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FRACTAL-FRACTIONAL DIFFERENTIAL AND INTEGRAL OPERATORS: DEFINITIONS, SOME PROPERTIES AND APPLICATIONS

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Abstract. In this paper, we prove some properties of the fractal and fractal-fractional integral and differential operators, then define the linear kinds of Abel integral equations of fractal and fractal-fractional orders. The existence of solutions of these kinds of Abel integral equations will be studied.
Two initial-value problems of fractal integro-differential Abel equations will be also discussed.

1. Introduction

This paper focuses on exploring the properties of the fractal and the fractal-fractional differential and integral operators, which serves as a crucial tool for studying various phenomena. Specifically, we examine the theoretical foundation of the fractal-fractional integral operator and delve into its application to the study of Abel integral equations [3] and [13].

The objective of this study is to investigate some of the key problems associated with the fractal and fractal-fractional Abel integral equations of the first and second kinds. By exploring these equations, we aim to expand the understanding of how fractal and fractional operators interact and their potential applications in fields such as physics, engineering, and mathematics [7], [9] and [16].

Our goal here, firstly, is to prove some properties of the fractal and fractal-fractional differential and integral operators. Secondly, define the first and second kinds fractal Abel integral equations and the first and second kind fractal-fractional Abel integral equations, then study their existence of solutions.

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2. Definitions and properties

In recent years, the concepts of fractal calculus, fractional calculus, and their combination fractal-fractional calculus have gained significant attention due to their applications in various scientific and engineering fields. These integrals provide powerful tools for modeling complex phenomena that can't be effectively described using classical calculus [1], [3], [10] and [14].
Below are some key properties of these integrals that highlight their significance and utility in various domains.
Let $\beta \in (0,1)$ and $\alpha \in (0,1]$, then we have the following definitions

Definition 2.1. *The fractal derivative of the function x at $t_0 \in (0,T)$ is defined by* [5]

$$D_\beta \; x(t) \; = \; \frac{dx(t)}{dt^\beta}\Big|_{t=t_0} = \lim_{t\to t_0} \frac{x(t)-x(t_0)}{t^\beta - t_0^\beta}$$

and if x is differentiable, then we can get

$$D_\beta \; x(t) = \frac{d}{dt^\beta} x(t) = \frac{t^{1-\beta}}{\beta} \; \frac{dx(t)}{dt}, \; t \in (0,T].$$

From which we can deduce that

$$\lim_{\beta\to 1} D_\beta x(t) \; = \; \frac{d}{dt} x(t).$$

Definition 2.2. *Let $x \in C[0,T]$ or bounded measurable on $[0,T]$. The fractal integral of the function x is defined by* [6]

$$I_\beta \; x(t) \; = \; \int_0^t \beta \; s^{\beta-1} \; x(s) \; ds$$

and we can deduce that

$$D_\beta I_\beta x(t) \; = \; \frac{t^{1-\beta}}{\beta} \frac{d}{dt} I_\beta \; x(t) \; = \; \frac{t^{1-\beta}}{\beta} \frac{d}{dt} \int_0^t \beta \; s^{\beta-1} \; x(s) \; ds \; = \; \frac{t^{1-\beta}}{\beta} \beta \; t^{\beta-1} \; x(t) = x(t).$$

Definition 2.3. *The Riemann–Liouville (R-L) fractional-order integral of the function x can be defined as* [4], [11] *and* [12]

$$I^\alpha \; x(t) \; = \; \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \; x(s) \; ds.$$

Definition 2.4. *The (R-L) fractal-fractional integral of the function x can be, simply, defined by*

$$I_\beta^\alpha \; x(t) \; = \; \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \beta \; s^{\beta-1} \; x(s) \; ds,$$

where $\Gamma(\alpha) = \int_0^\infty e^{-t} \; t^{\alpha-1} \; dt$.

Now, from the properties of the fractional calculus operators, [4], [8], [11] and [12], we can prove the following properties.

