Journal of Fractional Calculus and Applications
Vol. 16(2) July 2025, No.4.
ISSN: 2090-5858.
ISSN 2090-584X (print)
http://jfca.journals.ekb.eg/

ON UNIFORMLY STARLIKE AND CONVEX UNIVALENT FUNCTIONS

ABDELRAHMN M. YEHIA, SAMAR M. MADIAN AND MOHAMMED M.THARWAT

Abstract. In (1908) Jackson generalized the ordinary derivative by introducing the $q-$difference derivative, which became an essential tool in the study of q-calculus. Later, (2013) Brahim and Sidomon, introduced a further generalization known as the symmetric $q-$derivative operator, which has significant applications in various mathematical fields. This paper's primary goal is to investigate how the symmetric q-derivative can be used to define a novel class of convex and uniformly starlike univalent functions inside the complex plane's open unit disk. Numerous intriguing geometrical and analytical features are present in this recently described class of functions. In this study, we establish a wide range of characterizations for these functions, such as coefficient estimates, distortion theorems, some radii of starlikeness, convexity, close -to-convexity. Furthermore,, we determine sharp lower bounds for the ratios of the functions in this class and its partial sums $_v$ in the forms $\frac{(z)}{_v(z)}, \frac{_v(z)}{(z)}, \frac{'(z)}{'v(z)}$ and $\frac{'_v(z)}{'(z)}$. The findings reported in this study provide a substantial contribution to the fields of q-calculus and geometric function theory by providing fresh viewpoints and possible uses in complex analysis.

1. Introduction

Let N be the class of functions:

$$(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic and univalent in $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$.

2020 *Mathematics Subject Classification.* : 30C45, 30C50, 30C55.

Key words and phrases. Univalent functions, symmetric q-derivative operator, uniformly starlike and convex functions, partial sum.
Submitted March 23, 2025. Accepted May 13, 2025.

For $\lambda \geq 0$, $\in N$, $0 < q < 1$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we define the operator $\tilde{R}^n_{q,\lambda} : N \longrightarrow N$ by

$$\tilde{R}^0_{q,\lambda}(z) = (z), \tag{2}$$

$$\begin{aligned} \tilde{R}^1_{q,\lambda}(z) &= \tilde{R}_{q,\lambda}(z) = (1-\lambda)(z) + \lambda z \tilde{\bigtriangledown}_q(z) \\ &= z + \sum_{k=2}^{\infty} \left[1 + \lambda\left(\left[\tilde{k}\right]_q - 1\right)\right] a_k z^k \end{aligned} \tag{3}$$

and (in general)

$$\begin{aligned} \tilde{R}^n_{q,\lambda}(z) &= \tilde{R}_{q,\lambda}(z) = (1-\lambda)\tilde{R}^{n-1}_{q,\lambda}(z) + \lambda z \tilde{\bigtriangledown}_q\left(\tilde{R}^{n-1}_{q,\lambda}(z)\right) \\ &= z + \sum_{k=2}^{\infty} \left[1 + \lambda\left(\left[\tilde{k}\right]_q - 1\right)\right]^n a_k z^k, \end{aligned} \tag{4}$$

where the $q-$deference operator $\tilde{\bigtriangledown}_q(z)$ is given by

$$\tilde{\bigtriangledown}_q(z) = \begin{cases} \dfrac{(qz) - (q^{-1}z)}{(q - q^{-1})z} & \text{for } z \neq 0 \\ {}'(0) & \text{for } z = 0, \end{cases}$$

that is

$$\tilde{\bigtriangledown}_q(z) = 1 + \sum_{k=2}^{\infty} \left[\tilde{k}\right]_q a_k z^{k-1} \tag{5}$$

and

$$\left[\tilde{k}\right]_q = \frac{q^k - q^{-k}}{q - q^{-1}}, \quad \left[\tilde{0}\right]_q = 0, \tag{6}$$

which is defined by Brahim and Sidomou [5]. (see also [8], and [15])

We observe that for $q \to 1^-$, we obtain the differential operator D^n_λ defined by Al-Oboudi [1] and Frasin ([7], with $m = 1$). Also for $q \to 1^-$ and $\lambda = 1$, we get Sălăgean differential operator D^n [11] and for $\lambda = 1$, we get the symmetric Sălăgean q-differential operator D^n_q [16].

Definition 1. Using $\tilde{R}^n_{q,\lambda}$ and for $0 \leq \alpha < 1$, $0 \leq \beta \leq 1$, $\mu \geq 0$, $\lambda \geq 0$ and $n \in \mathbb{N}_0$, let $N_q(n, \lambda, \beta, \alpha, \mu)$ be the class of $\in N$ satisfying

$$\begin{aligned} &\Re\left\{\frac{(1-\beta) z\tilde{\bigtriangledown}_q\left(\tilde{R}^n_{q,\lambda}(z)\right) + \beta z\tilde{\bigtriangledown}_q\left(z\tilde{\bigtriangledown}_q\tilde{R}^n_{q,\lambda}(z)\right)}{(1-\beta)\left(\tilde{R}^n_{q,\lambda}(z)\right) + \beta z\tilde{\bigtriangledown}_q\left(\tilde{R}^n_{q,\lambda}(z)\right)} - \alpha\right\} \\ &\geq \mu\left|\frac{(1-\beta) z\tilde{\bigtriangledown}_q\left(\tilde{R}^n_{q,\lambda}(z)\right) + \beta z\tilde{\bigtriangledown}_q\left(z\tilde{\bigtriangledown}_q\tilde{R}^n_{q,\lambda}(z)\right)}{(1-\beta)\left(\tilde{R}^n_{q,\lambda}(z)\right) + \beta z\tilde{\bigtriangledown}_q\left(\tilde{R}^n_{q,\lambda}(z)\right)} - 1\right|. \end{aligned} \tag{7}$$

Not that:

(i) $N_q(0, \lambda, 0, \alpha, \mu) = \breve{S}_q(\alpha, \mu) = \left\{ : \Re\left\{\dfrac{z\tilde{\bigtriangledown}_q((z))}{(z)} - \alpha\right\}\right.$
$\left.\geq \mu\left|\dfrac{z\tilde{\bigtriangledown}_q((z))}{(z)} - 1\right|\right\}$, which defined by Kanas et al. [8];

