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A NEW SUBCLASS OF ANALYTIC FUNCTIONS CONNECTED WITH GALUE-TYPE FUNCTION

VIHAR S.SHINDE, S. B. CHAVHAN

Abstract. The study of the geometric properties of analytic functions and their numerous applications in a variety of mathematical fields, including fractional calculus, probability distributions, and special functions, has drawn significant and impressive attention to Geometric Function Theory (GFT), one of the most prominent branches of complex analysis, in recent years.In this work, we introduce and investigate a new subclass of analytic functions in the open unit disc E with negative coefficients defined by Galue-type Struve function. The object of the present paper is to determine the coefficient inequality, distortion properties, radii properties, extreme points and convolution results for this class.

1. Introduction

Let A denote the class of all functions $u(z)$ of the form

$$u(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions and satisfy the following usual normalization condition $u(0) = u'(0) - 1 = 0$. We denote by S the subclass of A consisting of functions $u(z)$ which are all univalent in E. A function $u \in A$ is a starlike function of the order $\upsilon, 0 \leq \upsilon < 1$, if it satisfy

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \upsilon, (z \in E). \tag{2}$$

We denote this class with $S^*(\upsilon)$.

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A function $u \in A$ is a convex function of the order $\upsilon, 0 \leq \upsilon < 1$, if it satisfy

$$\Re\left\{1 + \frac{zu''(z)}{u'(z)}\right\} > \upsilon, (z \in E). \quad (3)$$

We denote this class with $K(\upsilon)$.

Let T denote the class of functions analytic in E that are of the form

$$u(z) = z - \sum_{n=2}^{\infty} a_n z^n,\ a_n \geq 0\ (z \in E) \quad (4)$$

and let $T^*(\upsilon) = T \cap S^*(\upsilon),\ C(\upsilon) = T \cap K(\upsilon)$. The class $T^*(\upsilon)$ and allied classes possess some interesting properties and have been extensively studied by Silverman [10] and others. For $u \in A$ given by (1) and $g(z)$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (5)$$

their convolution (or Hadamard product), denoted by $(u * g)$, is defined as

$$(u * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * u)(z)\ (z \in E). \quad (6)$$

Note that $u * g \in A$.

The Galue-type Struve function (GTSF) was introduced in [4, 8] and defined by

$$\alpha\mathcal{W}^{\lambda,\mu}_{p,b,c,\xi}(z) = z + \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma(\lambda n + \mu)\Gamma(\alpha n + \frac{p}{\xi} + \frac{b+2}{2})} (\frac{z}{2})^{2n+p+1} \quad (z \in E). \quad (7)$$

where $\alpha \in \mathbb{N},\ z, p, b, c \in \mathbb{C},\ \lambda > 0,\ \xi > 0$ and μ is an arbitrary parameter. It is evident that when $\lambda = \alpha = 1, \mu = 3/2$ and $\xi = 1$ in (7) then we have the generalized Struve function (see [5, 6]) defined by

$$\mathcal{H}_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma(n + 3/2)\Gamma(n + p + \frac{b+2}{2})} (\frac{z}{2})^{2n+p+1} \quad (z \in E) \quad (8)$$

where $z, p, b, c \in \mathbb{C}$. Using (7), consider the function

$$\mathcal{U}_{p,b,c,\xi}(z) = 2^p \sqrt{\pi}\ \Gamma(\frac{p}{\xi} + \frac{b+c}{2}) z^{\frac{-(p+1)}{2}} \alpha\mathcal{W}^{\lambda,\mu}_{p,b,c,\xi}(\sqrt{z}) \quad (z \in E) \quad (9)$$

Using the Pochhammer symbol defined in terms of Euler's gamma function, Oyekan [7] presented the relation

$$(\gamma)_n = \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)} = \gamma(\gamma + 1)\cdots(\gamma + n - 1)$$

so that from (9), we have

$$\mathcal{V}_{p,b,c,\xi}(z) = z\mathcal{U}_{p,b,c,\xi}(z) = z + \sum_{n=2}^{\infty} \left(\frac{(\frac{-c}{4})^n}{(\mu)_{\lambda(n-1)(\gamma)_{\alpha(n-1)}}} \right) z^n \quad (z \in E). \quad (10)$$

