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CARATHÉODORY THEOREM FOR A NONLOCAL BOUNDARY VALUE PROBLEMS OF A FUNCTIONAL INTEGRO-FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we study the existence of solutions for a nonlocal two-point boundary value problem associated with an ordinary integro-fractional differential equation. Hyers-Ulam stability will be proved. Furthermore, the continuous dependence of the unique solution based on given parameters will be addressed. Several special cases and examples will be mentioned.

1. Introduction

Nonlocal boundary value problems for functional integro-fractional differential equations are an interesting and complex area of study in mathematical analysis. These problems combine fractional calculus with integro-differential equations. Adding nonlocal boundary conditions introduces further complexity, as these conditions typically involve values of the solution at points that are not limited to just the endpoints of the domain. naturally this equationsarise in many fields, such as physics, biology, and finance, where the rate of change depends not only on the present state of the system but also on the accumulated history of the system. [17, 2, 11, 5, 9].

Here, we are concerned with the following functional integro-fractional differential equation, which combines ordinary and fractional-order derivatives,

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$$\frac{\mathrm{d}y}{\mathrm{d}t} = a(t) + \lambda \int_0^t f(t,s,D^\alpha y(s))\mathrm{d}s, \quad t \in (0,1), \alpha \in (0,1), \lambda > 0, \tag{1.1}$$

subject to the two-point nonlocal boundary condition,

$$y(\tau) = \gamma y(\eta), \quad \tau, \eta \in [0,1], \gamma \neq 1, \tag{1.2}$$

Differential equations with fractional order have emerged as important tools for modeling diverse phenomena in science and engineering disciplines. Notably, there has been notable progress in the exploration of fractional differential equations and inclusions in recent times, as evidenced by scholarly works such as the publications by Kilbas et al. [15], Podlubny [19] and the comprehensive survey conducted by Agarwal et al. [3].

The paper is organized as follows: Section 2 contains the solvability of at least one solution $y \in C[0,T]$ applying the *Carathéodory* Theorem and the continuous dependence of the unique solution $y \in C[0,T]$ on the parameter λ and on the function a [24]. Moreover, the Hyers − Ulam stability [14] of (1.1)-(1.2). Some general discussion and examples are provided in Section 3.

2. Main Results

2.1. **Existence of Solutions.** Let $C[0,1]$ be the space of continuous functions on $[0,1]$, with the standard norm [16]

$$\|f\| = \sup_{t\in[0,1]} |f(t)|.$$

$f : [0,1]\times[0,1]\times\mathbb{R} \to \mathbb{R}$ is an L^1-*Carathéodory* function [8, 4], that is, the following properties are satisfied:

i) f is continuous in $y \in \mathbb{R}$ for each fixed (t,s) in $[0,1] \times [0,1]$.
ii) f is continuous in $t, \forall s \in [0,1], x \in \mathbb{R}$.
iii) f is measurable in $s, \forall t \in [0,1], x \in \mathbb{R}$.
iv) $|f(t,s,x(s))| \leq k(t,s)$.

Take into account the following assumptions:

v) $\sup_{t\in[0,1]} I^{1-\alpha} \int_0^s |k(s,\theta)| \,\mathrm{d}\theta < K, \quad K > 0.$
vi) $k(t,s)$ is continuous in $(t,s) \in [0,1] \times [0,1]$.
vii) $a(t)$ is continuous on $[0,1]$.

Now, we have the following lemma.

Lemma 2.1. *If the solution of (1.1)-(1.2) is exist, then it can be represented by*

$$y(t) = \frac{1}{1-\gamma}\left[-I^\alpha u|_{t=\tau} + \gamma I^\alpha u|_{t=\eta}\right] + I^\alpha u(t), \tag{2.1}$$

where $u(t)$ is given by

$$u(t) = I^{1-\alpha}a(t) + \lambda I^{1-\alpha}\int_0^t f(t,s,u(s))\mathrm{d}s, \quad t \in [0,1]. \tag{2.2}$$

proof. Operating by $I^{1-\alpha}$ on both sides of (1.1), we obtain

$$D^\alpha y(t) = I^{1-\alpha}a(t) + \lambda I^{1-\alpha}\int_0^t f(t,s,D^\alpha y(s))\mathrm{d}s. \tag{2.3}$$