(ii) $\lim_{q\to 1^-} N_q(0, \lambda, 0, \alpha, 0) = S^*(\alpha)$ (starlike of order α) and

$\lim_{q\to 1^-} N_q(0,\lambda,0,0,0) = S^*(0)$ (starlike of order 0) (Rebertson [9]);
(iii) $\lim_{q\to 1^-} N_q(0,\lambda,1,\alpha,0) = C(\alpha)$ (convex of order α) and
$\lim_{q\to 1^-} N_q(0,\lambda,1,0,0) = C(0)$ (convex of order 0) (Rebertson [9]);
(iv) $\lim_{q\to 1^-} N_q(0,\lambda,0,\alpha,\mu) = S(\alpha,\mu)$ ($\mu-$uniformly starlike of order α) and
$\lim_{q\to 1^-} N_q(0,\lambda,0,\alpha,1) = S(\alpha)$ (uniformly starlike of order α) (Shams et al. [11] and Owa et al. [10]);
(v) $\lim_{q\to 1^-} N_q(0,\lambda,1,\alpha,\mu) = K(\alpha,\mu)$ ($\mu-$uniformly convex of order α) and
$\lim_{q\to 1^-} N_q(0,\lambda,1,\alpha,1) = K(\alpha)$ (uniformly convex of order α) (Shams et al. [12] and Owa et al. [10]).

Also note that:

(i) $N_q(n,\lambda,0,\alpha,\mu) = N_q(n,\lambda,\alpha,\mu)$

$$= \left\{ : \Re\left\{ \frac{z\tilde{\bigtriangledown}_q\left(\tilde{R}^n_{q,\lambda}(z)\right)}{\tilde{R}^n_{q,\lambda}(z)} - \alpha \right\} \geq \mu \left| \frac{z\tilde{\bigtriangledown}_q\left(\tilde{R}^n_{q,\lambda}(z)\right)}{\tilde{R}^n_{q,\lambda}(z)} - 1 \right| \right\};$$

(ii) $N_q(n,\lambda,1,\alpha,\mu) = K_q(n,\lambda,\alpha,\mu) = \left\{ : \Re\left\{ \frac{\tilde{\bigtriangledown}_q\left(z\tilde{\bigtriangledown}_q\tilde{R}^n_{q,\lambda}(z)\right)}{\tilde{\bigtriangledown}_q\left(\tilde{R}^n_{q,\lambda}(z)\right)} - \alpha \right\}\right.$

$$\left. \geq \mu \left| \frac{\tilde{\bigtriangledown}_q\left(z\tilde{\bigtriangledown}_q\tilde{R}^n_{q,\lambda}(z)\right)}{z\tilde{\bigtriangledown}_q\left(\tilde{R}^n_{q,\lambda}(z)\right)} - 1 \right| \right\};$$

(iii) $N_q(n,\lambda,0,\alpha,0) = N_q(n,\lambda,\alpha) = \left\{ : \Re\left\{ \frac{z\tilde{\bigtriangledown}_q\left(\tilde{R}^n_{q,\lambda}(z)\right)}{\tilde{R}^n_{q,\lambda}(z)} \right\} \geq \alpha \right\};$

(iv)$N_q(n,\lambda,1,\alpha,0) = K_q(n,\lambda,\alpha) = \left\{ : \Re\left\{ \frac{\tilde{\bigtriangledown}_q\left(z\tilde{\bigtriangledown}_q\tilde{R}^n_{q,\lambda}(z)\right)}{\tilde{\bigtriangledown}_q\left(\tilde{R}^n_{q,\lambda}(z)\right)} \right\} \geq \alpha \right\}.$

Let

$$T = \left\{ \in N : (z) = z - \sum_{k=2}^{\infty} a_k z^k,\ a_k \geq 0 \right\} \tag{8}$$

and

$$T_q(n,\lambda,\beta,\alpha,\mu) = N_q(n,\lambda,\beta,\alpha,\mu) \cap T. \tag{9}$$

2. Main results

Unless indicated, we assume that $0 \leq \alpha < 1$, $0 \leq \beta \leq 1$, $\mu \geq 0$, $\lambda \geq 0$, $0 < q < 1$, $n \in \mathbb{N}_0$, $\in N$ and $z \in U$.

First, we obtain the coefficient estimates, which provide necessary and sufficient condition for a function to belong to the class $N_q(n, \lambda, \beta, \alpha, \mu)$.

Theorem 1. A function $\in N_q(n, \lambda, \beta, \alpha, \mu)$ if

$$\sum_{k=2}^{\infty}\left[1+\beta\left(\left[\tilde{k}\right]_q-1\right)\right]\left[(1+\mu)\left[\tilde{k}\right]_q-(\alpha+\mu)\right]\left[1+\lambda\left(\left[\tilde{k}\right]_q-1\right)\right]^n a_k \leq 1-\alpha. \tag{10}$$

Proof. It suffices to show that:

$$\mu\left|\frac{(1-\beta)\,z\tilde{\bigtriangledown}_q\left(\tilde{R}^n_{q,\lambda}(z)\right)+\beta z\tilde{\bigtriangledown}_q\left(z\tilde{\bigtriangledown}_q\tilde{R}^n_{q,\lambda}(z)\right)}{(1-\beta)\left(\tilde{R}^n_{q,\lambda}(z)\right)+\beta z\tilde{\bigtriangledown}_q\left(\tilde{R}^n_{q,\lambda}(z)\right)}-1\right|$$
$$-\Re\left\{\frac{(1-\beta)\,z\tilde{\bigtriangledown}_q\left(\tilde{R}^n_{q,\lambda}(z)\right)+\beta z\tilde{\bigtriangledown}_q\left(z\tilde{\bigtriangledown}_q\tilde{R}^n_{q,\lambda}(z)\right)}{(1-\beta)\left(\tilde{R}^n_{q,\lambda}(z)\right)+\beta z\tilde{\bigtriangledown}_q\left(\tilde{R}^n_{q,\lambda}(z)\right)}-1\right\}$$
$$\leq 1-\alpha.$$