Using the convolution principle, Oyekan [7] defined the function

$$\mathcal{L}^{\lambda,\mu,\xi}_{p,b,c}(z) = (f * \mathcal{V}_{p,b,c,\xi})(z) = z + \sum_{n=2}^{\infty} \left(\frac{(\frac{-c}{4})^n}{(\mu)_{\lambda(n-1)(\gamma)_{\alpha(n-1)}}} \right) a_n z^n \quad (z \in E). \quad (11)$$

$p, b, c \in \mathbb{C}, \gamma = \frac{p}{\xi} + \frac{b+2}{2} \neq 0, -1, -2, \cdots, \alpha \in \mathbb{N}, \lambda, \xi > 0$ and μ is an arbitrary parameter. Function $\mathcal{V}$ in (10) is the normalized form of Galue-type Struve function and is analytic in $\mathbb{C}$, while (10) is the simplied version.
A special function that occurs in probability, statistics, material science and partial differential equation is the error function. The error function is use in quantum mechanics to eliminate the probability of observing a particle in a specified region.The error function

$$eru(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^{n-1} z^{n+1}}{(2n+1)!} \tag{12}$$

was reported in [1] and for additional information see [2, 3]. In particular, Ramchandran et al. [9] made a slight modification to (12) and came up with the function

$$Eru(z) = z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(2n-1)(n-1)!} z^n \quad (z \in E). \tag{13}$$

where the function $Eru(z)$ was used to define a class of analytic functions and solved some coefficient problems.
Using the convolution concept, and in view of (11) and (13), we can deduce the function

$$\mathcal{G}(z) = (\mathcal{L}_{p,b,c}^{\lambda,\mu,\xi} * Eru)(z) = z + \sum_{n=2}^{\infty} \phi_n(\mu, \lambda, \gamma, c) a_n z^n \tag{14}$$

where

$$\phi_n = \phi_n(\mu, \lambda, \gamma, c) = \left(\frac{(\frac{-c}{4})^{n-1}}{(2n-1)(n-1)!(\mu)_{\lambda(n-1)(\gamma)_{\alpha(n-1)}}} \right)$$

Definition 1 For $0 \leq \omega < 1, 0 \leq \sigma < 1$ and $0 < \varsigma < 1$, we let $TS(\omega, \sigma, \varsigma)$ be the subclass of u consisting of functions of the form (4) and its geometrical condition satisfy

$$\left| \frac{\omega \left((\mathcal{G}(z))' - \frac{\mathcal{G}(z)}{z} \right)}{\sigma(\mathcal{G}(z))' + (1-\omega)\frac{\mathcal{G}(z)}{z}} \right| < \varsigma, \ z \in \mathbb{U}$$

where $\mathcal{G}(z)$, is given by (14).

2. Coefficient Inequality

In the following theorem, we obtain a necessary and sufficient condition for function to be in the class $TS(\omega, \sigma, \varsigma)$.

Theorem 1 Let the function u be defined by (4). Then $u \in TS(\omega, \sigma, \varsigma)$ if and only if

$$\sum_{n=2}^{\infty} [\omega(n-1) + \varsigma(n\sigma + 1 - \omega)] \phi(n) a_n \leq \varsigma(\sigma + (1-\omega)), \tag{15}$$

where $0 < \varsigma < 1, 0 \leq \omega < 1, 0 \leq \sigma < 1$ and $0 \leq \vartheta < 1$. The result (15) is sharp for the function

$$u(z) = z - \frac{\varsigma(\sigma + (1-\omega))}{[\omega(n-1) + \varsigma(n\sigma + 1 - \omega)]\phi(n)} z^n, \ n \geq 2.$$