Letting $D^{\alpha}y(t) = u(t)$, we obtain (2.2).
Additionally, we have [18]

$$y(t) = y(0) + I^{\alpha}u(t). \tag{2.4}$$

To determine $y(0)$ in (2.4), we evaluate $y(t)$ at $t = \tau$ and $t = \eta$, respectively

$$y(\tau) = y(0) + I^{\alpha}u|_{t=\tau} \tag{2.5}$$

and

$$y(\eta) = y(0) + I^{\alpha}u|_{t=\eta}\,. \tag{2.6}$$

From (2.5) and (2.6) in (1.2), a straightforward calculation yields

$$y(0) = \frac{1}{1-\gamma}\left[-I^{\alpha}u|_{t=\tau} + \gamma I^{\alpha}u|_{t=\eta}\right].$$

Replacing $y(0)$ in (2.4), we obtain (2.1).
Conversely, we operate by I^{α} on both sides of (2.2), we obtain

$$I^{\alpha}u(t) = \int_0^t a(s)\mathrm{d}s + \lambda\int_0^t\int_0^s f(s,\theta,u(\theta))\mathrm{d}\theta\ ds.$$

Then, we substitute in (2.4), we get

$$y(t) = y(0) + \int_0^t a(s)\mathrm{d}s + \lambda\int_0^t\int_0^s f(s,\theta,u(\theta))\mathrm{d}\theta\ ds.$$

Finally, differentiating with respect to t, from which we obtain (1.1).

Now, we have the following existences Theorem.

Theorem 2.1. *Let the assumptions $(i)-(vii)$ be satisfied, then the problem (1.1)–(1.2) has at least one solution $y \in C[0,1]$.*

proof. Firstly, we define the sequence $\{u_n(t)\} \subset C[0,1]$ satisfying the iterative formula

$$u_{n+1}(t) = I^{1-\alpha}a(t) + \frac{\lambda}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\left[\int_0^s f(s,\theta,u_n(\theta))\mathrm{d}\theta\right]\mathrm{d}s, \quad t\in(0,1), \alpha\in(0,1), \lambda>0. \tag{2.7}$$

Now, for all $n = 0,1,2,\ldots$, we have

$$\begin{aligned}|u_{n+1}(t)| &\le \left|I^{1-\alpha}a(t)\right| + \frac{\lambda}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\left[\int_0^s |f(s,\theta,u_n(\theta))|\mathrm{d}\theta\right]\mathrm{d}s\\ &\le M_1 + \frac{\lambda}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\left[\int_0^s k(s,\theta)\mathrm{d}\theta\right]\mathrm{d}s\\ &\le M_1 + \lambda\ \sup_t I^{1-\alpha}\left[\int_0^t |k(t,s)|\,\mathrm{d}s\right]\\ &\le M_1 + \lambda\ K = M^{\star}, \quad \forall t\in[0,1],\end{aligned}$$

This means that $\{u_n(t)\}$ uniformly bounded on $C[0,1]$.
Now, let $t_1, t_2 \in [0,1], t_1 < t_2$ and $|t_1 - t_2| < \delta, \quad \delta > 0$. Thus, for any $n = 0,1,2,\ldots,$

