^{t}\mathcal{I}_{q(\cdot),\tilde{\mathcal{M}}}^{\rho} P(t) &= {}_{a}^{t}\mathcal{I}_{q(\cdot),\tilde{\mathcal{M}}}^{\rho} \left[q(t)^{1-\rho} \vartheta'(t) \mathcal{M}(\vartheta) \right] \\ &= \tilde{\mathcal{M}}^{-1} \left(\int_a^t \vartheta'(t) \mathcal{M}(\vartheta)\, dx \right). \end{aligned}$$

On the other hand, we have

$$\int_a^t \vartheta'(t) \mathcal{M}(\vartheta)\, dx = \tilde{\mathcal{M}}(\vartheta(t)) - \tilde{\mathcal{M}}(\vartheta(a)).$$

Then, form this we get (3). □

Theorem 3.6. *The $\tilde{\mathcal{M}}$ conformable integral ${}_{a}^{t}\mathcal{I}_{q(\cdot),\tilde{\mathcal{M}}}^{\rho}(\cdot)$ exhibits the following properties*

a) ${}_{a}^{t}\mathcal{I}_{q(\cdot),\tilde{\mathcal{M}}}^{\rho}(\mu\vartheta + \nu\eth) = \tilde{\mathcal{M}}^{-1}\left(\mu \int_a^t \frac{\vartheta(x)}{q(x)^{1-\rho}} dx + \nu \int_a^t \frac{\eth(x)}{q(x)^{1-\rho}} dx\right)$*, for all* $\mu, \nu \in \mathbb{R}$.

b) Integration by parts

$$\int_a^b \eth\vartheta' \mathcal{M}(\vartheta\eth)\, dx = \tilde{\mathcal{M}}(\vartheta\eth(b)) - \tilde{\mathcal{M}}(\vartheta\eth(a)) - \int_a^b \vartheta\eth' \mathcal{M}(\vartheta\eth)\, dx,$$

and

$$\int_a^b \frac{\eth \mathfrak{D}_{q(\cdot),\mathcal{M}}^{\rho} \vartheta}{q(x)^{1-\rho} \mathcal{M}(\vartheta)} \mathcal{M}(\vartheta\eth)\, dx = \tilde{\mathcal{M}}(\vartheta\eth(b)) - \tilde{\mathcal{M}}(\vartheta\eth(a)) - \int_a^b \frac{\vartheta \mathfrak{D}_{q(\cdot),\mathcal{M}}^{\rho} \eth}{q(x)^{1-\rho} \mathcal{M}(\eth)} \mathcal{M}(\vartheta\eth)\, dx.$$

Proof. The first formula easily follows the definition (2). To prove the formula of integration by parts, we consider the following identity

$$\mathfrak{D}_{q(\cdot),\mathcal{M}}^{\rho}(\vartheta\eth) = q(\cdot)^{1-\rho} (\eth\vartheta' + \vartheta\eth') \mathcal{M}(\vartheta\eth),$$

by integrating both sides, we find

$${}_{a}^{b}\mathcal{I}_{q(\cdot),\mathcal{M}}^{\rho} \left[\mathfrak{D}_{q(\cdot),\mathcal{M}}^{\rho} \vartheta\eth \right] = {}_{a}^{b}\mathcal{I}_{q(\cdot),\tilde{\mathcal{M}}}^{\rho} \left[q(x)^{1-\rho} (\eth\vartheta' + \vartheta\eth') \mathcal{M}(\vartheta\eth) \right].$$

Using the formula (3), we obtain

$$\tilde{\mathcal{M}}^{-1}\left(\tilde{\mathcal{M}}(\vartheta\eth(b)) - \tilde{\mathcal{M}}(\vartheta\eth(a)) \right) = \tilde{\mathcal{M}}^{-1}\left(\int_a^b (\eth\vartheta' + \vartheta\eth') \mathcal{M}(\vartheta\eth)\, dx \right),$$

then

$$\tilde{\mathcal{M}}(\vartheta\eth(b)) - \tilde{\mathcal{M}}(\vartheta\eth(a)) = \int_a^b \eth\vartheta' \mathcal{M}(\vartheta\eth)\, dx + \int_a^b \vartheta\eth' \mathcal{M}(\vartheta\eth)\, dx,$$

this implies that

$$\int_a^b \eth\vartheta' \mathcal{M}(\vartheta\eth)\, dx = \tilde{\mathcal{M}}(\vartheta\eth(b)) - \tilde{\mathcal{M}}(\vartheta\eth(a)) - \int_a^b \vartheta\eth' \mathcal{M}(\vartheta\eth)\, dx. \tag{4}$$