Proof. Suppose that the inequality (15) holds true and $|z| = 1$. Then we obtain

$$\begin{aligned}
&\left|\omega\left((\mathcal{G}(z))' - \frac{\mathcal{G}(z)}{z}\right)\right| - \varsigma\left|\sigma\left(\mathcal{G}(z))' + (1-\omega)\frac{\mathcal{G}(z)}{z}\right)\right| \\
=&\left|-\omega\sum_{n=2}^{\infty}(n-1)\phi(n)a_n z^{n-1}\right| \\
&-\varsigma\left|\sigma + (1-\omega) - \sum_{n=2}^{\infty}(n\sigma + 1 - \omega)\phi(n)a_n z^{n-1}\right| \\
\le&\sum_{n=2}^{\infty}[\omega(n-1) + \varsigma(n\sigma + 1 - \omega)]\phi(n)a_n - \varsigma(\sigma + (1-\omega)) \\
\le&\ 0.
\end{aligned}$$

Hence, by maximum modulus principle,$u \in TS(\omega, \sigma, \varsigma)$. Now assume that $u \in TS(\omega, \sigma, \varsigma)$ so that

$$\left|\frac{\omega\left((\mathcal{G}(z))' - \frac{\mathcal{G}(z)}{z}\right)}{\sigma(\mathcal{G}(z))' + (1-\omega)\frac{\mathcal{G}(z)}{z}}\right| < \varsigma,\ z \in \mathbb{U}.$$

Hence,

$$\left|\omega\left((\mathcal{G}(z))' - \frac{\mathcal{G}(z)}{z}\right)\right| < \varsigma\left|\sigma\left(\mathcal{G}(z))' + (1-\omega)\frac{\mathcal{G}(z)}{z}\right)\right|.$$

Therefore, we get

$$\begin{aligned}
&\left|-\sum_{n=2}^{\infty}\omega(n-1)\phi(n)a_n z^{n-1}\right| \\
&< \varsigma\left|\sigma + (1-\omega) - \sum_{n=2}^{\infty}(n\sigma + 1 - \omega)\phi(n)a_n z^{n-1}\right|.
\end{aligned}$$

Thus,

$$\sum_{n=2}^{\infty}[\omega(n-1) + \varsigma(n\sigma + 1 - \omega)]\phi(n)a_n \le \varsigma(\sigma + (1-\omega))$$

and this completes the proof. □

Corollary 1 Let the function $u \in TS(\omega, \sigma, \varsigma)$.Then

$$a_n \le \frac{\varsigma(\sigma + (1-\omega))}{[\omega(n-1) + \varsigma(n\sigma + 1 - \omega)]\phi(n)} z^n,\ n \ge 2.$$

3. Distortion and Covering Theorem

We introduce the growth and distortion theorems for the functions in the class $TS(\omega, \sigma, \varsigma)$.

Theorem 2 Let the function $u \in TS(\omega, \sigma, \varsigma)$. Then

$$|z| - \frac{\varsigma(\sigma+(1-\omega))}{\phi(2)[\omega+\varsigma(2\sigma+1-\omega)]}|z|^2 \leq |u(z)| \leq |z| + \frac{\varsigma(\sigma+(1-\omega))}{\phi(2)[\omega+\varsigma(2\sigma+1-\omega)]}|z|^2.$$

The result is sharp and attained

$$u(z) = z - \frac{\varsigma(\sigma+(1-\omega))}{\phi(2)[\omega+\varsigma(2\sigma+1-\omega)]}z^2.$$

Proof.

$$|u(z)| = \left| z - \sum_{n=2}^{\infty} a_n z^n \right| \leq |z| + \sum_{n=2}^{\infty} a_n |z|^n \leq |z| + |z|^2 \sum_{n=2}^{\infty} a_n.$$

By Theorem 1, we get

$$\sum_{n=2}^{\infty} a_n \leq \frac{\varsigma(\sigma+(1-\omega))}{[\omega+\varsigma(2\sigma+1-\omega)]\phi(n)}. \tag{16}$$

Thus,

$$|u(z)| \leq |z| + \frac{\varsigma(\sigma+(1-\omega))}{\phi(2)[\omega+\varsigma(2\sigma+1-\omega)]}|z|^2.$$