we have

$$\begin{aligned}
u_{n+1}(t_2)-u_{n+1}(t_1) \;&=\; I^{1-\alpha}a(t_2) \;+\; \frac{\lambda}{\Gamma(1-\alpha)}\int_0^{t_2}(t_2-s)^{-\alpha}\left[\int_0^s f(s,\theta,u_n(\theta))\mathrm{d}\theta\right]\mathrm{d}s\\
&\quad-\; I^{1-\alpha}\;a(t_1)\;-\;\frac{\lambda}{\Gamma(1-\alpha)}\int_0^{t_1}(t_1-s)^{-\alpha}\left[\int_0^s f(s,\theta,u_n(\theta))\mathrm{d}\theta\right]\mathrm{d}s\\
&= I^{1-\alpha}\Big(a(t_2)-a(t_1)\Big)+\frac{\lambda}{\Gamma(1-\alpha)}\int_0^{t_1}(t_2-s)^{-\alpha}\left[\int_0^s f(s,\theta,u_n(\theta))\mathrm{d}\theta\right]\mathrm{d}s\\
&\quad+\;\frac{\lambda}{\Gamma(1-\alpha)}\int_{t_1}^{t_2}(t_2-s)^{-\alpha}\left[\int_0^s f(s,\theta,u_n(\theta))\mathrm{d}\theta\right]\mathrm{d}s\\
&\quad-\frac{\lambda}{\Gamma(1-\alpha)}\int_0^{t_1}(t_1-s)^{-\alpha}\left[\int_0^s f(s,\theta,u_n(\theta))\mathrm{d}\theta\right]\mathrm{d}s\\
&= I^{1-\alpha}\Big(a(t_2)-a(t_1)\Big)+\;\frac{\lambda}{\Gamma(1-\alpha)}\int_{t_1}^{t_2}(t_2-s)^{-\alpha}\left[\int_0^s f(s,\theta,u_n(\theta))\mathrm{d}\theta\right]\mathrm{d}s\\
&\quad+\;\frac{\lambda}{\Gamma(1-\alpha)}\int_0^{t_1}\Big((t_2-s)^{-\alpha}-(t_1-s)^{-\alpha}\Big)\cdot\left[\int_0^s f(s,\theta,u_n(\theta))\mathrm{d}\theta\right]\mathrm{d}s\\
&= I^{1-\alpha}\Big(a(t_2)-a(t_1)\Big)+\;\frac{\lambda}{\Gamma(1-\alpha)}\int_{t_1}^{t_2}(t_2-s)^{-\alpha}\left[\int_0^s f(s,\theta,u_n(\theta))\mathrm{d}\theta\right]\mathrm{d}s\\
&\quad+\;\frac{\lambda}{\Gamma(1-\alpha)}\int_0^{t_1}\left(\frac{1}{(t_2-s)^{\alpha}}-\frac{1}{(t_1-s)^{\alpha}}\right)\cdot\left[\int_0^s f(s,\theta,u_n(\theta))\mathrm{d}\theta\right]\mathrm{d}s\\
&= I^{1-\alpha}\Big(a(t_2)-a(t_1)\Big)+\;\frac{\lambda}{\Gamma(1-\alpha)}\int_{t_1}^{t_2}(t_2-s)^{-\alpha}\left[\int_0^s f(s,\theta,u_n(\theta))\mathrm{d}\theta\right]\mathrm{d}s\\
&\quad+\;\frac{\lambda}{\Gamma(1-\alpha)}\int_0^{t_1}\left(\frac{(t_1-s)^{\alpha}\;-\;(t_2-s)^{\alpha}}{(t_2-s)^{\alpha}\;(t_1-s)^{\alpha}}\right)\cdot\left[\int_0^s f(s,\theta,u_n(\theta))\mathrm{d}\theta\right]\mathrm{d}s,
\end{aligned}$$

then

$$\begin{aligned}
|u_{n+1}(t_2)-u_{n+1}(t_1)| \leq\; & I^{1-\alpha}\Big|a(t_2)-a(t_1)\Big|+\;\frac{\lambda}{\Gamma(1-\alpha)}\int_{t_1}^{t_2}|(t_2-s)^{-\alpha}|\left[\int_0^s |f(s,\theta,u_n(\theta))|\mathrm{d}\theta\right]\mathrm{d}s\\
&+\;\frac{\lambda}{\Gamma(1-\alpha)}\int_0^{t_1}\left|\frac{(t_2-s)^{\alpha}\;-\;(t_1-s)^{\alpha}}{(t_2-s)^{\alpha}\;(t_1-s)^{\alpha}}\right|\cdot\left[\int_0^s |f(s,\theta,u_n(\theta))|\mathrm{d}\theta\right]\mathrm{d}s.
\end{aligned}$$

This means that the sequence $\{u_n(t)\}$ is equicontinuous on $[0,1]$. According to the Arzela-Ascoli Theorem [12], the sequence $\{u_n(t)\}$ is relatively compact in $C[0,1]$. Thus, there exists a subsequence $\{u_{n_k}(t)\}$ that converges uniformly to a function $u \in C[0,1]$. Now,

$$u_{n_k}(t) = I^{1-\alpha}a(t)+\lambda\cdot I^{1-\alpha}\left[\int_0^t f(t,s,u_{n_k}(s))\mathrm{d}s\right],$$

since $\{u_{n_k}(t)\}\subset\{u_n(t)\}$.