Theorem 2.1. *Let $f : [0,T] \to R$ be continuous or measurable and bounded on $[0,T]$ with $|f(t)| \le r$, then*

(i) $|I_\beta\ f(t)| \le r\ t^\beta,\ \Rightarrow\ I_\beta\ f(t)|_{t=0} = 0.$

(ii) $|I^\alpha\ f(t)| \le r\ \frac{t^\alpha}{\Gamma(1+\alpha)}\ \Rightarrow\ I^\alpha\ f(t)|_{t=0} = 0.$

(iii) $|I^\alpha_\beta\ f(t)| \le r\ \frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)},\ \Rightarrow\ I^\alpha_\beta\ f(t)|_{t=0} = 0,\ \alpha+\beta > 1.$

Proof. From definitions 1-4, we have

(i) $|I_\beta\ f(t)|\ \le\ \int_0^t \beta\ s^{\beta-1}\ |f(s)|\ ds\ \le r\ \int_0^t \beta\ s^{\beta-1}\ ds\ \le r\ t^\beta.$

Then $|I_\beta\ f(t)|_{t=0} \le 0$, which implies $|I_\beta\ f(t)|_{t=0} = 0$ and $I_\beta\ f(t)|_{t=0}\ =\ 0.$

(ii) $|I^\alpha\ f(t)|\ \le\ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\ |f(s)|\ ds\ \le\ r\ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\ ds\ =\ r\ \frac{t^\alpha}{\Gamma(\alpha+1)}.$

Then $|I^\alpha\ f(t)|_{t=0} \le 0$, which implies $|I^\alpha\ f(t)|_{t=0} = 0$ and $I^\alpha\ f(t)|_{t=0}\ =\ 0.$

(iii) $|I^\alpha_\beta\ f(t)|\ \le\ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\beta\ s^{\beta-1}\ |f(s)|\ ds\ \le r\ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\beta\ s^{\beta-1}\ ds\ =\ r\ \beta\ I^\alpha\ t^{\beta-1}$

$=\ r\ \frac{\beta\ \Gamma(\beta)}{\Gamma(\alpha+\beta)}\ t^{\alpha+\beta-1}\ =\ r\ \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta)}\ t^{\alpha+\beta-1}\ \le r\ \frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)},\ \alpha+\beta > 1.$

Then $|I^\alpha_\beta\ f(t)|_{t=0} \le 0$, which implies $|I^\alpha_\beta\ f(t)|_{t=0} = 0$ and $I^\alpha_\beta\ f(t)|_{t=0}\ =\ 0.$

Theorem 2.2. *Let* $f \in C[0,T]$ *be continuous and* $|f(t)| \le r$, *then the operators* $F_i : C[0,T] \to\ C[0,T],\ i = 1,2,3$ *where*

(1) $F_1(t)\ =\ I_\beta\ f(t),\ \beta \in (0,1).$
(2) $F_2(t)\ =\ I^\alpha\ f(t)\ ,\alpha \in (0,1].$
(3) $F_3(t)\ =\ I^\alpha_\beta\ f(t),\ \alpha+\beta > 1.$

Proof. Let $t_2 > t_1 \in [0,T]$ such that $|t_2 - t_1| < \delta$. Then we have

(1) $|F_1(t_2) - F_1(t_1)|\ =\ |\int_{t_1}^{t_2}\ \beta\ s^{\beta-1}\ f(s)\ ds|\ \le r(t_2^\beta - t_1^\beta).$

(2) $|F_2(t_2) - F_2(t_1)|\ =\ |\ \int_0^{t_2}\ \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\ f(s)\ ds\ -\ \int_0^{t_1}\ \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\ f(s)\ ds|$

$=\ |\int_0^{t_1}\ \frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\ f(s)ds\ +\ \int_{t_1}^{t_2}\ \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\ f(s)ds|$

$\le\ r\int_0^{t_1}\ |\frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}|ds\ +\ r\ \int_{t_1}^{t_2}\ \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds$

$\le\ \frac{r}{\Gamma(\alpha+1)}\ (t_1^\alpha - t_2^\alpha)\ +\ \frac{2r}{\Gamma(\alpha+1)}(t_2-t_1)^\alpha\ \le\ \frac{2r}{\Gamma(\alpha+1)}(t_2-t_1)^\alpha.$

(3) Let $0 < \gamma < \alpha$ be given such that $\alpha - \gamma + \beta \ge 1$, then from the properties of the fractional-order integral, we have $I^\alpha f(t)\ =\ I^\gamma I^{\alpha-\gamma} f(t)$ and

$I^\alpha_\beta\ f(t)\ =\ I^\alpha\ (\beta t^{\beta-1} f(t))\ =\ I^\gamma I^{\alpha-\gamma}\ (\beta t^{\beta-1} f(t))\ =\ I^\gamma I^{\alpha-\gamma}_\beta f(t).$