Also,

$$|u(z)| \geq |z| - \sum_{n=2}^{\infty} a_n |z|^2 \geq |z| - |z|^2 \sum_{n=2}^{\infty} a_n \geq |z| - \frac{\varsigma(\sigma+(1-\omega))}{\phi(2)[\omega+\varsigma(2\sigma+1-\omega)]}|z|^2.$$

□

Theorem 3 Let $u \in TS(\omega, \sigma, \varsigma)$. Then

$1 - \frac{2\varsigma(\sigma+(1-\omega))}{\phi(2)[\omega+\varsigma(2\sigma+1-\omega)]}|z| \leq |u'(z)| \leq 1 + \frac{2\varsigma(\sigma+(1-\omega))}{\phi(2)[\omega+\varsigma(2\sigma+1-\omega)]}|z|$

with equality for

$$u(z) = z - \frac{2\varsigma(\sigma+(1-\omega))}{\phi(2)[\omega+\varsigma(2\sigma+1-\omega)]}z^2.$$

Proof. Notice that

$$\begin{aligned}&\phi(2)[\omega+\varsigma(2\sigma+1-\omega)]\sum_{n=2}^{\infty}na_n\\ \leq&\sum_{n=2}^{\infty}n[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\phi(n)a_n\\ \leq&\varsigma(\sigma+(1-\omega)),\end{aligned} \tag{17}$$

from Theorem 1. Thus,

$$\begin{aligned}|u'(z)|&=\left|1-\sum_{n=2}^{\infty}na_nz^{n-1}\right|\\ &\leq 1+\sum_{n=2}^{\infty}na_n|z|^{n-1}\\ &\leq 1+|z|\sum_{n=2}^{\infty}na_n\\ &\leq 1+|z|\frac{2\varsigma(\sigma+(1-\omega))}{\phi(2)[\omega+\varsigma(2\sigma+1-\omega)]}.\end{aligned} \tag{18}$$

On the other hand

$$\begin{aligned}|u'(z)|&=\left|1-\sum_{n=2}^{\infty}na_nz^{n-1}\right|\\ &\geq 1-\sum_{n=2}^{\infty}na_n|z|^{n-1}\\ &\geq 1-|z|\sum_{n=2}^{\infty}na_n\\ &\geq 1-|z|\frac{2\varsigma(\sigma+(1-\omega))}{\phi(2)[\omega+\varsigma(2\sigma+1-\omega)]}.\end{aligned} \tag{19}$$

Combining (18) and (19), we get the result. □

4. Radii of Starlikeness, Convexity and Close-to-Convexity

In the following theorems, we obtain the radii of starlikeness, convexity and close-to-convexity for the class $TS(\omega,\sigma,\varsigma)$.

Theorem 4 Let $u\in TS(\omega,\sigma,\varsigma)$. Then u is starlike in $|z|<R_1$ of order $\wp$, $0\leq\wp<1$, where

$$R_1=\inf_n\left\{\frac{(1-\wp)(\omega(n-1)+\varsigma(n\sigma+1-\omega))\phi(n)}{(n-\wp)\varsigma(\sigma+(1-\omega))}\right\}^{\frac{1}{n-1}},\ n\geq 2. \tag{20}$$

Proof. u is starlike of order $\wp, 0\leq\wp<1$ if

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\}>\wp.$$

Thus, it is enough to show that

$$\left|\frac{zu'(z)}{u(z)}-1\right| = \left|\frac{-\sum\limits_{n=2}^{\infty}(n-1)a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty}a_n z^{n-1}}\right| \leq \frac{\sum\limits_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}a_n|z|^{n-1}}.$$

Thus,

$$\left|\frac{zu'(z)}{u(z)}-1\right| \leq 1-\wp \ if \ \sum_{n=2}^{\infty}\frac{(n-\wp)}{(1-\wp)}a_n|z|^{n-1} \leq 1. \tag{21}$$