Taking the limit as $k\to\infty$ for both sides, we obtain

$$\begin{aligned}
\lim_{k\to\infty} u_{n_k}(t) =\; & \lim_{k\to\infty}\left(I^{1-\alpha}\;a(t)+\lambda I^{1-\alpha}\left[\int_0^t f(t,s,u_{n_k}(s))\mathrm{d}s\right]\right)\\
=\; & I^{1-\alpha}\;a(t)+\lambda\;I^{1-\alpha}\lim_{k\to\infty}\left[\int_0^t f(t,s,u_{n_k}(s))\mathrm{d}s\right].
\end{aligned} \tag{2.8}$$

Taking into consideration that the conditions of the Lebesgue Dominated Convergence Theorem [12] (which state: pointwise convergence and existence of an integrable dominating function) are satisfied with the assumptions (i)-(iv), we deduce that

$$\begin{aligned}\lim_{k\to\infty}\int_0^t f(t,s,u_{n_k}(s))\mathrm{d}s &= \int_0^t \lim_{k\to\infty} f(t,s,u_{n_k}(s))\mathrm{d}s\\ &= \int_0^t f(t,s,u(s))\mathrm{d}s.\end{aligned}$$

We substitute in (2.8), we arrive at

$$\begin{aligned}\lim_{k\to\infty} u_{n_k}(t) &= I^{1-\alpha}\, a(t) + \lambda\, I^{1-\alpha} \int_0^t f(t,s,u(s))\mathrm{d}s\\ &= u(t).\end{aligned}$$

As a result, there exists at least one solution $u \in C[0,1]$ of (2.2) which is

$$u(t) = \lim_{n_k\to\infty} u_{n_k}(t).$$

Consequently, there exists at least one solution $y \in C[0,1]$ of (1.1)-(1.2).

2.2. **Uniqueness of Solution.** The following key assumptions must be satisfied

($\grave{i}$) $f : [0,1]^2 \times \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous [20, 23] in its third argument, i.e., $\exists c_f > 0$ such that

$$|f(t,s,u_1(s)) - f(t,s,u_2(s))| \le c_f\, |u_1(s) - u_2(s)|\,.$$

($\grave{ii}$) $\lambda c_f < 1$.

Theorem 2.2. *Let the assumptions of Theorem (2.1) and* ($\grave{i}$)–($\grave{ii}$) *be satisfied, then the solution of the problem (1.1)-(1.2) is unique continuous on* $[0,1]$.

proof. Let u_1, u_2 be two solutions of the implicit formula (2.2), then

$$\begin{aligned}|u_1(t) - u_2(t)| &= \lambda \left| I^{1-\alpha}\int_0^t f(t,s,u_1(s)) - f(t,s,u_2(s))\mathrm{d}s \right|\\ &\le \lambda I^{1-\alpha}\int_0^t c_f \cdot |u_1(s) - u_2(s)|\,\mathrm{d}s\\ &\le \lambda c_f \cdot \|u_1 - u_2\| \cdot I^{1-\alpha}\int_0^t \mathrm{d}s\\ &\le \lambda c_f \cdot \|u_1 - u_2\| \cdot I^{1-\alpha} t\\ &\le \lambda c_f \cdot \|u_1 - u_2\| \cdot \frac{t^{2-\alpha}}{\Gamma(3-\alpha)}\\ &\le \lambda c_f \cdot \|u_1 - u_2\|,\end{aligned}$$

then

$$\|u_1 - u_2\| \le \lambda c_f \cdot \|u_1 - u_2\|.$$

Hence

$$(1 - \lambda c_f) \cdot \|u_1 - u_2\| \le 0.$$

then $u_1 = u_2$, ∀t∈ $[0,1]$. Building on this, if y_1 and y_2 are two corresponding solutions of (2.4) related with u_1 and u_2, respectively, and are obtained by

$$y_1(t) = y(0) + I^\alpha u_1(t),$$
$$y_2(t) = y(0) + I^\alpha u_2(t).$$

This allows us to conclude that

$$\begin{aligned}|y_1(t) - y_2(t)| &\leq I^\alpha |u_1 - u_2| \\ &\leq I^\alpha \|u_1 - u_2\| \\ &\leq 0, \quad \forall t \in [0,1].\end{aligned}$$

It is clear that we must have

$$|y_1(t) - y_2(t)| = 0, \quad \forall t \in [0,1].$$

.