Now, by the formula (4), we find

$$\int_a^b \frac{\eth \mathfrak{D}_{q(\cdot),\mathcal{M}}^{\rho} \vartheta}{q(x)^{1-\rho} \mathcal{M}(\vartheta)} \mathcal{M}(\vartheta\eth)\, dx = \tilde{\mathcal{M}}(\vartheta\eth(b)) - \tilde{\mathcal{M}}(\vartheta\eth(a)) - \int_a^b \frac{\vartheta \mathfrak{D}_{q(\cdot),\mathcal{M}}^{\rho} \eth}{q(x)^{1-\rho} \mathcal{M}(\eth)} \mathcal{M}(\vartheta\eth)\, dx.$$

□

Now we present the integration of some functions.

Proposition 2. *Let $q(t) := t$, we have*

1. ${}_0^t\mathcal{I}^{\rho}_{t,\tilde{\mathcal{M}}}\lambda = \tilde{\mathcal{M}}^{-1}\left(\lambda \frac{t^{\rho}}{\rho}\right)$.
2. ${}_0^t\mathcal{I}^{\rho}_{t,\tilde{\mathcal{M}}}e^{-t} = \tilde{\mathcal{M}}^{-1}\left(\gamma(\rho,t)\right)$, *where $\gamma(\cdot,\cdot)$ is incomplete gamma function.*
3. ${}_0^t\mathcal{I}^{\rho}_{t,\tilde{\mathcal{M}}}(t^n) = \tilde{\mathcal{M}}^{-1}\left(\frac{t^{\rho+n}}{\rho+n}\right)$.
4. ${}_0^t\mathcal{I}^{\rho}_{t,\tilde{\mathcal{M}}}\log(t) = \tilde{\mathcal{M}}^{-1}\left(\frac{t^{\rho}\log(t)}{\rho} - \frac{t^{\rho}}{\rho^2}\right)$.

4. Some Important Theorems on $\mathcal{M}$-Conformable Derivative

In this section, we prove Rolle's theorem, the Mean Value theorem and the L'Hôpital's rule for the $\mathcal{M}$-conformable fractional derivative.

Theorem 4.7 (Rolle's theorem for $\mathcal{M}$-Conformable Fractional Differentiable Functions)**.** *Let $\vartheta : [a,b] \to \mathbb{R}$ be a function with the properties that*

1. *ϑ is continuous on $[a,b]$.*
2. *ϑ is ρ-differentiable on $]a,b[$ for some $\rho \in \,]0,1[$.*
3. *$\vartheta(a) = \vartheta(b)$.*

Then, there exists $c \in \,]a,b[$, such that $\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\vartheta(c) = 0$.

Proof. According to Theorem 2.3, we have ϑ is differentiable. On the other hand, since ϑ is continuous on $[a,b]$ and $\vartheta(a) = \vartheta(b)$, there exists $c \in \,]a,b[$ at which the function has a local extrema. This means that

$$\vartheta'(c) = 0,$$

then

$$q\left(c\right)^{1-\rho}\vartheta'(c)\mathcal{M}\left(\vartheta(c)\right) = 0.$$

Hence

$$\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\vartheta(c) = 0.$$

□

Theorem 4.8 (Mean Value Theorem for $\mathcal{M}$-Conformable Fractional Differentiable Functions)**.** *Let $\vartheta : [a,b] \to \mathbb{R}$ be a function with the properties that*

- *ϑ is continuous on $[a,b]$.*
- *ϑ is ρ-differentiable on $]a,b[$ for some $\rho \in \,]0,1[$.*

Then, there exists $c \in \,]a,b[$ such that $\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\vartheta(c) = \frac{\vartheta(b)-\vartheta(a)}{b-a}q\left(c\right)^{1-\rho}\mathcal{M}\left(\vartheta(c)\right)$.