Hence, by Theorem 1, (21) will be true if

$$\frac{n-\wp}{1-\wp}|z|^{n-1} \leq \frac{(\omega(n-1)+\varsigma(n\sigma+1-\omega))\phi(n)}{\varsigma(\sigma+(1-\omega)}$$

or if

$$|z| \leq \left[\frac{(1-\wp)(\omega(n-1)+\varsigma(n\sigma+1-\omega))\phi(n)}{(n-\wp)\varsigma(\sigma+(1-\omega))}\right]^{\frac{1}{n-1}}, n \geq 2. \tag{22}$$

The theorem follows easily from (22). □

Theorem 5 Let $u \in TS(\omega,\sigma,\varsigma)$. Then u is convex in $|z| < R_2$ of order $\wp, 0 \leq \wp < 1$, where

$$R_2 = \inf_n \left\{\frac{(1-\wp)(\omega(n-1)+\varsigma(n\sigma+1-\omega))\phi(n)}{n(n-\wp)\varsigma(\sigma+(1-\omega))}\right\}^{\frac{1}{n-1}}, \ n \geq 2. \tag{23}$$

Proof. u is convex of order $\wp, 0 \leq \wp < 1$ if

$$\Re\left\{1+\frac{zu''(z)}{u'(z)}\right\} > \wp.$$

Thus, it is enough to show that

$$\left|\frac{zu''(z)}{u'(z)}\right| = \left|\frac{-\sum\limits_{n=2}^{\infty}n(n-1)a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty}na_n z^{n-1}}\right| \leq \frac{\sum\limits_{n=2}^{\infty}n(n-1)a_n|z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}na_n|z|^{n-1}}.$$

Thus,

$$\left|\frac{zu''(z)}{u'(z)}\right| \leq 1-\wp \ if \ \sum_{n=2}^{\infty}\frac{n(n-\wp)}{(1-\wp)}a_n|z|^{n-1} \leq 1. \tag{24}$$

Hence, by Theorem 1, (24) will be true if

$$\frac{n(n-\wp)}{1-\wp}|z|^{n-1} \leq \frac{(\omega(n-1)+\varsigma(n\sigma+1-\omega))\phi(n)}{\varsigma(\sigma+(1-\omega)}$$

or if

$$|z| \leq \left[\frac{(1-\wp)(\omega(n-1)+\varsigma(n\sigma+1-\omega))\phi(n)}{n(n-\wp)\varsigma(\sigma+(1-\omega))}\right]^{\frac{1}{n-1}}, n \geq 2. \tag{25}$$

The theorem follows easily from (25). □

Theorem 6 Let $u \in TS(\omega,\sigma,\varsigma)$. Then u is close-to-convex in $|z| < R_3$ of order $\wp, \ \ 0 \leq \wp < 1$, where

$$R_3 = \inf_n \left\{\frac{(1-\wp)(\omega(n-1)+\varsigma(n\sigma+1-\omega))\phi(n)}{n\varsigma(\sigma+(1-\omega))}\right\}^{\frac{1}{n-1}}, \ n \geq 2. \tag{26}$$

Proof. u is close-to-convex of order $\wp, 0 \leq \wp < 1$ if

$$\Re\{u'(z)\} > \wp.$$

Thus, it is enough to show that

$$|u'(z) - 1| = \left| -\sum_{n=2}^{\infty} n a_n z^{n-1} \right| \leq \sum_{n=2}^{\infty} n a_n |z|^{n-1}.$$

Thus,

$$|u'(z) - 1| \leq 1 - \wp \ if \ \sum_{n=2}^{\infty} \frac{n}{(1-\wp)} a_n |z|^{n-1} \leq 1. \tag{27}$$

Hence, by Theorem 1, (27) will be true if

$$\frac{n}{1-\wp}|z|^{n-1} \leq \frac{(\omega(n-1) + \varsigma(n\sigma + 1 - \omega))\phi(n)}{\varsigma(\sigma + (1-\omega)}$$

or if

$$|z| \leq \left[\frac{(1-\wp)(\omega(n-1) + \varsigma(n\sigma + 1 - \omega))\phi(n)}{n\varsigma(\sigma + (1-\omega))} \right]^{\frac{1}{n-1}}, n \geq 2. \tag{28}$$