This gives

$$y_1(t) = y_2(t), \quad \forall t \in [0,1].$$

Then the solution of the problem (1.1)-(1.2) is unique.

2.3. **Hyers-Ulam Stability.**

Definition 2.1. [7, 22] *Let the solution $y \in C[0,1]$ of (1.1)–(1.2) be exists. The problem (1.1)-(1.2) is Hyers-Ulam stable if, for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that for any $\delta-$approximate solution $y_s(t)$ of (1.1)–(1.2) satisfying*

$$\left|\frac{\mathrm{d}y_s}{\mathrm{d}t} - a(t) - \lambda \int_0^t f(t,s,D^\alpha y_s(s))\mathrm{d}s\right| < \delta,$$

it follows that

$$\|y - y_s\| < \epsilon.$$

Theorem 2.3. *Let the assumptions of Theorem (2.2) be satisfied, then the problem (1.1)-(1.2) is Hyers-Ulam stable.*

proof. Suppose $y_s(t)$ is an $\delta-$ approximate solution of (1.1)-(1.2) satisfying

$$\left|\frac{\mathrm{d}y_s}{\mathrm{d}t} - a(t) - \lambda \int_0^t f(t,s,D^\alpha y_s(s))\mathrm{d}s\right| < \delta.$$

We can write this inequality as

$$-\delta < \frac{\mathrm{d}y_s}{\mathrm{d}t} - a(t) - \lambda \int_0^t f(t,s,D^\alpha y_s(s))\mathrm{d}s < \delta.$$

$$-I^{1-\alpha}\delta < D^\alpha y_s(t) - I^{1-\alpha}a(t) - \lambda I^{1-\alpha}\int_0^t f(t,s,D^\alpha y_s(s))\mathrm{d}s < I^{1-\alpha}\delta.$$

Let $\delta_1 = I^{1-\alpha}\delta$ and $D^\alpha y_s(t) = u_s(t)$. Then, the above inequality becomes

$$-\delta_1 < u_s(t) - I^{1-\alpha}a(t) - \frac{\lambda}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\left[\int_0^s f(s,\theta,u_s(\theta))\mathrm{d}\theta\right]\mathrm{d}s < \delta_1.$$

This leads to

$$\left|u_s(t) - I^{1-\alpha}a(t) - \frac{\lambda}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\left[\int_0^s f(s,\theta,u_s(\theta))\mathrm{d}\theta\right]\mathrm{d}s\right| < \delta_1.$$

Now, for all $t \in [0,1]$

$$\begin{aligned}
|u(t) - u_s(t)| &= \left| I^{1-\alpha} a(t) + \frac{\lambda}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \left[\int_0^s f(s,\theta,u(\theta)) \mathrm{d}\theta \right] \mathrm{d}s - u_s(t) \right| \\
&= \left| I^{1-\alpha} a(t) + \frac{\lambda}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \left[\int_0^s f(s,\theta,u(\theta)) \mathrm{d}\theta \right] \mathrm{d}s \right. \\
&\quad + \frac{\lambda}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \left[\int_0^s f(s,\theta,u_s(\theta)) \mathrm{d}\theta \right] \mathrm{d}s \\
&\quad \left. - \frac{\lambda}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \left[\int_0^s f(s,\theta,u_s(\theta)) \mathrm{d}\theta \right] \mathrm{d}s - u_s(t) \right| \\
&\le \left| I^{1-\alpha} a(t) + \frac{\lambda}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \left[\int_0^s f(s,\theta,u_s(\theta)) \mathrm{d}\theta \right] \mathrm{d}s - u_s(t) \right| \\
&\quad + \lambda I^{1-\alpha} \int_0^t |f(t,s,u(s)) - f(t,s,u_s(s))| \mathrm{d}s \\
&\le \delta_1 + \lambda c_f \cdot |u(t) - u_s(t)| \cdot I^{1-\alpha} \int_0^t \mathrm{d}s \\
&< \delta_1 + \lambda c_f \cdot |u(t) - u_s(t)| \cdot \frac{t^{2-\alpha}}{\Gamma(3-\alpha)} \\
&< \delta_1 + \lambda c_f \cdot |u(t) - u_s(t)| .
\end{aligned}$$