Now
$|F_3(t_2) - F_3(t_1)|\ =\ |I^\gamma I^{\alpha-\gamma}\ (\beta t_2^{\beta-1} f(t_2)) - I^\gamma I^{\alpha-\gamma}\ (\beta t_1^{\beta-1} f(t_1))|$

$=\ |\int_0^{t_2}\ \frac{(t_2-s)^{\gamma-1}}{\Gamma(\gamma)}\ I^{\alpha-\gamma}(\beta s^{\beta-1} f(s))ds\ -\ \int_0^{t_1}\ \frac{(t_1-s)^{\gamma-1}}{\Gamma(\gamma)}\ I^{\alpha-\gamma}(\beta s^{\beta-1} f(s))ds|$

$$= |\int_0^{t_1} \frac{(t_2-s)^{\gamma-1}-(t_1-s)^{\gamma-1}}{\Gamma(\gamma)} I^{\alpha-\gamma}(\beta s^{\beta-1} f(s))ds - \int_{t_1}^{t_2} \frac{(t_2-s)^{\gamma-1}}{\Gamma(\gamma)} I^{\alpha-\gamma}(\beta s^{\beta-1} f(s))ds|.$$

$$\leq \int_0^{t_1} |\frac{(t_2-s)^{\gamma-1}-(t_1-s)^{\gamma-1}}{\Gamma(\gamma)}| I^{\alpha-\gamma}(\beta s^{\beta-1}|f(s)|)ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\gamma-1}}{\Gamma(\gamma)} I^{\alpha-\gamma}(\beta s^{\beta-1}|f(s)|)ds.$$

But

$$I^{\alpha-\gamma}(\beta s^{\beta-1}|f(s)|) \leq rI^{\alpha-\gamma}(\beta s^{\beta-1}) = \frac{r}{\Gamma(\alpha-\gamma)}\int_0^s (s-\theta)^{\alpha-\gamma-1}\beta\theta^{\beta-1}d\theta$$

$$= \frac{r}{\Gamma(\alpha-\gamma)}\frac{\beta\Gamma(\beta)}{\Gamma(\alpha-\gamma+\beta)} s^{\alpha-\gamma+\beta-1} \leq \frac{r}{\Gamma(\alpha-\gamma)}\frac{T^{\alpha-\gamma+\beta-1}}{\Gamma(\alpha-\gamma+\beta)} = M.$$

Then

$$|F_3(t_2) - F_3(t_1)| \leq M \int_0^{t_1} |\frac{(t_2-s)^{\gamma-1}-(t_1-s)^{\gamma-1}}{\Gamma(\gamma)}|ds + M \int_{t_1}^{t_2} \frac{(t_2-s)^{\gamma-1}}{\Gamma(\gamma)}ds$$

$$\leq \frac{2\ M}{\Gamma(\gamma+1)}\ (t_2-t_1)^{\gamma}.$$

Hence, the results follow.

Now, we have the following corollaries.

Corollary 2.0. $\lim_{\alpha\to 1} I^{\alpha}_{\beta} f(t) = I_{\beta} f(t)$.

Proof. From the fractional calculus properties, [4], [8] and [12], we can get

$$\lim_{\alpha\to 1} I^{\alpha}_{\beta} f(t) = \lim_{\alpha\to 1} I^{\alpha}\ \beta\ t^{\beta-1} f(t) = I\ \beta\ t^{\beta-1} f(t) = I_{\beta} f(t).$$

Corollary 2.0. $\lim_{\beta\to 1} I^{\alpha}_{\beta} f(t) = I^{\alpha} f(t)$.

Proof. Consider

$$I^{\alpha}_{\beta} f(t) - I^{\alpha} f(t) = I^{\alpha}\ \beta\ t^{\beta-1}\ f(t) - I^{\alpha} f(t) = I^{\alpha}\ (\beta\ t^{\beta-1} - 1)\ f(t),$$

$$|\ I^{\alpha}_{\beta} f(t) - I^{\alpha} f(t)\ | \leq I^{\alpha}\ |\beta\ t^{\beta-1} - 1|\ |f(t)|.$$

But $\lim_{\beta\to 1} \beta t^{\beta-1} = 1$, then

$$\lim_{\beta\to 1}\ |\ I^{\alpha}_{\beta} f(t) - I^{\alpha} f(t)\ | \leq I^{\alpha}\ \lim_{\beta\to 1}\ |\beta\ t^{\beta-1} - 1|\ |f(t)| = 0$$

and

$$\lim_{\beta\to 1}\ I^{\alpha}_{\beta} f(t) = I^{\alpha} f(t).$$

Lemma 2.1. *Let $f \in C[0,T]$. If $\alpha, \beta \in (0,1)$, then $I^{\alpha}\ I_{\beta} f(t) = I\ I^{\alpha}_{\beta} f(t) = I^{1+\alpha}_{\beta} f(t)$.*