We have

$$\mu\left|\frac{(1-\beta)\,z\tilde{\bigtriangledown}_q\left(\tilde{R}^n_{q,\lambda}(z)\right)+\beta z\tilde{\bigtriangledown}_q\left(z\tilde{\bigtriangledown}_q\tilde{R}^n_{q,\lambda}(z)\right)}{(1-\beta)\left(\tilde{R}^n_{q,\lambda}(z)\right)+\beta z\tilde{\bigtriangledown}_q\left(\tilde{R}^n_{q,\lambda}(z)\right)}-1\right|$$
$$-\Re\left\{\frac{(1-\beta)\,z\tilde{\bigtriangledown}_q\left(\tilde{R}^n_{q,\lambda}(z)\right)+\beta z\tilde{\bigtriangledown}_q\left(z\tilde{\bigtriangledown}_q\tilde{R}^n_{q,\lambda}(z)\right)}{(1-\beta)\left(\tilde{R}^n_{q,\lambda}(z)\right)+\beta z\tilde{\bigtriangledown}_q\left(\tilde{R}^n_{q,\lambda}(z)\right)}-1\right\}$$
$$\leq (1+\mu)\left|\frac{(1-\beta)\,z\tilde{\bigtriangledown}_q\left(\tilde{R}^n_{q,\lambda}(z)\right)+\beta z\tilde{\bigtriangledown}_q\left(z\tilde{\bigtriangledown}_q\tilde{R}^n_{q,\lambda}(z)\right)}{(1-\beta)\left(\tilde{R}^n_{q,\lambda}(z)\right)+\beta z\tilde{\bigtriangledown}_q\left(\tilde{R}^n_{q,\lambda}(z)\right)}-1\right|$$
$$\leq \frac{(1+\mu)\sum\limits_{k=2}^{\infty}\left(\left[\tilde{k}\right]_q-1\right)\left[1+\beta\left(\left[\tilde{k}\right]_q-1\right)\right]\left[1+\lambda\left(\left[\tilde{k}\right]_q-1\right)\right]^n|a_k|}{1-\sum\limits_{k=2}^{\infty}\left[1+\beta\left(\left[\tilde{k}\right]_q-1\right)\right]\left[1+\lambda\left(\left[\tilde{k}\right]_q-1\right)\right]^n|a_k|},$$

which is bounded above by $(1-\alpha)$ if

$$\sum_{k=2}^{\infty}\left[1+\beta\left(\left[\tilde{k}\right]_q-1\right)\right]\left[(1+\mu)\left[\tilde{k}\right]_q-(\alpha+\mu)\right]\left[1+\lambda\left(\left[\tilde{k}\right]_q-1\right)\right]^n a_k \leq 1-\alpha.$$

Second, we obtain the coefficient estimates, which provide necessary and sufficient condition for a function to belong to the class $T_q(n,\lambda,\beta,\alpha,\mu)$.

Theorem 2. A function $\in T_q(n,\lambda,\beta,\alpha,\mu)$ if and only if

$$\sum_{k=2}^{\infty}\left[1+\beta\left(\left[\tilde{k}\right]_q-1\right)\right]\left[(1+\mu)\left[\tilde{k}\right]_q-(\alpha+\mu)\right]\left[1+\lambda\left(\left[\tilde{k}\right]_q-1\right)\right]^n a_k \leq 1-\alpha. \tag{11}$$

Proof. From Theorem 1, we need to prove the only part. If $\in T_q(n,\lambda,\beta,\alpha,\mu)$ and z is real, then

$$\frac{1-\sum\limits_{k=2}^{\infty}\left[1+\lambda\left(\left[\tilde{k}\right]_q-1\right)\right]^n\left\{\left[\tilde{k}\right]_q\left[1+\beta\left(\left[\tilde{k}\right]_q-1\right)\right]\right\}a_kz^{k-1}}{1-\sum\limits_{k=2}^{\infty}\left[1+\lambda\left(\left[\tilde{k}\right]_q-1\right)\right]^n\left[1+\beta\left(\left[\tilde{k}\right]_q-1\right)\right]a_kz^{k-1}}-\alpha$$

$$\geq \mu\left|\frac{\sum\limits_{k=2}^{\infty}\left(\left[\tilde{k}\right]_q-1\right)\left[1+\beta\left(\left[\tilde{k}\right]_q-1\right)\right]\left[1+\lambda\left(\left[\tilde{k}\right]_q-1\right)\right]^n a_kz^{k-1}}{1-\sum\limits_{k=2}^{\infty}\left[1+\lambda\left(\left[\tilde{k}\right]_q-1\right)\right]^n\left[1+\beta\left(\left[\tilde{k}\right]_q-1\right)\right]a_kz^{k-1}}\right|.$$

Letting $z\longrightarrow 1^-$, we get (11).

Corollary 1. Let $\in T_q(n,\lambda,\beta,\alpha,\mu)$. Then

$$a_k \leq \frac{1-\alpha}{\left[1+\beta\left(\left[\tilde{k}\right]_q-1\right)\right]\left[(1+\mu)\left[\tilde{k}\right]_q-(\alpha+\mu)\right]\left[1+\lambda\left(\left[\tilde{k}\right]_q-1\right)\right]^n}, \tag{12}$$

$k\geq 2$. Equality holds for

$$(z)=z-\frac{1-\alpha}{\left[1+\beta\left(\left[\tilde{k}\right]_q-1\right)\right]\left[(1+\mu)\left[\tilde{k}\right]_q-(\alpha+\mu)\right]\left[1+\lambda\left(\left[\tilde{k}\right]_q-1\right)\right]^n}z^k, \tag{13}$$

$k\geq 2$.

Remark 1. Taking $n=0$ and $\beta=0$ in the previous results, we get the results given by Kanas et al. [8], Theorem 2.1, Theorem 2.2 and Corollary 2.1, respectively.