Taking the supremum for any $t \in [0,1]$,

$$\|u - u_s\| < \delta_1 + \lambda c_f \cdot \|u - u_s\|.$$

We rewrite it and use the condition (ii) to obtain

$$\|u - u_s\| < \frac{\delta_1}{1 - \lambda c_f} = \epsilon^*.$$

Hence, from (2.4), we have

$$\begin{aligned}
|y(t) - y_s(t)| &= |y(0) + I^\alpha u(t) - y(0) - I^\alpha u_s(t)| \\
&\le I^\alpha |u(t) - u_s(t)| \\
&\le \frac{\epsilon^\star}{\Gamma(\alpha+1)} = \epsilon.
\end{aligned}$$

Since this estimation valid for any $t \in [0,1]$, it follows that

$$\|y - y_s\| \le \epsilon.$$

2.4. **Continuous Dependence.** This section investigates the continuous dependence of the unique solution to (1.1)–(1.2) on the function $a \in C[0,1]$ and parameter $\lambda > 0$.

Definition 2.2. [6, 21] *The solution $y \in C[0,1]$ to (1.1)-(1.2) depends continuously on the function $u \in C[0,1]$ if, for every $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ satisfying*

$$\|u - u^*\| < \delta$$

implies

$$\|y - y^*\| < \epsilon.$$

Where, $u^*(t)$ and $y^*(t)$ are two unique solutions of

$$u^*(t) = I^{1-\alpha}a(t) + \lambda I^{1-\alpha}\int_0^t f(t,s,u^*(s))\mathrm{d}s,$$

$$y^*(t) = y(0) + I^{\alpha}u^*(t), \tag{2.9}$$

respectively.

Theorem 2.4. *If the assumptions of Theorem (2.2) hold, then the solution $y \in C[0,1]$ of (1.1)-(1.2) depends continuously on $u \in C[0,1]$.*

proof. Let $y,\ y^*$ be two solutions defined by (2.1) and (2.9), respectively. Assume $\|u - u^*\| < \delta$. Then, for all $t \in [0,1]$, we obtain

$$\begin{aligned} |y(t) - y^*(t)| &= |y(0) + I^{\alpha}u(t) - y(0) - I^{\alpha}u^*(t)| \\ &\le I^{\alpha}|u(t) - u^*(t)| \\ &\le \|u - u^*\| \cdot \frac{t^{\alpha}}{\Gamma(\alpha+1)} \\ &\le \frac{\delta}{\Gamma(\alpha+1)} = \epsilon. \end{aligned}$$

Hence

$$\|y - y^*\| < \epsilon.$$

Definition 2.3. [6, 21] *The solution $u \in C[0,1]$ of (2.2) depends continuously on the function $a \in C[0,1]$ and the parameter $\lambda > 0$ if, for any $\epsilon > 0$, we can choose $\delta(\epsilon) > 0$ such that*

$$\max\{|\lambda - \lambda^*|, |a(t) - a^*(t)|\} < \delta, \quad \forall t \in [0,1].$$

It follows that

$$\|u - u^*\| < \epsilon,$$

where $u^*(t)$ is the unique solution of

$$u^*(t) = I^{1-\alpha}a(t) + \lambda I^{1-\alpha}\int_0^t f(t,s,u^*(s))\mathrm{d}s. \tag{2.10}$$

Theorem 2.5. *Assume the assumptions of Theorem (2.2) are satisfied, then the solution $u \in C[0,1]$ of (2.2) depends continuously on the function $a(t)$ and the parameter $\lambda > 0$.*