Proof. We have

$$I_{\beta} f(t) = \int_0^t \beta\ s^{\beta-1} f(s)\ ds = I\beta\ t^{\beta-1} f(t),$$

then

$$I^{\alpha}\ I_{\beta} f(t) = I^{\alpha}\ I\beta\ t^{\beta-1} f(t) = I^{1+\alpha}\ \beta\ t^{\beta-1} f(t) = I^{1+\alpha}_{\beta} f(t).$$

And

$$I\ I^{\alpha}_{\beta} f(t) = I\ I^{\alpha}\beta\ t^{\beta-1} f(t) = I^{1+\alpha}\ \beta\ t^{\beta-1} f(t) = I^{1+\alpha}_{\beta}\ f(t).$$

Corollary 2.0. $\lim_{\alpha\to 0} I_\beta^{1+\alpha} f(t) = I_\beta\ f(t)$.

Proof. From the fractional calculus properties, [4], [8] and [12], we can get

$$\lim_{\alpha\to 0} I_\beta^{1+\alpha} f(t) = \lim_{\alpha\to 0} I^\alpha\ I_\beta f(t)\ =\ I_\beta f(t).$$

Lemma 2.2. *Let* $f \in C[0,T]$. *If* $\alpha+\beta \in (0,1)$, *then* $I_\beta\ I_\gamma f(t) = t^\beta\ I_\gamma f(t) - \frac{\gamma}{\beta+\gamma}\ I_{\beta+\gamma} f(t)$.

Proof. We have

$$\begin{aligned}
I_\beta\ I_\gamma f(t) &= I_\beta \int_0^t \gamma\ s^{\gamma-1}\ f(s)\ ds \\
&= \int_0^t \beta\ s^{\beta-1} \int_0^s \gamma\ \theta^{\gamma-1}\ f(\theta)\ d\theta\ ds \\
&= \int_0^t \gamma\ \theta^{\gamma-1}\ f(\theta)\ (\int_\theta^t \beta\ s^{\beta-1}\ ds)\ d\theta \\
&= \int_0^t \gamma\ \theta^{\gamma-1}\ f(\theta)\ (t^\beta - \theta^\beta)\ d\theta \\
&= t^\beta \int_0^t \gamma\ \theta^{\gamma-1}\ f(\theta)\ d\theta - \int_0^t \gamma\ \theta^{\beta+\gamma-1}\ f(\theta)\ d\theta \\
&= t^\beta\ I_\gamma f(t) - \frac{\gamma}{\beta+\gamma} \int_0^t (\beta+\gamma)\ \theta^{(\beta+\gamma)-1}\ f(\theta)\ d\theta \\
&= t^\beta\ I_\gamma f(t) - \frac{\gamma}{\beta+\gamma}\ I_{\beta+\gamma}\ f(t).
\end{aligned}$$

So, we obtain

$$I_\beta\ I_\gamma f(t) = t^\beta\ I_\gamma f(t) - \frac{\gamma}{\beta+\gamma}\ I_{\beta+\gamma} f(t).$$

Also, we can get

$$I_\gamma\ I_\beta f(t) = t^\gamma\ I_\beta f(t) - \frac{\beta}{\gamma+\beta}\ I_{\gamma+\beta} f(t).$$

Then, we obtain

$$I_\gamma\ I_\beta\ f(t)\ \neq\ I_\beta\ I_\gamma f(t).$$

Corollary 2.0. 1. $\lim_{\beta\to 1} I_\beta\ I_\gamma f(t)\ =\ I\ I_\gamma f(t) = t\ I_\gamma f(t) - \frac{\gamma}{1+\gamma}\ I_{1+\gamma} f(t)$.
2. $\lim_{\gamma\to 1} I_\beta\ I_\gamma f(t)\ =\ I_\beta\ I f(t) = t^\beta\ I f(t) - \frac{1}{\beta+1}\ I_{\beta+1} f(t)$.