The theorem follows easily from (28). □

5. Extreme Points

In the following theorem, we obtain extreme points for the class $TS(\omega, \sigma, \varsigma)$.
Theorem 7 Let $u_1(z) = z$ and

$$u_k(z) = z - \frac{\varsigma(\sigma + (1-\omega))}{[\omega(n-1) + \varsigma(n\sigma + 1 - \omega)]\phi(n)} z^n, \text{ for } n = 2, 3, \cdots.$$

Then $u \in TS(\omega, \sigma, \varsigma)$ if and only if it can be expressed in the form

$$u(z) = \sum_{n=1}^{\infty} \theta_n u_n(z), \ where \ \theta_n \geq 0 \ and \ \sum_{n=1}^{\infty} \theta_n = 1.$$

Proof. Assume that $u(z) = \sum\limits_{n=1}^{\infty} \theta_n u_n(z)$, hence we get

$$u(z) = z - \sum_{n=2}^{\infty} \frac{\varsigma(\sigma + (1-\omega))\theta_n}{[\omega(n-1) + \varsigma(n\sigma + 1 - \omega)]\phi(n)} z^n.$$

Now, $u \in TS(\omega, \sigma, \varsigma)$, since

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{[\omega(n-1) + \varsigma(n\sigma + 1 - \omega)]\phi(n)}{\varsigma(\sigma + (1-\omega))} \\ &\quad \times \frac{\varsigma(\sigma + (1-\omega))\theta_n}{[\omega(n-1) + \varsigma(n\sigma + 1 - \omega)]\phi(n)} \\ &= \sum_{n=2}^{\infty} \theta_n = 1 - \theta_1 \leq 1. \end{aligned}$$

Conversely, suppose $u \in TS(\omega, \sigma, \varsigma)$. Then we show that u can be written in the form $\sum_{n=1}^{\infty} \theta_n u_n(z)$.
Now, $u \in TS(\omega, \sigma, \varsigma)$ implies from Theorem 1

$$a_n \leq \frac{\varsigma(\sigma + (1-\omega))}{[\omega(n-1) + \varsigma(n\sigma + 1 - \omega)]\phi(n)}.$$

Setting $\theta_n = \frac{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\phi(n)}{\varsigma(\sigma+(1-\omega))} a_n, n = 2, 3, \cdots$
and $\theta_1 = 1 - \sum_{n=2}^{\infty} \theta_n$, we obtain $u(z) = \sum_{n=1}^{\infty} \theta_n u_n(z)$. □

6. Hadamard product

In the following theorem, we obtain the convolution result for functions belongs to the class $TS(\omega, \sigma, \varsigma)$.

Theorem 8 Let $u, g \in TS(\omega, \sigma, \varsigma, \vartheta)$. Then $u * g \in TS(\omega, \sigma, \zeta, \vartheta)$ for

$$u(z) = z - \sum_{n=2}^{\infty} a_n z^n, g(z) = z - \sum_{n=2}^{\infty} b_n z^n \text{ and } (u*g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n,$$

where

$$\zeta \geq \frac{\varsigma^2(\sigma + (1-\omega))\omega(n-1)}{[\omega(n-1) + \varsigma(n\sigma + 1 - \omega)]^2\phi(n) - \varsigma^2(\sigma + (1-\omega))(n\sigma + 1 - \omega)}.$$

Proof. $u \in TS(\omega, \sigma, \varsigma)$ and so

$$\sum_{n=2}^{\infty} \frac{[\omega(n-1) + \varsigma(n\sigma + 1 - \omega)]\phi(n)}{\varsigma(\sigma + (1-\omega))} a_n \leq 1, \tag{29}$$

and

$$\sum_{n=2}^{\infty} \frac{[\omega(n-1) + \varsigma(n\sigma + 1 - \omega)]\phi(n)}{\varsigma(\sigma + (1-\omega))} b_n \leq 1. \tag{30}$$

We have to find the smallest number ζ such that

$$\sum_{n=2}^{\infty} \frac{[\omega(n-1) + \zeta(n\sigma + 1 - \omega)]\phi(n)}{\zeta(\sigma + (1-\omega))} a_n b_n \leq 1. \tag{31}$$