Proof. Consider the function

$$\eth(t) = \vartheta(t) - \vartheta(a) - \frac{\vartheta(b)-\vartheta(a)}{b-a}t.$$

Then, the function $\eth$ satisfies the conditions of the fractional Rolle's theorem. Hence, there exists $c \in \,]a,b[$ such that

$$\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\eth(c) = 0.$$

Using the fact that

$$\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\eth(t) = \left[\frac{\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\vartheta(t)}{\mathcal{M}\left(\vartheta(t)\right)} - \frac{\vartheta(b)-\vartheta(a)}{b-a}\frac{\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}t}{\mathcal{M}\left(t\right)}\right]\mathcal{M}\left(\eth(t)\right),$$

and $\mathcal{M}\left(\eth(c)\right) \neq 0$, we get

$$\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\vartheta(c) = \frac{\vartheta(b)-\vartheta(a)}{b-a}q\left(c\right)^{1-\rho}\mathcal{M}\left(\vartheta(c)\right).$$

□

Theorem 4.9. *Let $G : [a,b] \to \mathbb{R}$, be a given function that satisfies*
1. *G is continuous on $[a,b]$.*
2. *G is ρ-differentiable on $]a,b[$.*
3. $\sup\limits_{[a,b]} \left| \mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}} G \right| = G_0$ *and* $\min\limits_{[a,b]} q\left(\cdot\right)^{1-\rho} \left|\mathcal{M}\left(G\right)\right| = q_0$, *with* $q_0 > G_0$.

Then, G is a contractive.

Proof. Let $t, s \in [a,b]$. By Theorem 4.8, there exists $c \in\,]a,b[$ such that

$$\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}} G(c) = \frac{G(t)-G(s)}{t-s} q\left(c\right)^{1-\rho} \mathcal{M}\left(G\left(c\right)\right).$$

Therefore, we have

$$\begin{aligned} |G(t)-G(s)| &\leq \left| \frac{\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}} G(c)}{q\left(c\right)^{1-\rho} \mathcal{M}\left(G\left(c\right)\right)} \right| |t-s| \\ &\leq \frac{G_0}{q_0} \left|t-s\right|. \end{aligned}$$

Then, G is a contractive. □

Proposition 3. *Let ϑ and $\eth : [a,b] \to \mathbb{R}$ be two given functions such that*
- *ϑ and $\eth$ are continuous on $[a,b]$.*
- *ϑ and $\eth$ are ρ-differentiable on $]a,b[$ for some $\rho \in\,]0,1[$.*

Then, there exists $c \in\,]a,b[$ such that

$$\left[\frac{\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}} \eth\left(c\right)}{\mathcal{M}\left(\eth\left(c\right)\right)} \right] \left(\vartheta\left(a\right) - \vartheta\left(b\right)\right) = \left[\frac{\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}} \vartheta\left(c\right)}{\mathcal{M}\left(\vartheta\left(c\right)\right)} \right] \left(\eth\left(a\right) - \eth\left(b\right)\right).$$

Proof. Consider the function Λ defined by

$$\Lambda\left(t\right) = \vartheta(t)\left(\eth(b) - \eth(a)\right) - \eth(t)\left(\vartheta(b) - \vartheta(a)\right),$$

then, the function Λ satisfies the conditions of Theorem 4.7. Hence, there exists $c \in\,]a,b[$ such that

$$\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}} \Lambda\left(c\right) = 0,$$

where

$$\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}} \Lambda\left(c\right) = q^{1-\rho}\left(t\right) \Lambda'\left(c\right) \mathcal{M}\left(\Lambda\left(c\right)\right).$$

Then, there exists $c \in\,]a,b[$ such that

$$\left(\left(\eth(b) - \eth(a)\right) \frac{\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}} \vartheta\left(c\right)}{\mathcal{M}\left(\vartheta\left(c\right)\right)} - \left(\vartheta(b) - \vartheta(a)\right) \frac{\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}} \eth\left(c\right)}{\mathcal{M}\left(\eth\left(c\right)\right)} \right) \mathcal{M}\left(\Lambda\left(c\right)\right) = 0.$$

We have $\mathcal{M}\left(\Lambda\left(c\right)\right) \neq 0$, then we get the result. □

The following result provide a generalization of L'Hôpital's rule.

Theorem 4.10. *Let $\vartheta, \eth : [a,b] \longrightarrow \mathbb{R}$ be two given functions such that*
1. *ϑ and $\eth$ are continuous on $[a,b]$.*
2. *ϑ and $\eth$ are ρ-differentiable on $]a,b[$, for some $\rho \in\,]0,1[$.*
3. *$\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}} \eth$ is never zero on $]a,b[$, and that* $\lim_{t\to a^+} \frac{\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}} \vartheta(t)}{\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}} \eth(t)} \cdot \frac{\mathcal{M}(\eth(t))}{\mathcal{M}(\vartheta(t))} = l$.