Thus, we obtain

$$I\ I_\gamma f(t) \neq I_\gamma\ I f(t).$$

From Lemma 2.2, when $\beta=\gamma=\alpha$, we can obtain the following corollary

Corollary 2.0. *Let* $f \in C[0,T]$. *If* $2\alpha \in (0,1)$, *then* $I_\alpha\ I_\alpha f(t) = t^\alpha\ I_\alpha f(t) - \frac{1}{2}\ I_{2\alpha}\ f(t)$.

3. Abel integral equations

3.1. **Abel integral equations via fractional calculus.** The first and second kinds of the Abel's integral equations via fractional calculus are given by [7] and [16]

$$\frac{1}{\Gamma(\alpha)}\int_a^t \frac{x(s)}{(t-s)^{1-\alpha}}\,ds = f(t),\ t\in[a,b] \tag{1}$$

and

$$x(t)\ +\ \frac{\lambda}{\Gamma(\alpha)}\int_a^t \frac{x(s)}{(t-s)^{1-\alpha}}\,ds = f(t),\ x\in[a,b] \tag{2}$$

where $x(t)$ is the unknown function and $\alpha\in(0,1)$.
From the properties of the fractional calculus, [4], [8] and [12], we have the following lemma

Lemma 3.3. *1) If $I^{1-\alpha}f(t)$ is differentiable on $[0,T]$, then the solution of (1) is given by*

$$x(t)\ =\ \frac{d}{dt}I^{1-\alpha}f(t)\ =\ {}^{R}D^{\alpha}f(t)$$

where the operator ${}^{R}D^{\alpha}$ is the R-L fractional order derivative.

2) If $f\in C[0,T]$ and $|\lambda|T^{\alpha}<\Gamma(1+\alpha)$ then the solution of equation (2) can be given by

$$x(t)\ =\ (1+\lambda I^{\alpha})^{-1}f(t)\ =\ \sum_{n=0}^{\infty}(-\lambda)^n I^{n\alpha}f(t).$$

3.2. **Fractal Abel integral equations.** Now, we can define the linear first kind Abel fractal integral equation as

$$\int_0^t \frac{\beta}{s^{1-\beta}}x(s)\,ds\ =\ f(t)\ t\in[0,T] \tag{3}$$

or

$$I_\beta\ x(t)\ =\ \int_0^t \beta s^{\beta-1}x(s)\,ds\ =\ f(t),\ t\in[0,T] \tag{4}$$

and the linear second kind Abel's fractal integral equation as

$$x(t)\ +\ \lambda\int_0^t \beta s^{\beta-1}x(s)\,ds\ =\ f(t),\ t\in[0,T]. \tag{5}$$

Lemma 3.4. *1) Let $\beta\in(0,1)$. If the function f is differentiable on $[0,T]$, then the solution of (4) is given by*

$$x(t)\ =\ \frac{d}{dt^{\beta}}f(t)\ =\ D_\beta f(t)$$

2) Let $f\in C[0,T]$. If $|\lambda|T^{\beta}<1$ then the solution of equation (5) can be given by

$$x(t)\ =\ (1+\lambda I_\beta)^{-1}f(t)\ =\ \sum_{n=0}^{\infty}(-\lambda)^n(I_\beta)^n f(t)\in C[0,T].$$

Proof. 1) Consider the integral equation (4), then

$$\frac{d}{dt}f(t)\ =\ \frac{d}{dt}\int_0^t \beta s^{\beta-1}x(s)\,ds\ =\ \beta t^{\beta-1}x(t).$$

This proves that

$$\frac{1}{\beta}t^{1-\beta}\frac{d}{dt}f(t) \;=\; x(t),\;\; \frac{d}{dt^{\beta}}f(t) \;=\; x(t) \;\text{ and }\; D_{\beta}f(t) \;=\; x(t).$$

2) For the integral equation (5) we have

$$(1+\lambda I_{\beta})x(t) \;=\; f(t).$$

But

$$|\lambda I_{\beta}f(t)| \;=\; |\lambda|\;|\int_0^t \beta s^{\beta-1}f(s)ds| \;\leq\; |\lambda|\;||f||\;T^{\beta},\;\text{ where } ||f|| = \sup_{t\in[0,T]}|f(t)|$$

then $|\lambda|T^{\beta} < 1$ implies that $||\lambda I_{\beta}|| < 1$ and by the Neumann expansion [2] we can get

$$x(t) \;=\; (1+\lambda I_{\beta})^{-1}f(t) \;=\; \sum_{n=0}^{\infty}(-\lambda)^n\;(I_{\beta})^n f(t) \in C[0,T].$$