By Cauchy-Schwarz inequality

$$\sum_{n=2}^{\infty} \frac{[\omega(n-1) + \varsigma(n\sigma + 1 - \omega)]\phi(n)}{\varsigma(\sigma + (1-\omega))} \sqrt{a_n b_n} \leq 1. \tag{32}$$

Therefore, it is enough to show that

$$\begin{aligned} &\frac{[\omega(n-1) + \zeta(n\sigma + 1 - \omega)]\phi(n)}{\zeta(\sigma + (1-\omega))} a_n b_n \\ \leq &\frac{[\omega(n-1) + \varsigma(n\sigma + 1 - \omega)]\phi(n)}{\varsigma(\sigma + (1-\omega))} \sqrt{a_n b_n}. \end{aligned}$$

That is

$$\sqrt{a_n b_n} \leq \frac{[\omega(n-1) + \varsigma(n\sigma + 1 - \omega)]\zeta}{[\omega(n-1) + \zeta(n\sigma + 1 - \omega)]\varsigma}. \tag{33}$$

From (32)

$$\sqrt{a_n b_n} \leq \frac{\varsigma(\sigma+(1-\omega))}{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\phi(n)}.$$

Thus, it is enough to show that

$$\frac{\varsigma(\sigma+(1-\omega))}{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\phi(n)} \leq \frac{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\zeta}{[\omega(n-1)+\zeta(n\sigma+1-\omega)]\varsigma},$$

which simplifies to

$$\zeta \geq \frac{\varsigma^2(\sigma+(1-\omega))\omega(n-1)}{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]^2\phi(n)-\varsigma^2(\sigma+(1-\omega))(n\sigma+1-\omega)}.$$

□

7. Closure Theorems

We shall prove the following closure theorems for the class $TS(\omega,\sigma,\varsigma)$.

Theorem 9 Let $u_j \in TS(\omega,\sigma,\varsigma), j=1,2,\ldots,s$. Then

$$g(z)=\sum_{j=1}^{s} c_j u_j(z) \in TS(\omega,\sigma,\varsigma).$$

For $u_j(z)=z-\sum_{n=2}^{\infty} a_{n,j} z^n$, where $\sum_{j=1}^{s} c_j = 1$.

Proof.

$$\begin{aligned} g(z) &= \sum_{j=1}^{s} c_j u_j(z) \\ &= z-\sum_{n=2}^{\infty}\sum_{j=1}^{s} c_j a_{n,j} z^n \\ &= z-\sum_{n=2}^{\infty} e_n z^n, \end{aligned}$$

where $e_n=\sum_{j=1}^{s} c_j a_{k,j}$. Thus $g(z) \in TS(\omega,\sigma,\varsigma)$ if

$$\sum_{n=2}^{\infty} \frac{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\phi(n)}{\varsigma(\sigma+(1-\omega))} e_n \leq 1,$$

that is, if

$$\begin{aligned} &\sum_{n=2}^{\infty}\sum_{j=1}^{s} \frac{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\phi(n)}{\varsigma(\sigma+(1-\omega))} c_j a_{n,j} \\ =&\sum_{j=1}^{s} c_j \sum_{n=2}^{\infty} \frac{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\phi(n)}{\varsigma(\sigma+(1-\omega))} a_{n,j} \\ \leq&\sum_{j=1}^{s} c_j = 1. \end{aligned}$$

□

Theorem 10 Let $u, g \in TS(\omega, \sigma, \varsigma)$. Then

$$h(z) = z - \sum_{n=2}^{\infty}(a_n^2 + b_n^2)z^n \in TS(\omega, \sigma, \varsigma), where$$

$$\zeta \geq \frac{2\omega(n-1)\varsigma^2(\sigma + (1-\omega))}{[\omega(n-1) + \varsigma(n\sigma + 1 - \omega)]^2\phi(n) - 2\varsigma^2(\sigma + (1-\omega))(n\sigma + 1 - \omega)}.$$