Then

$$\lim_{t\to a^+}\frac{\vartheta(t)-\vartheta(a)}{\eth(t)-\eth(a)}=\lim_{t\to a^+}\frac{\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\vartheta(t)}{\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\eth(t)}.\frac{\mathcal{M}(\eth(t))}{\mathcal{M}(\vartheta(t))}=l.$$

Proof. By th hypotheses (1) and (2), we can apply Proposition 3. So, for all $t\in[a,b]$ there exists $c_t\in\,]a,t[$ such that

$$\left[\frac{\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\eth(c_t)}{\mathcal{M}(\eth(c_t))}\right](\vartheta(t)-\vartheta(a))=\left[\frac{\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\vartheta(c_t)}{\mathcal{M}(\vartheta(c_t))}\right](\eth(t)-\eth(a)).$$

It is clear that if $t\to a$, so $c_t\to a$, then, we have

$$\frac{\vartheta(t)-\vartheta(a)}{\eth(t)-\eth(a)}=\frac{\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\vartheta(t)}{\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\eth(t)}.\frac{\mathcal{M}(\eth(t))}{\mathcal{M}(\vartheta(t))}.$$

Finally by (3), we find

$$\lim_{t\to a^+}\frac{\vartheta(t)-\vartheta(a)}{\eth(t)-\eth(a)}=\lim_{t\to a^+}\frac{\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\vartheta(t)}{\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\eth(t)}.\frac{\mathcal{M}(\eth(t))}{\mathcal{M}(\vartheta(t))}=l.$$

□

5. Applications to $\mathcal{M}$-conformable differential equations

Now, we solve some fractional differential equations using the $\mathcal{M}$-conformable derivative operator $\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}(\cdot)$. In the first examples, we discuss methods for solving both homogeneous and non-homogeneous fractional differential equations. In the last, we address the Cauchy problem for $\mathcal{M}$-conformable fractional differential equations.

5.1. Examples for $\mathcal{M}$-conformable differential equations.

Example 1. *Consider the $\mathcal{M}$-conformable differential equation*

$$\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}y(t)+r(t)y(t)=0,$$

where $r(\cdot)$ is continuous. Using the expression given in (1), the equation gets transformed

$$q(t)^{1-\rho}y'(t)\mathcal{M}(y(t))+r(t)y(t)=0,$$

this implies that

$$\left(\tilde{\mathcal{M}}(y(t))\right)'=\frac{r(t)}{q(t)^{1-\rho}}.$$

By integration, we get

$$y(t)=\tilde{\mathcal{M}}^{-1}\left(c+\int\frac{r(t)}{q(t)^{1-\rho}}dt\right),$$

where c is arbitrary constant such that $\left(c+\int\frac{r(t)}{q(t)^{1-\rho}}dt\right)\in Dom\left(\tilde{\mathcal{M}}^{-1}\right)$.

Example 2. *We now consider a non-homogeneous $\mathcal{M}$-conformable differential equation*

$$\mathfrak{D}^{\rho}_{t,\mathcal{M}} y(t) = t, \quad t > 0,$$

where $\mathcal{M}(\cdot) = e^{-(\cdot)^2}$.

We have

$$\mathfrak{D}^{\rho}_{t,\mathcal{M}} y(t) = t^{1-\rho} y'(t) e^{-(y(t))^2},$$

then, by integration, we find

$$e^{-y(t)^2} = \frac{2t^{\rho+1}}{1+\rho} + 2c,$$

where $c > 0$ *is arbitrary constant.*

So, the solution is

$$y(t) = \pm\sqrt{\ln\left(\frac{2t^{\rho+1}}{1+\rho} + 2c\right)^{-1}},$$

with $t \leq \left[(\rho+1)(1/2 - c)\right]^{1/(1+\rho)}$ *and* $0 < c < 1/2$.