proof. Let $u,\ u^*$ be two solutions of (2.2),(2.10), respectively. Suppose

$$\max\{|\lambda - \lambda^*|, |a(t) - a^*(t)|\} < \delta.$$

For all $t \in [0,1]$, we obtain

$$
\begin{aligned}
|u(t)-u^*(t)| &= \left| I^{1-\alpha}a(t) + \lambda I^{1-\alpha}\int_0^t f(t,s,u(s))\mathrm{d}s - I^{1-\alpha}a^*(t) - \lambda^* I^{1-\alpha}\int_0^t f(t,s,u^*(s))\mathrm{d}s \right| \\
&\leq I^{1-\alpha}|a(t)-a^*(t)| + \left| \lambda I^{1-\alpha}\int_0^t f(t,s,u(s) - \lambda^* I^{1-\alpha}\int_0^t f(t,s,u^*(s))\mathrm{d}s)\mathrm{d}s \right| \\
&\leq \frac{\delta}{\Gamma(2-\alpha)} + \left| \lambda I^{1-\alpha}\int_0^t f(t,s,u(s))\mathrm{d}s - \lambda^* I^{1-\alpha}\int_0^t f(t,s,u(s))\mathrm{d}s \right. \\
&\quad \left. + \lambda^* I^{1-\alpha}\int_0^t f(t,s,u(s))\mathrm{d}s - \lambda^* I^{1-\alpha}\int_0^t f(t,s,u^*(s))\mathrm{d}s \right| \\
&< \frac{\delta}{\Gamma(2-\alpha)} + |\lambda-\lambda^*|\cdot \sup_t I^{1-\alpha}\int_0^t |f(t,s,u(s))|\mathrm{d}s \\
&+ \lambda^*\cdot I^{1-\alpha}\int_0^t |f(t,s,u(s)) - f(t,s,u^*(s))|\,\mathrm{d}s \\
&< \frac{\delta}{\Gamma(2-\alpha)} + \delta\cdot K + \lambda^*\cdot I^{1-\alpha}\int_0^t c_f\cdot|u(s)-u^*(s)|\,\mathrm{d}s \\
&< \left(\frac{1}{\Gamma(2-\alpha)} + K\right)\cdot\delta + \lambda^* c_f\cdot|u(t)-u^*(t)|\cdot I^{1-\alpha}t \\
&< B\cdot\delta + \lambda^* c_f\cdot|u(t)-u^*(t)|\cdot\frac{t^{2-\alpha}}{\Gamma(3-\alpha)} \\
&< B\cdot\delta + \lambda^* c_f\cdot|u(t)-u^*(t)|,
\end{aligned}
$$

where $B = \frac{1}{\Gamma(2-\alpha)} + K$. Rearranging terms, we have

$$(1-\lambda^* c_f)\,|u(t)-u^*(t)| < B\cdot\delta.$$

Under the condition $\lambda^* c_f < 1$, it implies that

$$|u(t)-u^*(t)| < \frac{B\delta}{(1-\lambda^* c_f)} = \epsilon.$$

This inequality holds for any t over $[0,1]$, we conclude that

$$\|u-u^*\| < \epsilon.$$

Corollary 2.0. *Using Theorem (2.4) and Theorem (2.5), we deduce that the solution $y \in C[0,1]$ depends continuously on the function $a \in C[0,1]$ and the parameter $\lambda > 0$.*

3. Special cases and Examples

Case 1:: $\tau = 0$ and $\eta = 1$.
The problem given by (1.1) is now linked with the boundary condition:

$$y(0) = \gamma y(1), \quad \gamma \neq 1.$$

This represents a standard two-point boundary value problem.

Case 2:: $\tau = 1$ and $\eta = 0$. Our problem is now given by (1.1) and the condition

$$y(1) = \gamma y(0), \quad \gamma \neq 1.$$

This is another form of a two-point boundary value problem.

Case 3: $\tau = \eta = 0$.**:** This case, the problem represents by (1.1), subject to the condition:

$$y(0) = 0.$$

This is the usual initial value problem.

Case 4: $\tau = \eta = 1$.**:** Here, the problem defined by (1.1) and the boundary condition:

$$y(1) = 0.$$

This is referred a backward initial value problem.

Case 5: $\eta = 1 - \tau$.**:** This case, the problem link the equation (1.1) with the condition:

$$y(\tau) = \gamma y(1-\tau), \quad \gamma \neq 1.$$

This is an anti-periodic boundary value problem, which has applications in the study of oscillatory systems [1, 13].