Also, we obtain distortion theorems, which provide sharp bounds for functions and their derivatives in the class $T_q(n,\lambda,\beta,\alpha,\mu)$. These bounds are crucial in understanding the growth and distortion properties of functions in this class.

Theorem 3. Let $\in T_q(n,\lambda,\beta,\alpha,\mu)$. Then

$$|z| - \frac{1-\alpha}{\left[1+\beta\left(\left[\tilde{2}\right]_q - 1\right)\right]\left[(1+\mu)\left[\tilde{2}\right]_q - (\alpha+\mu)\right]\left[1+\lambda\left(\left[\tilde{2}\right]_q - 1\right)\right]^n}|z|^2 \leq |(z)|$$
$$\leq |z| + \frac{1-\alpha}{\left[1+\beta\left(\left[\tilde{2}\right]_q - 1\right)\right]\left[(1+\mu)\left[\tilde{2}\right]_q - (\alpha+\mu)\right]\left[1+\lambda\left(\left[\tilde{2}\right]_q - 1\right)\right]^n}|z|^2 \tag{14}$$

and

$$1 - \frac{2(1-\alpha)}{\left[1+\beta\left(\left[\tilde{2}\right]_q - 1\right)\right]\left[(1+\mu)\left[\tilde{2}\right]_q - (\alpha+\mu)\right]\left[1+\lambda\left(\left[\tilde{2}\right]_q - 1\right)\right]^n}|z| \leq |'(z)|$$
$$\leq 1 + \frac{2(1-\alpha)}{\left[1+\beta\left(\left[\tilde{2}\right]_q - 1\right)\right]\left[(1+\mu)\left[\tilde{2}\right]_q - (\alpha+\mu)\right]\left[1+\lambda\left(\left[\tilde{2}\right]_q - 1\right)\right]^n}|z| \tag{15}$$

The bounds in (14) and (15) are attained for

$$(z) = |z| + \frac{1-\alpha}{\left[1+\beta\left([2]_q - 1\right)\right]\left[(1+\mu)[2]_q - (\alpha+\mu)\right]\left[1+\lambda\left([2]_q - 1\right)\right]^n}|z|^2. \tag{16}$$

Proof. First of all, for $\in T_q(n,\lambda,\beta,\alpha,\mu)$, it follows from (11) that

$$\sum_{k=2}^{\infty} a_k \leq \frac{1-\alpha}{\left[1+\beta\left(\left[\tilde{2}\right]_q - 1\right)\right]\left[(1+\mu)\left[\tilde{2}\right]_q - (\alpha+\mu)\right]\left[1+\lambda\left(\left[\tilde{2}\right]_q - 1\right)\right]^n},$$

which, in view of (8), yields

$$|(z)| \geq |z| - |z|^2 \sum_{k=2}^{\infty} a_k$$
$$\geq |z| - \frac{1-\alpha}{\left[1+\beta\left(\left[\tilde{2}\right]_q - 1\right)\right]\left[(1+\mu)\left[\tilde{2}\right]_q - (\alpha+\mu)\right]\left[1+\lambda\left(\left[\tilde{2}\right]_q - 1\right)\right]^n}|z|^2,$$

and

$$|(z)| \leq |z| + |z|^2 \sum_{k=2}^{\infty} a_k$$
$$\leq |z| + \frac{1-\alpha}{\left[1+\beta\left(\left[\tilde{2}\right]_q - 1\right)\right]\left[(1+\mu)\left[\tilde{2}\right]_q - (\alpha+\mu)\right]\left[1+\lambda\left(\left[\tilde{2}\right]_q - 1\right)\right]^n}|z|^2.$$

Next, we see from (11) that:

$$\frac{\left[1+\beta\left(\left[\tilde{2}\right]_q - 1\right)\right]\left[(1+\mu)\left[\tilde{2}\right]_q - (\alpha+\mu)\right]\left[1+\lambda\left(\left[\tilde{2}\right]_q - 1\right)\right]^n}{2}\sum_{k=2}^{\infty} k a_k \leq$$
$$\sum_{k=2}^{\infty}\left[1+\beta\left(\left[\tilde{k}\right]_q - 1\right)\right]\left[(1+\mu)\left[\tilde{k}\right]_q - (\alpha+\mu)\right]\left[1+\lambda\left(\left[\tilde{k}\right]_q - 1\right)\right]^n a_k$$
$$\leq 1-\alpha,$$

then

$$\sum_{k=2}^{\infty} ka_k \leq \frac{2(1-\alpha)}{\left[1+\beta\left(\left[\tilde{2}\right]_q-1\right)\right]\left[(1+\mu)\left[\tilde{2}\right]_q-(\alpha+\mu)\right]\left[1+\lambda\left(\left[\tilde{2}\right]_q-1\right)\right]^n},$$

which, again in view of (8), yields

$$\left|{}'(z)\right| \geq 1-|z|\sum_{k=2}^{\infty} ka_k$$

$$\geq 1-\frac{2(1-\alpha)}{\left[1+\beta\left(\left[\tilde{2}\right]_q-1\right)\right]\left[(1+\mu)\left[\tilde{2}\right]_q-(\alpha+\mu)\right]\left[1+\lambda\left(\left[\tilde{2}\right]_q-1\right)\right]^n}|z|,$$

and

$$\left|{}'(z)\right| \leq 1+|z|\sum_{k=2}^{\infty} ka_k$$

$$\leq 1+\frac{2(1-\alpha)}{\left[1+\beta\left(\left[\tilde{2}\right]_q-1\right)\right]\left[(1+\mu)\left[\tilde{2}\right]_q-(\alpha+\mu)\right]\left[1+\lambda\left(\left[\tilde{2}\right]_q-1\right)\right]^n}|z|.$$

Finally, the bounds in (14) and (15) are attained for given by (16).

Remark 2. Taking $n=0$ and $\beta=0$ in Theorem 3, we get the results given by Kanas et al. [8], Theorem 2.3, Theorem 2.4.