Example

Let $f(t) = x_0$ in (5), then the solution of the linear second kind Abel's fractal integral equation (5) is given by

$$x(t) \;=\; \sum_{n=0}^{\infty}(-\lambda)^n\;(I_{\beta})^n\;x_0 \;=\; \sum_{n=0}^{\infty}(-\lambda)^n\frac{t^{n\beta}}{n!}\;x_0 \in C[0,T].$$

Lemma 3.5. *Let* $f:[0,T]\rightarrow R$ *be such that* $F(t) = \beta\, t^{\beta-1}f(t) \in L_1[0,T]$. *If* $|\lambda|T^{\beta} < 1$, *then the solution of equation (5) can be given by*

$$x(t) \;=\; \frac{t^{1-\beta}}{\beta}\sum_{n=0}^{\infty}(-\lambda)^n(\beta t^{\beta-1}I)^n\;F(t) \in L_1[0,T]$$

where $IF(t) = \int_0^T F(s)ds$ *and* $||F||_1 = \int_0^T |F(t)|dt$.

Proof. Let $Y(t) = \beta t^{\beta-1}x(t)$ and multiply the integral equation (5) by $\beta t^{\beta-1}$,
then we get

$$Y(t) \;+\; \lambda\;\beta t^{\beta-1}IY(t) \;=\; \beta t^{\beta-1}f(t),$$

which can be written as

$$(1+\;\lambda\;\beta t^{\beta-1}I)\;Y(t) \;=\; F(t),$$

where

$$IY(t) \;=\; \int_0^t Y(s)\;ds \;=\; \int_0^t \beta s^{\beta-1}x(s)\;ds.$$

But

$$|\;\lambda\;\beta t^{\beta-1}I\;F(t)| \leq |\lambda|\;||F||_1\;\beta t^{\beta-1}\;\text{ and }\;||\;\lambda\;\beta t^{\beta-1}I\;F||_1 \leq |\lambda|\;T^{\beta}\;||F||_1,$$

then $|\lambda|T^{\beta} < 1$ implies that $||\;\lambda\;\beta t^{\beta-1}I||_1 < 1$ and by the Neumann expansion [2] we can get

$$Y(t) \;=\; (1\;+\;\lambda\;\beta t^{\beta-1}I)^{-1}F(t) \;=\; \sum_{n=0}^{\infty}(-\lambda)^n(\beta t^{\beta-1}I)^n\;F(t) \in L_1[0,T]$$

and

$$x(t) \ = \ \frac{t^{1-\beta}}{\beta} \ Y(t) \ \in L_1[0,T].$$

3.3. **Fractal integro-differential Abel equations.** 1- Let $\gamma, \delta \in (0,1)$ be such that $\gamma < \delta$. Consider the initial-value problem of the fractal integro-differential equation

$$I_\gamma \ D_\delta x(t) \ = \ f(t), \ t \in (0,T], \ x(0) \ = \ x_o. \tag{6}$$

We can write (6) as

$$\int_0^t \gamma s^{\gamma-1} \ \frac{s^{1-\delta}}{\delta} \frac{d}{ds} x(s) ds \ = \ \int_0^t \frac{\gamma}{\delta} s^{\gamma-\delta} \ \frac{d}{ds} x(s) ds$$

$$= \ \frac{\gamma}{\delta \ \beta} \int_0^t \ \beta s^{\beta-1} y(s) \ ds \ = \ f(t)$$

and obtain the first kind fractal Abel integral equation

$$I_\beta \ y(t) \ = \ F(t) \tag{7}$$

where

$$y(t) \ = \ \frac{d}{dt} x(t), \ F(t) \ = \ \frac{\delta \ \beta}{\gamma} f(t) \text{ and } \beta \ = \ 1 \ + \ \gamma - \delta \ \in (0,1).$$

Now, if the function $f(t) \in AC[0,T]$ is absolutely continuous on $[0,T]$, then the solution of (7) is given by

$$y(t) \ = \ D_\beta \ F(t) \ = \ \frac{\delta \ \beta}{\gamma} \ D_\beta \ f(t) \ \in L_1[0,T]$$

and finally the solution of (6) is given by

$$x(t) \ = \ x_o \ + \ \int_0^t \ y(s) \ ds \ \in AC[0,T].$$

2- Let $\beta \in (0,1)$. Consider the initial-value problem of fractal differential equation

$$D_\beta \ x(t) \ + \ \lambda \ x(t) \ = \ f(t), \ t \in (0,T] \ \text{ and } x(0) \ = \ x_0. \tag{8}$$