Example 3. *We consider following $\mathcal{M}$-conformable problem*

$$\begin{cases} \mathfrak{D}^{1/2}_{t,\sin(\cdot)}\left(\mathfrak{D}^{1/2}_{t,\cos(\cdot)} y(t)\right) = -t^{1/2}\pi \sin(\pi t), & t > 0, \\ \mathfrak{D}^{1/2}_{1,\cos(\cdot)} y(1) = \pi. \end{cases} \tag{5}$$

We have

$$\mathfrak{D}^{1/2}_{t,\cos(\cdot)} y(t) = t^{1/2} y'(t) \cos y(t).$$

So the first equation of (5) *becomes as follows*

$$t^{1/2}\left(t^{1/2} y'(t) \cos y(t)\right)' \sin\left(t^{1/2} y'(t) \cos y(t)\right) = -t^{1/2}\pi \sin(\pi t),$$

this implies that

$$\left[\cos\left(t^{1/2} y'(t) \cos y(t)\right)\right]' = -\pi \sin(\pi t).$$

Now, by integration, we find

$$\cos\left(t^{1/2} y'(t) \cos y(t)\right) = \cos(\pi t) + c,$$

using the fact that $\mathfrak{D}^{1/2}_{1,\cos(\cdot)} y(1) = \pi$, *we get* $c = 0$. *Then, we have*

$$y'(t) \cos y(t) = \pi t^{1/2},$$

then

$$\sin y(t) = \frac{2\pi}{3} t^{3/2} + c,$$

from this, we get

$$y(t) = \arcsin\left(\frac{2\pi}{3} t^{3/2} + c\right),$$

where $0 < t \leq \left[\frac{3}{2\pi}(1-c)\right]^{2/3}$ *and* $0 \leq c < 1$ *is arbitrary constant.*

5.2. Cauchy problem for $\mathcal{M}$-conformable differential equations.

Example 4. *We consider the following Cauchy problem*

$$\begin{cases} \mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}} y(t) = \mathfrak{F}\left(t, y(t)\right), \; t \in \left]0, T\right], \\ \\ y(0) = y_0. \end{cases} \tag{6}$$

To study this problem, we denote by $\mathcal{H} := C([0,T];\mathbb{R})$ the Banach space of all real-valued continuous functions defined on $[0,T]$. The norm in this space will be denoted by $\|y\|_\infty = \sup\limits_{[0,T]} |y(\cdot)|$. We also use the following notations

$$\mathcal{B}_r := \left\{ y \in \mathcal{H} \colon \; \|y\|_\infty \leq r \right\},$$

and

$$K_0 = \min_{[0,T]} q\left(\cdot\right)^{1-\rho}.$$

Proposition 4. *Let $\tilde{\mathcal{M}}$ be the primitive function of $\mathcal{M}$ and $\tilde{\mathcal{M}}^{-1}$ the inverse of $\tilde{\mathcal{M}}$. If*

$$\left(\int_0^t \frac{\mathfrak{F}\left(x, y(x)\right)}{q\left(x\right)^{1-\rho}} dx + \tilde{\mathcal{M}}\left(y_0\right) \right) \in Dom\left(\tilde{\mathcal{M}}^{-1}\right),$$

then, the system (6) *is equivalent to the following $\tilde{\mathcal{M}}$-conformable integral equation*

$$y\left(t\right) = \tilde{\mathcal{M}}^{-1} \left(\int_0^t \frac{\mathfrak{F}\left(x, y(x)\right)}{q\left(x\right)^{1-\rho}} dx + \tilde{\mathcal{M}}\left(y_0\right) \right). \tag{7}$$

Proof. By integrating both sides of (6), we find

$${}_0^t\mathcal{I}^{\rho}_{q(\cdot),\mathcal{M}} \left[\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}} y(t) \right] = {}_0^t\mathcal{I}^{\rho}_{q(\cdot),\mathcal{M}} \; \mathfrak{F}\left(t, y(t)\right).$$

Using the formula (3), we have

$$\tilde{\mathcal{M}}^{-1} \left(\tilde{\mathcal{M}}\left(y\left(t\right)\right) - \tilde{\mathcal{M}}\left(y\left(0\right)\right) \right) = \tilde{\mathcal{M}}^{-1} \left(\int_0^t \frac{\mathfrak{F}\left(x, y(x)\right)}{q\left(x\right)^{1-\rho}} dx \right),$$

this implies that

$$\tilde{\mathcal{M}}\left(y\left(t\right)\right) = \int_0^t \frac{\mathfrak{F}\left(x, y(x)\right)}{q\left(x\right)^{1-\rho}} dx + \tilde{\mathcal{M}}\left(y\left(0\right)\right),$$

then

$$y\left(t\right) = \tilde{\mathcal{M}}^{-1} \left(\int_0^t \frac{\mathfrak{F}\left(x, y(x)\right)}{q\left(x\right)^{1-\rho}} dx + \tilde{\mathcal{M}}\left(y\left(0\right)\right) \right).$$