Let ${}_j(z)$ be defined, for $j=1,2,...,m$ by

$$_v(z)=z-\sum_{k=2}^{\infty} a_{k,j}z^k \qquad (a_{k,j}\geq 0,\ z\in U). \tag{17}$$

Now, we show that any function defined using basis functions already present in the class $T_q(n,\lambda,\beta,\alpha,\mu)$ also belongs to the class $T_q(n,\lambda,\beta,\alpha,\mu)$. This result is essential for proving that the class is closed under such representations, simplifying function analysis and decomposition.

Theorem 4. Let ${}_j(z)\in T_q(n,\lambda,\beta,\alpha,\mu)$ for $j=1,2,...,m$. Then $h(z)\in T_q(n,\lambda,\beta,\alpha,\mu)$, where

$$h(z)=\sum_{j=1}^{m} b_{j\,j}(z), \quad b_j\geq 0 \text{ and } \sum_{j=1}^{m} b_j=1. \tag{18}$$

Proof. By (18), we have

$$h(z)=\sum_{k=2}^{\infty}\left(\sum_{j=1}^{m} b_j a_{k,j}\right)z^k.$$

Further, since ${}_j(z)\in T_q(n,\lambda,\beta,\alpha,\mu)$, we get

$$\sum_{k=2}^{\infty}\left[1+\beta\left(\left[\tilde{k}\right]_q-1\right)\right]\left[(1+\mu)\left[\tilde{k}\right]_q-(\alpha+\mu)\right]\left[1+\lambda\left(\left[\tilde{k}\right]_q-1\right)\right]^n a_{k,j}$$

$$\leq 1-\alpha.$$

Hence

$$\sum_{k=2}^{\infty}\left[1+\beta\left(\left[\tilde{k}\right]_q-1\right)\right]\left[(1+\mu)\left[\tilde{k}\right]_q-(\alpha+\mu)\right]\left[1+\lambda\left(\left[\tilde{k}\right]_q-1\right)\right]^n$$

$$\times\left(\sum_{j=1}^{m}b_j a_{k,j}\right)=\sum_{j=1}^{m}b_j\left[\sum_{k=2}^{\infty}\left[1+\beta\left(\left[\tilde{k}\right]_q-1\right)\right]\left[(1+\mu)\left[\tilde{k}\right]_q-(\alpha+\mu)\right]\right.$$

$$\left.\times\left[1+\lambda\left(\left[\tilde{k}\right]_q-1\right)\right]^n a_{k,j}\right]\leq\sum_{j=1}^{m}b_j(1-\alpha)=1-\alpha,$$

which implies that $h(z)\in T_q(n,\lambda,\beta,\alpha,\mu)$. Thus we have the theorem.

Taking $b_1=\tau$ and $b_2=1-\tau$ in Theorem 4, we have:

Corollary 3. The class $T_q(n,\lambda,\beta,\alpha,\mu)$ is closed under convex linear combination.

Also, we obtain the extreme points of the class $T_q(n,\lambda,\beta,\alpha,\mu)$, which are crucial in understanding the convex structure of the class.

Theorem 5. Let ${}_1(z)=z$ and

$$_k(z)=z-\frac{1-\alpha}{\left[1+\beta\left(\left[\tilde{k}\right]_q-1\right)\right]\left[(1+\mu)\left[\tilde{k}\right]_q-(\alpha+\mu)\right]\left[1+\lambda\left(\left[\tilde{k}\right]_q-1\right)\right]^n}z^k,$$

$k\geq 2$. Then $\in T_q(n,\lambda,\beta,\alpha,\mu)$ if and only if

$$(z)=\sum_{k=1}^{\infty}\tau_{k\,k}(z),$$

where $\tau_k\geq 0\ \ (k\geq 1)$ and $\sum\limits_{k=1}^{\infty}\tau_k=1$.

Proof. Suppose that

$$(z)=\sum_{k=1}^{\infty}\tau_{k\,k}(z)=z-\frac{1-\alpha}{\left[1+\beta\left(\left[\tilde{k}\right]_q-1\right)\right]\left[(1+\mu)\left[\tilde{k}\right]_q-(\alpha+\mu)\right]\left[1+\lambda\left(\left[\tilde{k}\right]_q-1\right)\right]^n}\tau_k z^k. \tag{19}$$

Then it follows that:

$$\sum_{k=2}^{\infty}\frac{\left[1+\beta\left(\left[\tilde{k}\right]_q-1\right)\right]\left[(1+\mu)\left[\tilde{k}\right]_q-(\alpha+\mu)\right]\left[1+\lambda\left(\left[\tilde{k}\right]_q-1\right)\right]^n}{1-\alpha}\times$$

$$\frac{1-\alpha}{\left[1+\beta\left(\left[\tilde{k}\right]_q-1\right)\right]\left[(1+\mu)\left[\tilde{k}\right]_q-(\alpha+\mu)\right]\left[1+\lambda\left(\left[\tilde{k}\right]_q-1\right)\right]^n}\tau_k$$

$$= \sum_{k=2}^{\infty} \tau_k = 1 - \tau_1 \leq 1.$$

So by Theorem 2, $\in T_q(n, \lambda, \beta, \alpha, \mu)$. Conversely, let $\in T_q(n, \lambda, \beta, \alpha, \mu)$. Then

$$a_k \leq \frac{1-\alpha}{\left[1+\beta\left(\left[\tilde{k}\right]_q - 1\right)\right]\left[(1+\mu)\left[\tilde{k}\right]_q - (\alpha+\mu)\right]\left[1+\lambda\left(\left[\tilde{k}\right]_q - 1\right)\right]^n},$$

$k \geq 2$. Setting

$$\tau_k = \frac{\left[1+\beta\left(\left[\tilde{k}\right]_q - 1\right)\right]\left[(1+\mu)\left[\tilde{k}\right]_q - (\alpha+\mu)\right]\left[1+\lambda\left(\left[\tilde{k}\right]_q - 1\right)\right]^n}{1-\alpha} a_k,$$

$k \geq 2$ and

$$\tau_1 = 1 - \sum_{k=2}^{\infty} \tau_k,$$

we see that can be expressed in the form (19).

Remark 3. Taking $n = 0$ and $\beta = 0$ in Theorem 5, we get the results given by Kanas et al. [8], Theorem 2.5.