From (8), we can get

$$\frac{d}{dt} x(t) \ + \ \lambda \ \beta \ t^{\beta-1} \ x(t) \ = \ \beta \ t^{\beta-1} \ f(t).$$

Integrating, we obtain the second kind Abel fractal integral equation

$$x(t) \ + \lambda \int_0^t \ \beta \ s^{\beta-1} \ x(s) \ ds \ = \ g(t),$$

where

$$g(t) \ = \ x_0 \ + \ I_\beta f(t).$$

Then if $|\lambda| \ T^\beta < 1$ and $f \in C[0,T]$, then the solution of (8) is given by

$$x(t) \ = \ (1 + \lambda \ I_\beta)^{-1} \ g(t) \ \in AC[0,T]$$

and

$$x(t) \ = \ \sum_{n=0}^{\infty} \ (-\lambda)^n \ (I_\beta)^n \ x_0 \ + \ \sum_{n=0}^{\infty} \ (-\lambda)^n \ (I_\beta)^{n+1} f(t) \ \in AC[0,T].$$

4. Fractal-fractional Abel integral equations

Let $\alpha \in (0,1]$, and $\beta \in (0,1)$. Here, we can define the linear first and second kinds Abel fractal-fractional integral equations respectively as

$$I_\beta^\alpha \, x(t) \; = \; \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\beta s^{\beta-1}}{(t-s)^{1-\alpha}} \, x(s) \, ds \; = \; f(t), \;\; t \in [0,T] \tag{9}$$

$$x(t) \; + \; \frac{\lambda}{\Gamma(\alpha)} \int_0^t \frac{\beta s^{\beta-1}}{(t-s)^{1-\alpha}} \, x(s) \, ds \; = \; f(t), \;\; t \in [0,T]. \tag{10}$$

Definition 4.5. *Let* $I^{1-\alpha} f$ *be differentiable on* $[0,T]$ *and* $\beta \in (0,1)$*, then the fractal-fractional order derivative is given by*

$${}^{R}D_\beta^\alpha \, f(t) = \frac{d}{dt^\beta} I^{1-\alpha} f(t).$$

For the solution of the integral equation (9) we have the following lemma.

Lemma 4.6. *If the function* $I^{1-\alpha} f$ *is differentiable on* $[0,T]$*, then the solution of (9) is given by*

$$x(t) \; = \; \frac{d}{dt^\beta} I^{1-\alpha} f(t) \; = \; {}^{R}D_\beta^\alpha \, f(t)$$

where the operator ${}^{R}D_\beta^\alpha$ *defines the R-L fractal-fractional order derivative.*

Proof. From (9) we have

$$I_\beta^\alpha \, x(t) \; = \; f(t) \; \Rightarrow \; I_\beta \, x(t) \; = \; I^{1-\alpha} \, I_\beta^\alpha \, x(t) \; = \; I^{1-\alpha} f(t)$$

and

$$x(t) = \frac{d}{dt^\beta} I^{1-\alpha} f(t) \; = \; {}^{R}D_\beta^\alpha \, f(t).$$

Now, we can prove the following properties of the R-L fractal-fractional order derivative ${}^{R}D_\beta^\alpha$.

Theorem 4.3. *Let* $x \in C[0,T]$ *or (x be bounded and measurable), then* ${}^{R}D_\beta^\alpha \, I_\beta^\alpha \, x(t) \; = \; x(t)$*.*

Proof. We have

$${}^{R}D_\beta^\alpha \, I_\beta^\alpha \, x(t) \quad = \quad \frac{d}{dt^\beta} \, I^{1-\alpha} \, I_\beta^\alpha \, x(t) \; a.e, \;\; t \in [0,T]$$

using Theorem 1, we have $I^{1-\alpha} \, I_\beta^\alpha = I_\beta$, then

$$\begin{aligned}
{}^{R}D_\beta^\alpha \, I_\beta^\alpha \, x(t) \quad &= \quad \frac{d}{dt^\beta} \, I_\beta \, x(t) \\
&= \quad \frac{t^{1-\beta}}{\beta} \frac{d}{dt} \int_0^t \beta \, s^{\beta-1} \, x(s) \, ds \; a.e, \;\; t \in [0,T] \\
&= \quad \frac{t^{1-\beta}}{\beta} \, \beta \, t^{\beta-1} \, x(t) \; = \; x(t),
\end{aligned}$$

which means that the inverse of the operator I_β^α will be ${}^{R}D_\beta^\alpha$.