Proof. Since $u, g \in TS(\omega, \sigma, \varsigma)$, so Theorem 1 yields

$$\sum_{n=2}^{\infty}\left[\frac{(\omega(n-1) + \varsigma(n\sigma + 1 - \omega))\phi(n)}{\varsigma(\sigma + (1-\omega))}a_n\right]^2 \leq 1$$

and

$$\sum_{n=2}^{\infty}\left[\frac{(\omega(n-1) + \varsigma(n\sigma + 1 - \omega))\phi(n)}{\varsigma(\sigma + (1-\omega))}b_n\right]^2 \leq 1.$$

We obtain from the last two inequalities

$$\sum_{n=2}^{\infty}\frac{1}{2}\left[\frac{(\omega(n-1) + \varsigma(n\sigma + 1 - \omega))\phi(n)}{\varsigma(\sigma + (1-\omega))}\right]^2(a_n^2 + b_n^2) \leq 1. \tag{34}$$

But $h(z) \in TS(\omega, \sigma, \zeta, q, m)$, if and only if

$$\sum_{n=2}^{\infty}\frac{[\omega(n-1) + \zeta(n\sigma + 1 - \omega)]\phi(n)}{\zeta(\sigma + (1-\omega))}(a_n^2 + b_n^2) \leq 1, \tag{35}$$

where $0 < \zeta < 1$, however (34) implies (35) if

$$\frac{[\omega(n-1) + \zeta(n\sigma + 1 - \omega)]\phi(n)}{\zeta(\sigma + (1-\omega))} \leq \frac{1}{2}\left[\frac{(\omega(n-1) + \varsigma(n\sigma + 1 - \omega))\phi(n)}{\varsigma(\sigma + (1-\omega))}\right]^2.$$

Simplifying, we get

$$\zeta \geq \frac{2\omega(n-1)\varsigma^2(\sigma + (1-\omega))}{[\omega(n-1) + \varsigma(n\sigma + 1 - \omega)]^2\phi(n) - 2\varsigma^2(\sigma + (1-\omega))(n\sigma + 1 - \omega)}.$$

□

References

[1] Abramowitz M., Stegun I.A. (eds). Handbook of Mathmatical Functions with Formulas, Graphs and Mathematical Tables, Dover Publications Inc., (1965).

[2] Coman D. The radius of starlikeness for error function, Stud. Univ. Babes Bolyal Math., 36, (1991): 13-16.

[3] Elbert A., Laforgia A. The zeros of the complementary error function, Numer. Algorithms, 49 (1-4), (2008): 153-157.

[4] Nisar K.S., Baleanu D., Qurashi M.A. Fractional calculus and application of generalized Struve function, SpringerPlus J., 5 (910), (2016): 13 pages.

[5] Orhan H., Yagmur N. Geometric properties of generalized struve functions. In: The International Congress in honour of Professor H.M. Srivastava, 23-26, Bursa, Turkey, (2012).

[6] Orhan H, Yagmur N. Starlikeness and convexity of generalized Struve functions. Abstr Appl Anal. Art. ID 954513: 6, 2013.

[7] Oyekan E.A. Certain geometric properties of functions involving Galue type Struve function, Ann. Math. Comput. Sci., 8, (2022): 43-53.

[8] Oyekan E.A., Lasode A.O., Olatunji T.A. Intial bounds for analytic functions classes characterized by certain special functions and bell numbers,JMMCS 4(120),(2023): 41-51.

[9] Ramachandran K., Dhanalakshmi C., Vanitha L. Hankel determinant for a subclass of analytic functions associated with error functions bounded by conical regions, Internat. J. Math. Anal., 11 (2), (2017): 571-581.

[10] Silverman H. Univalent functions with negative coefficients, Proc. Amer. Math. Soc. 51(1) (1975): 109-116.

Vihar S. Shinde
Department of Mathematics,, K. N. B. A. C. & V. P. S. College, Bhosare (Kurduwadi) Dist. Solapur- 413208, Maharastra, India.
Email address: `viharss@gmail.com`

S. B. Chavhan
Department of Mathematics,, D.B.A.C. & S. College, Bhokar, Dist.-Nanded Maharashtra, India.
Email address: `sbcmath2015@gmail.com`