Corollary 4. The extreme points of $T_q(n, \lambda, \beta, \alpha, \mu)$ are ${}_k(z)$ $(k \geq 1)$ given by Theorem 5.

Now, we obtain the radii of starlikeness, convexity, and close-to-convexity in the class $T_q(n, \lambda, \beta, \alpha, \mu)$, which are essential in understanding the geometric properties of functions in this class and their behavior in specific regions of the unit disk.

Theorem 6. Let $\in T_q(n, \lambda, \beta, \alpha, \mu)$. Then for $0 \leq \sigma < 1$, is
(i) Close -to- convex of order σ in $|z| < r_1$,

$$r_1 = r_1(n, \lambda, \beta, \alpha, \mu, \sigma) :=$$

$$\inf_k \left[\frac{(1-\sigma)\left[1+\beta\left(\left[\tilde{k}\right]_q - 1\right)\right]\left[(1+\mu)\left[\tilde{k}\right]_q - (\alpha+\mu)\right]\left[1+\lambda\left(\left[\tilde{k}\right]_q - 1\right)\right]^n}{k(1-\alpha)}\right]^{\frac{1}{k-1}}, \tag{20}$$

$k \geq 2$.
(ii) Starlike of order σ in $|z| < r_2$,

$$r_2 = r_2(n, \lambda, \beta, \alpha, \mu, \sigma) :=$$

$$\inf_k \left[\frac{(1-\sigma)\left[1+\beta\left(\left[\tilde{k}\right]_q - 1\right)\right]\left[(1+\mu)\left[\tilde{k}\right]_q - (\alpha+\mu)\right]\left[1+\lambda\left(\left[\tilde{k}\right]_q - 1\right)\right]^n}{(k-\sigma)(1-\alpha)}\right]^{\frac{1}{k-1}}.$$

$k \geq 2$.
(iii) Convex of order σ in $|z| < r_3$,

$$r_3 = r_3\left(n, \lambda, \beta, \alpha, \mu, \sigma\right) :=$$

$$\inf_k \left[\frac{(1-\sigma)\left[1+\beta\left(\left[\tilde{k}\right]_q - 1\right)\right]\left[(1+\mu)\left[\tilde{k}\right]_q - (\alpha+\mu)\right]\left[1+\lambda\left(\left[\tilde{k}\right]_q - 1\right)\right]^n}{k\left(k-\sigma\right)\left(1-\alpha\right)} \right]^{\frac{1}{k-1}},$$

$k \geq 2$. The results are sharp, for given by (13).
Proof. To prove (i) we must show that:

$$\left|'\left(z\right) - 1\right| \leq 1-\sigma \quad \text{for } |z| < r_1\left(n, \lambda, \beta, \alpha, \mu, \sigma\right).$$

From (2), we have

$$\left|'\left(z\right) - 1\right| \leq \sum_{k=2}^{\infty} k a_k \left|z\right|^{k-1}.$$

Thus

$$\left|'\left(z\right) - 1\right| \leq 1-\sigma,$$

if

$$\sum_{k=2}^{\infty} \left(\frac{k}{1-\sigma}\right) a_k \left|z\right|^{k-1} \leq 1. \tag{21}$$

By Theorem 2, (21) will be true if

$$\left(\frac{k}{1-\sigma}\right)\left|z\right|^{k-1} \leq$$

$$\frac{\left[1+\beta\left(\left[\tilde{k}\right]_q - 1\right)\right]\left[(1+\mu)\left[\tilde{k}\right]_q - (\alpha+\mu)\right]\left[1+\lambda\left(\left[\tilde{k}\right]_q - 1\right)\right]^n}{(1-\alpha)},$$

that is, if

$$|z| \leq$$

$$\left[\frac{(1-\sigma)\left[1+\beta\left(\left[\tilde{k}\right]_q - 1\right)\right]\left[(1+\mu)\left[\tilde{k}\right]_q - (\alpha+\mu)\right]\left[1+\lambda\left(\left[\tilde{k}\right]_q - 1\right)\right]^n}{k\left(1-\alpha\right)} \right]^{\frac{1}{k-1}},$$

$(k \geq 2)$, which gives (20).
To prove (ii) and (iii) it suffices to show that

$$\left|\frac{z''\left(z\right)}{'\left(z\right)} - 1\right| < 1-\sigma \quad \text{for } |z| < r_3,$$

$$\left|\frac{z'\left(z\right)}{\left(z\right)} - 1\right| \leq 1-\sigma \quad \text{for } |z| < r_2,$$

respectively, by using arguments as in proving (i).

For $\in N$, its partial sums are given by

$$_v(z) = z + \sum_{k=2}^{v} a_k z^k.$$

Silverman [14] determined sharp lower bounds for the real part of $\frac{(z)}{_v(z)}, \frac{_v(z)}{(z)}, \frac{'(z)}{'_v(z)}$ and $\frac{'_v(z)}{'(z)}$ for some subclasses of N.

We will follow the works of [2, 3, 4, 6, 13, 14] on partial sums of analytic functions, to obtain our results of this section. We let

$$\Psi_{q,k}^{n} = \left[1 + \beta\left(\left[\tilde{k}\right]_q - 1\right)\right]\left[(1+\mu)\left[\tilde{k}\right]_q - (\alpha+\mu)\right]\left[1 + \lambda\left(\left[\tilde{k}\right]_q - 1\right)\right]^n. \quad (22)$$

Finally, we establish bounds on partial sums, which are significant for approximating functions and analyzing their convergence.