Theorem 4.4. *Let* $x \in C[0,T]$ *or (x be bounded and measurable), then* $I_\beta^\alpha \, {}^{R}D_\beta^\alpha \, x(t) = x(t)$*.*

Proof. Since x is bounded and measurable, then we have $I^{1-\alpha}\ x(t)|_{t=0}\ =\ 0$

$$\begin{aligned} I^{\alpha}_{\beta}\ {}^{R}D^{\alpha}_{\beta}\ x(t) &= I^{\alpha}\ \beta\ t^{\beta-1}\ {}^{R}D^{\alpha}_{\beta}\ x(t) \\ &= I^{\alpha}\ \beta\ t^{\beta-1}\ \frac{d}{dt^{\beta}}\ I^{1-\alpha}\ x(t) \\ &= I^{\alpha}\ \beta\ t^{\beta-1}\ \frac{t^{1-\beta}}{\beta}\frac{d}{dt}\ I^{1-\alpha}\ x(t) \\ &= I^{\alpha}\ \frac{d}{dt}\ I^{1-\alpha}\ x(t). \end{aligned}$$

But, $I^{1-\alpha}\ x(t)|_{t=0} = 0\ \Rightarrow\ I^{\alpha}\ \frac{d}{dt} = \frac{d}{dt}\ I^{\alpha}$, then

$$\begin{aligned} I^{\alpha}_{\beta}\ {}^{R}D^{\alpha}_{\beta}\ x(t) &= \frac{d}{dt}\ I^{\alpha}\ I^{1-\alpha}\ x(t) \\ &= \frac{d}{dt}\ I\ x(t)\ =\ x(t), \end{aligned}$$

which means that the inverse of the operator ${}^{R}D^{\alpha}_{\beta}$ will be I^{α}_{β}.

4.1. **Second kind Abel equation.** The linear second kind fractal-fractional Abel's integral equation (10) can be written as

$$x(t) + \lambda\ I^{\alpha}_{\beta}x(t) = f(t). \tag{11}$$

Theorem 4.5. *Let* $f \in C[0,T],\ \alpha + \beta > 1$. *If* $\frac{|\lambda|}{\Gamma(\alpha+\beta-1)}\ T^{\alpha+\beta-1} < 1$, *then the solution* $x \in C[0,T]$ *of (11) is given by*

$$x(t) = \sum_{n=0}^{\infty}(-\lambda)^{n}\ (I^{\alpha}_{\beta})^{n}\ f(t) \in C[0,T]. \tag{12}$$

Proof. From (11) we have

$$(1 + \lambda\ I^{\alpha}_{\beta})\ x(t)\ =\ f(t).$$

Now

$$\begin{aligned} |\ \lambda\ I^{\alpha}_{\beta}\ f(t)\ | &= |\ \lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\beta\ s^{\beta-1}f(s)\ ds\ | \\ &\le |\lambda| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\beta\ s^{\beta-1}\ |f(s)|\ ds \\ &\le |\lambda|\ ||f||\ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\beta\ s^{\beta-1}\ ds \\ &\le |\lambda|\ ||f||\ \beta\ I^{\alpha}\ t^{\beta-1} \\ &\le |\lambda|\ ||f||\ \beta\ \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta-1)}\ t^{\alpha+\beta-1},\ \alpha+\beta>1 \\ &\le ||f||\ \frac{|\lambda|}{\Gamma(\alpha+\beta-1)}\ T^{\alpha+\beta-1}. \end{aligned}$$

Then by the Neumann expansion [2] we have

$$\begin{aligned} x(t) &= \sum_{n=0}^{\infty}(-\lambda)^{n}\ (I^{\alpha}_{\beta})^{n}\ f(t) \\ &= f(t) - \lambda\ I^{\alpha}_{\beta}\ f(t) + \lambda^{2}\ (I^{\alpha}_{\beta})^{2}\ f(t) - \lambda^{3}\ (I^{\alpha}_{\beta})^{3}\ f(t) + ... \in C[0,T]. \end{aligned}$$

5. Conclusion

In this paper, we studied some of fundamental properties of fractal and fractal-fractional integral and differential operators and examined some various types of fractal-fractional Abel integral equations. The analysis of these equations provides valuable insights into their mathematical structure and potential applications in complex systems. Our findings contribute to the growing field of fractal and fractal-fractional calculus, paving the way for further research and applications in scientific and engineering disciplines.

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