Now, using the condition $y(0) = y_0$, we get

$$y(t) = \tilde{\mathcal{M}}^{-1} \left(\int_0^t \frac{\mathfrak{F}\left(x, y(x)\right)}{q\left(x\right)^{1-\rho}} dx + \tilde{\mathcal{M}}\left(y_0\right) \right).$$

Conversely, assuming (7) and applying the $\mathcal{M}$-conformable derivative operator $\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}(\cdot)$ to both sides of the equation, we find

$$\begin{aligned}
&\mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}y(t) = \mathfrak{D}^{\rho}_{q(\cdot),\mathcal{M}}\left[\tilde{\mathcal{M}}^{-1}\left(\int_0^t \frac{\mathfrak{F}(x,y(x))}{q(x)^{1-\rho}}dx + \tilde{\mathcal{M}}(y_0)\right)\right]\\
&= q(t)^{1-\rho}\left[\tilde{\mathcal{M}}^{-1}\left(\int_0^t \frac{\mathfrak{F}(x,y(x))}{q(x)^{1-\rho}}dx + \tilde{\mathcal{M}}(y_0)\right)\right]' \mathcal{M}\left(\tilde{\mathcal{M}}^{-1}\left(\int_0^t \frac{\mathfrak{F}(x,y(x))}{q(x)^{1-\rho}}dx + \tilde{\mathcal{M}}(y_0)\right)\right)\\
&= q(t)^{1-\rho}\left[\tilde{\mathcal{M}}\left(\tilde{\mathcal{M}}^{-1}\left(\int_0^t \frac{\mathfrak{F}(x,y(x))}{q(x)^{1-\rho}}dx + \tilde{\mathcal{M}}(y_0)\right)\right)\right]'\\
&= q(t)^{1-\rho}\left(\int_0^t \frac{\mathfrak{F}(x,y(x))}{q(x)^{1-\rho}}dx + \tilde{\mathcal{M}}(y_0)\right)'\\
&= \mathfrak{F}(t,y(t)).
\end{aligned}$$

□

Theorem 5.11. *Under the following assumptions*

1. $\mathfrak{F} : [0,T]\times\mathbb{R} \longrightarrow \mathbb{R}$ *is continuous.*
2. *There exists a constant* $\mu > 0$ *such that*

$$|\mathfrak{F}(t,u(t)) - \mathfrak{F}(t,y(t))| \leq \mu\,|u(t) - y(t)|,$$

for all $t \in [0,T]$ *and* $u, y : [0,T] \longrightarrow \mathbb{R}$.

3. *There exists a constant* $c_{\tilde{\mathcal{M}}} > 0$ *such that* $\left|\tilde{\mathcal{M}}^{-1}(u) - \tilde{\mathcal{M}}^{-1}(\eth)\right| \leq c_{\tilde{\mathcal{M}}}\,|u - \eth|$.
4. $\tilde{\mathcal{M}}^{-1}(0) = 0$.
5. $\mu T c_{\tilde{\mathcal{M}}} < K_0$ *and* $\frac{\left(A\,T + \left|\tilde{\mathcal{M}}(y_0)\right|\right)c_{\tilde{\mathcal{M}}}}{K_0 + \mu T c_{\tilde{\mathcal{M}}}} < r$, *where* $A := \sup_{[0,T]}|\mathfrak{F}(\cdot,0)|$.

The Cauchy problem (6) *has a unique solution.*

Proof. Firstly, we define the mapping $\mathcal{J} : \mathcal{H} \longrightarrow \mathcal{H}$ by

$$\mathcal{J}(y(t)) = \tilde{\mathcal{M}}^{-1}\left(\int_0^t \frac{\mathfrak{F}(x,y(x))}{q(x)^{1-\rho}}dx + \tilde{\mathcal{M}}(y_0)\right),$$

then, we show that $\mathcal{J}(\mathcal{B}_r) \subset \mathcal{B}_r$. Let $y \in \mathcal{B}_r$, we have

$$\mathcal{J}(y(t)) = \tilde{\mathcal{M}}^{-1}\left(\int_0^t \frac{\mathfrak{F}(x,y(x))}{q(x)^{1-\rho}}dx + \tilde{\mathcal{M}}(y_0)\right).$$