Theorem 7. If satisfies (10), then

$$\Re\left(\frac{(z)}{_v(z)}\right) \geq \frac{\Psi_{q,v+1}^{n} - 1 + \alpha}{\Psi_{q,v+1}^{n}} \quad (z \in U), \quad (23)$$

where

$$\Psi_{q,k}^{n} \geq \begin{cases} 1-\alpha, & \text{if } k = 2,3,...v \\ \Psi_{q,v+1}^{n}, & \text{if } k \geq v+1. \end{cases} \quad (24)$$

The result (23) is sharp for

$$(z) = z + \frac{1-\alpha}{\Psi_{q,v+1}^{n}} z^k. \quad (25)$$

Proof. Define $h(z)$ by

$$\frac{1+h(z)}{1-h(z)} = \frac{\Psi_{q,v+1}^{n}}{1-\alpha}\left[\frac{(z)}{_v(z)} - \frac{\Psi_{q,v+1}^{n} - 1 + \alpha}{\Psi_{q,v+1}^{n}}\right]$$

$$= \frac{1 + \sum\limits_{k=2}^{v} a_k z^{k-1} + \left(\frac{\Psi_{q,v+1}^{n}}{1-\alpha}\right)\sum\limits_{k=v+1}^{\infty} a_k z^{k-1}}{1 + \sum\limits_{k=2}^{v} a_k z^{k-1}}. \quad (26)$$

It suffices to show that $|h(z)| \leq 1$. Now from (26) we have

$$h(z) = \frac{\left(\frac{\Psi_{q,v+1}^{n}}{1-\alpha}\right)\sum\limits_{k=v+1}^{\infty} a_k z^{k-1}}{2 + 2\sum\limits_{k=2}^{v} a_k z^{k-1} + \left(\frac{\Psi_{q,v+1}^{n}}{1-\alpha}\right)\sum\limits_{k=v+1}^{\infty} a_k z^{k-1}}.$$

Hence we obtain

$$|h(z)| \leq \frac{\left(\frac{\Psi^{v}_{n+1}}{1-\alpha}\right)\sum_{k=v+1}^{\infty} a_k}{2-2\sum_{k=2}^{v} a_k - \left(\frac{\Psi^{v}_{n+1}}{1-\alpha}\right)\sum_{k=v+1}^{\infty} a_k}.$$

Now $|h(z)| \leq 1$ if and only if

$$2\left(\frac{\Psi^{n}_{q,v+1}}{1-\alpha}\right)\sum_{k=v+1}^{\infty} a_k \leq 2-2\sum_{k=2}^{v} a_k.$$

or, equivalently,

$$\sum_{k=2}^{v} a_k + \sum_{k=n+1}^{\infty} \frac{\Psi^{n}_{q,v+1}}{1-\alpha} a_k \leq 1.$$

By (10), we get

$$\sum_{k=2}^{v} a_k + \sum_{k=v+1}^{\infty} \frac{\Psi^{n}_{q,v+1}}{1-\alpha} a_k \leq \sum_{k=2}^{\infty} \frac{\Psi^{n}_{q,k}}{1-\alpha} a_k,$$

which is equivalent to

$$\sum_{k=2}^{v}\left(\frac{\Psi^{n}_{q,k}-1+\alpha}{1-\alpha}\right) a_k + \sum_{k=v+1}^{\infty}\left(\frac{\Psi^{n}_{q,k}-\Psi^{n}_{q,v+1}}{1-\alpha}\right) a_k \geq 0.$$

For $z = re^{i\pi/n}$ and $r \to 1^{-}$ we have

$$\frac{(z)}{{}_{v}(z)} = 1 + \frac{1-\alpha}{\Psi^{n}_{q,v+1}} z^k \longrightarrow 1 - \frac{1-\alpha}{\Psi^{n}_{q,v+1}} = \frac{\Psi^{n}_{q,v+1}-1+\alpha}{\Psi^{n}_{q,v+1}},$$

then (25) gives the sharpness.

Theorem 8. If subject to (10) and $\Psi^{n}_{q,v+1}$ as in (24), then

$$\Re\left(\frac{{}_{v}(z)}{(z)}\right) \geq \frac{\Psi^{n}_{q,v+1}}{\Psi^{n}_{q,v+1}+1-\alpha}\ ,$$

(25) gives the sharpness.
Proof. Letting

$$\frac{1+h(z)}{1-h(z)} = \frac{\Psi^{n}_{q,v+1}+1-\alpha}{1-\alpha}\left[\frac{{}_{v}(z)}{(z)} - \frac{\Psi^{n}_{q,v+1}}{\Psi^{n}_{q,v+1}+1-\alpha}\right]$$

and much akin to similar arguments in Theorem 7. So, we omit it.

Theorem 9. If satisfies (10), then

$$\Re\left(\frac{{}'(z)}{{}'_{v}(z)}\right) \geq \frac{\Psi^{n}_{q,v+1}-(v+1)(1-\alpha)}{\Psi^{n}_{q,v+1}} \quad (z \in U),$$

and

$$\Re\left(\frac{{}'_{v}(z)}{{}'(z)}\right) \geq \frac{\Psi^{n}_{q,v+1}}{\Psi^{n}_{q,v+1}+(v+1)(1-\alpha)} \quad (z \in U), \tag{27}$$

where $\Psi^n_{q,v+1} \geq (v+1)(1-\alpha)$ and

$$\Psi^n_{q,k} \geq \begin{cases} k(1-\alpha), & \text{if } \ k = 2,3,...v \\ k\left(\dfrac{\Psi^n_{q,v+1}}{v+1}\right), & \text{if } \ k \geq v+1, v+2, ..., \end{cases}$$

(25) gives the sharpness.

Proof. We write

$$\frac{1+h(z)}{1-h(z)} = \frac{\Psi^n_{q,v+1}}{(v+1)(1-\alpha)}\left[\frac{'(z)}{'_v(z)} - \frac{\Psi^n_{q,v+1} - (v+1)(1-\alpha)}{\Psi^n_{q,v+1}}\right],$$

where

$$h(z) = \frac{\left(\dfrac{\Psi^n_{q,v+1}}{(v+1)(1-\alpha)}\right)\sum\limits_{k=v+1}^{\infty} a_k z^{k-1}}{2 + 2\sum\limits_{k=2}^{v} k a_k z^{k-1} + \left(\dfrac{\Psi^n_{q,v+1}}{(v+1)(1-\alpha)}\right)\sum\limits_{k=v+1}^{\infty} k a_k z^{k-1}}.$$

Now $|h(z)| \leq 1$ if and only if

$$\sum_{k=2}^{v} k a_k + \left(\frac{\Psi^n_{v+1}}{(v+1)(1-\alpha)}\right)\sum_{k=v+1}^{\infty} k a_k \leq 1.$$

From (10), we get

$$\sum_{k=2}^{v} k a_k + \left(\frac{\Psi^n_{q,v+1}}{(v+1)(1-\alpha)}\right)\sum_{k=v+1}^{\infty} k a_k \leq \sum_{k=2}^{\infty} \frac{\Psi^n_{q,k}}{1-\alpha} a_k,$$

which is equivalent to

$$\sum_{k=2}^{v} \frac{\Psi^n_{q,k} - k(1-\alpha)}{1-\alpha} a_k + \sum_{k=v+1}^{\infty}\left(\frac{(v+1)\Psi^n_{q,k} - k\Psi^n_{q,v+1}}{(v+1)(1-\alpha)}\right) a_k \geq 0.$$

To prove the result (27), define $h(z)$ by

$$\frac{1+h(z)}{1-h(z)} = \frac{(v+1)(1-\alpha) + \Psi^n_{q,v+1}}{(v+1)(1-\alpha)}\left[\frac{'_v(z)}{'(z)} - \frac{\Psi^n_{q,v+1}}{(v+1)(1-\alpha) + \Psi^n_{q,v+1}}\right],$$

we get the desired result through similar arguments in the first part.

3. Conclusion

In the present paper, a new class of uniformly starlike and convex functions is defined by using a new symmetric $q-$ derivative operator. For this class of functions, we obtain many characterizations such as coefficient estimates, distortion theorems, some radii of starlikeness, convexity, close -to- convexity. Also, we determine sharp lower bounds for the ratios of the functions in this class and its partial sums $_v$.

Open Problem

The authors suggest investigating the quasi Hadamard product for functions $_{1},\ _{2},...,\ _{k}\in T_{q}\left(n,\lambda,\beta,\alpha,\mu\right)$. Studying this product is significant because it can:
(i) Provide deeper insights into the structural properties of functions in the class $T_{q}\left(n,\lambda,\beta,\alpha,\mu\right);$
(ii) Extend known results on Hadamard products to more generalized settings, potentially leading to new applications in geometric function theory or related areas;
(iii) Unify or generalize existing theorems by considering finite combinations of functions within this class.
Further research in this direction may also open avenues for exploring coefficient estimates, distortion theorems, or subordination properties involving the quasi Hadamard product.

Acknowledgments

The authors wish to thank Prof. Dr. M. K. Aouf for his kind encouragement and help in the preparation of this paper.

References

References

[1] Al-Oboudi, F. M., On univalent functions defined by a generalized Sălăgean operator, Int. J. Math. Math. Sci., (2004), no. 25-28, 1429 − 1436.
[2] Aouf, M. K., Mostafa, A. O., Lashin, A. Y. and Munassar, B. M., Partial sums for a certain subclass of meromorphic univalent functions, Sarajevo J. Math., 10 (2014), no. 23, 161 − 169.
[3] Aouf, M. K., Shamandy, A., Mostafa, A. O. and Adwan, E. A., Partial sums of certain of analytic functions defined by Dziok-Srivastava operator, Acta Univ. Apulensis, (2012), no. 30, 65 − 76.
[4] Aouf, M. K. and Srivastava, H. M., Some families of starlike functions with negative coefficients, J. Math. Anal. Appl., 203 (1996), Art. no. 0411, 762 − 790.
[5] Brahim, K. L. and Sidomon, Y., On some symmetric q-special functions, Matematiche, 68 (2013), 107 − 122.
[6] Rosy, T., Subramanian, K. G. and Murugusundaramoorthy, G., Neighborhoods and partial sums of starlike functions based on Ruscheweyeh derivatives, J. Ineq. Pure Appl. Math., 64 (2003), no. 4, Art., 4, 1 − 8.
[7] Frasin, B. A., A new differential operator of analytic functions involving binomial series, Bol. Soc. Paran Mat., 38 (2020), no. 5, 205 − 213.
[8] Kanas, S., Altinkaya, S. and Yalcin, S., Subclass of k-uniformly starlike functions defined by symmetric q-derivative operator, Ukrainian Math. J., 70 (2019), no.11, 1727 − 1740.
[9] Robertson, M. S., On the theory of univalent functions, Ann. Math., 37 (1936), 374 − 408.
[10] Owa, S., Polatoglu, Y. and Yavuz, E., Coefficient inequalities for classes of uniformly starlike and convex functions, J. Inequal. Pure Appl. Math., 7 (2006), no. 5, 1 − 5.
[11] Sălăgean, G., Subclasses of univalent functions, Lecture note in Math., Springer-Verlag, 1013 (1983), 362 − 372.
[12] Shams, S., Kulkarni, S. R. and Jahangiri, J. M., Classes of uniformly starlike and convex functions, Internat. J. Math. Math. Sci., 55 (2004), 2959 − 2961.

[13] Sheil-Small, T., A note on partial sums of convex schlicht functions, Bull. London Math. Soc., 2 (1970), $165 - 168$.
[14] Silverman, H., Partial sums of starlike and convex functions, J. Math. Appl., 209 (1997), $221 - 227$.
[15] Srivastava, H. M., Hari, M., Altinkaya, S. and Yalcin, S., Hankel determinant for a subclass of bi-univalent functions by using a symmetric $q-$derivative operator, Filomat, 32(2018), no. 2, $503 - 56$.
[16] Zhang, C., Khan, S., Hussain, A., Khan, N., Hussain, S. and Khan, N., Applications of q-difference symmetric operator in harmonic univalent functions, AIMS Math., 7 (2021), $667 - 680$.

Abdelrahmn M. Yehia, Mathematics and Computer Science Department, Faculty of Science,, Beni-Suef University, Egypt
Email address: abdelrahmnyehia@gmail.com

Samar M. Madian, Basic Science Department, Higher Institute of Engineering and Technology, New Damietta, Egypt
Email address: samar_math@yahoo.com

Mohammed M.Tharwat, Mathematics and Computer Science Department, Faculty of Science,, Beni-Suef University, Egypt
Email address: zahraa26@yahoo.com