Using hypothesis (3) and (4), we find

$$\begin{aligned}
|\mathcal{J}(y(t))| &\leq c_{\tilde{\mathcal{M}}}\left|\tilde{\mathcal{M}}(y_0)\right| + c_{\tilde{\mathcal{M}}}\int_0^t \frac{|\mathfrak{F}(x,y(x))|}{q(x)^{1-\rho}}dx\\
&\leq c_{\tilde{\mathcal{M}}}\left|\tilde{\mathcal{M}}(y_0)\right| + c_{\tilde{\mathcal{M}}}\int_0^t q(x)^{\rho-1}\left(|\mathfrak{F}(x,y(x)) - \mathfrak{F}(x,0)| + |\mathfrak{F}(x,0)|\right)dx\\
&\leq c_{\tilde{\mathcal{M}}}\left|\tilde{\mathcal{M}}(y_0)\right| + \frac{c_{\tilde{\mathcal{M}}}(\mu r + A)T}{K_0}\\
&\leq r.
\end{aligned}$$

Now, for all $u, y \in \mathcal{B}_r$, we get

$$\begin{aligned}
&\left|\mathcal{J}\left(u(t)\right)-\mathcal{J}\left(y(t)\right)\right| \\
&\leq \left|\tilde{\mathcal{M}}^{-1}\left(\int_0^t \frac{\mathfrak{F}\left(x,u(x)\right)}{q\left(x\right)^{1-\rho}}dx+\tilde{\mathcal{M}}\left(y_0\right)\right)-\tilde{\mathcal{M}}^{-1}\left(\int_0^t \frac{\mathfrak{F}\left(x,u(x)\right)}{q\left(x\right)^{1-\rho}}dx+\tilde{\mathcal{M}}\left(y_0\right)\right)\right| \\
&\leq c_{\tilde{\mathcal{M}}}\int_0^t \frac{1}{q\left(x\right)^{1-\rho}}\left|\mathfrak{F}\left(x,u(x)\right)-\mathfrak{F}\left(x,y(x)\right)\right|dx \\
&\leq \frac{\mu T c_{\tilde{\mathcal{M}}}}{K_0}\left\|u-y\right\|_\infty,
\end{aligned}$$

this implies that

$$\left\|\mathcal{J}\left(u(\cdot)\right)-\mathcal{J}\left(y(\cdot)\right)\right\|_\infty \leq \frac{\mu T c_{\tilde{\mathcal{M}}}}{K_0}\left\|u-y\right\|_\infty.$$

Since $\frac{\mu T c_{\tilde{\mathcal{M}}}}{K_0} < 1$, the mapping $\mathcal{J}$ is a contraction. Then, $\mathcal{J}$ has a unique fixed point which is the solution of the Cauchy problem. □

Remark 4. *For example, we can take* $\mathcal{M}(\cdot) := \cos(\cdot)$*, in this case we find* $\tilde{\mathcal{M}}(\cdot) := \sin(\cdot)$ *and* $\tilde{\mathcal{M}}^{-1}(\cdot) := \arcsin(\cdot)$.

6. Conclusion

This paper introduces a new fractional derivative, the $\mathcal{M}$-conformable derivative, along with its inverse operator. This derivative represents a generalization of the conformable derivative proposed by Khalil et al [17]. The examples provided illustrate that this operator opens up promising avenues for modeling various phenomena that cannot be adequately addressed using the classical derivative. By adjusting the kernel function $\mathcal{M}$, the $\mathcal{M}$-conformable derivative offers an effective and flexible tool for studying a wide range of local phenomena that cannot be treated by traditional differential models.

Several open questions naturally arise from this study:

(a) What is the geometric interpretation and physical relevance of the $\mathcal{M}$-conformable derivative?

(b) Can the function $\mathcal{M}$ be regarded as an operator, and what are the implications of this perspective for the analytical properties of the definition?

In addition to these questions, potential avenues for further research include exploring the application of the $\mathcal{M}$-conformable derivative in defining generalized integral transforms. For example, one could develop a generalized Laplace transform within this framework. Additionally, the $\mathcal{M}$-conformable derivative may provide a foundation for extending classical fractional derivatives, such as the Caputo and Riemann-Liouville types, particularly in contexts like epidemiological modeling (see for example ([4])).

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