Example1. Suppose the following functional integro-fractional differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}s} = \cos(t) + \frac{1}{4}\int_0^t s\sin(D^\alpha y(s))\mathrm{d}s, \tag{3.1}$$

with the two-point boundary condition:

$$y(0.2) = 0.4y(0.8). \tag{3.2}$$

Then, we justify $f(t,s,x(s)) = s\sin(x(s))$ is an L^1-Carathéodory function.
Since f is continuous in $x(s)$ for each $(t,s) \in [0,1]^2$. It is continuous in t and measurable in s for each $x(s) \in \mathbb{R}$ and bounded by s, because $|f(t,s,(x(s)))| = |s \cdot \sin(x(s))| \leq s = k(t,s)$. With,

$$\sup_{t\in[0,1]} I^{1-\alpha}\int_0^t |k(t,s)|\,\mathrm{d}s = \sup_{t\in[0,1]} I^{1-\alpha}\int_0^t s\mathrm{d}s = \frac{1}{\Gamma(4-\alpha)} = K > 0.$$

Moreover, $k(t,s) = s$ is continuous in $(t,s) \in [0,1]^2$, and $a(t) = \cos(t) \in C[0,1]$.
Next, we obtain that the function $f(t,s,x(s)) = s\sin(x(s))$ is Lipschitz continuous in $x \in \mathbb{R}$. For any $x_1, x_2 \in \mathbb{R}$, we have:

$$|f(t,s,x_1(s)) - f(t,s,x_2(s))| = |s\sin(x_1) - s\sin(x_2)| = s\,|\sin(x_1) - \sin(x_2)| \leq s\,|x_1(s) - x_2(s)|\,.$$

Taking the constant $c_f = \sup_{(t,s)\in[0,1]^2} s = 1$, we conclude

$$\lambda \cdot c_f = \frac{1}{4}\cdot 1 = \frac{1}{4} < 1.$$

Thus, the assumptions of the Theorem (2.2) are met. Then, the solution to the problem (3.1)–(3.2) is unique.

Example2. Consider the next functional integro-fractional differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}s} = e^{-t} + \frac{1}{6}\int_0^t [\cos(s)\sin(D^\alpha y(s)) + h(s)]\,\mathrm{d}s, \tag{3.3}$$

subject to the two-point boundary condition:

$$y(0.3) = 0.6y(0.7), \tag{3.4}$$

where the function $h(s)$ is given as:

$$h(s) = \begin{cases} 1, & s \in [0, 0.4), \\ 0, & s \in [0.4, 1]. \end{cases}$$

Now, we justify $f(t, s, x(s)) = \cos(s) \sin(x(s)) + h(s)$ is an L^1-Carathéodory function.

Since f is continuous in $x(s)$ for each $(t, s) \in [0, 1]^2$. It is continuous in t for each $s \in [0, 1]$ and $x \in \mathbb{R}$ and measurable discontinuous at 0.4 only in s for each $t \in [0, 1]$ and $x(s) \in \mathbb{R}$ and bounded by $1 + |h(s)|$, because $|f(t, s, (x(s)))| = |\cos(s) \cdot \sin(x(s))| + |h(s)| \leq 1 + 1 = 2 = k(t, s)$. With,

$$\sup_{t\in[0,1]} I^{1-\alpha} \int_0^t |k(t, s)| \, \mathrm{d}s = \sup_{t\in[0,1]} I^{1-\alpha} \int_0^t 2\mathrm{d}s \leq \frac{2}{\Gamma(3 - \alpha)} = K > 0, \quad 0 < \alpha < 1.$$

Moreover, $k(t, s) = 2$ is continuous in $(t, s) \in [0, 1]^2$, and $a(t) = e^{-t} \in C[0, 1]$. Next, we obtain that the function $f(t, s, x(s)) = \cos(s) \cdot \sin(x(s)) + h(s)$ is Lipschitz continuous in $x \in \mathbb{R}$. For any $x_1, x_2 \in \mathbb{R}$, we have:

$$\begin{aligned} |f(t, s, x_1(s)) - f(t, s, x_2(s))| &= |\cos(s) \sin(x_1) - \cos(s) \sin(x_2)| = |\cos(s)| \, |\sin(x_1) - \sin(x_2)| \\ &\leq |x_1(s) - x_2(s)| \, . \end{aligned}$$

Taking the constant $c_f = \sup_{(t,s)\in[0,1]^2} 1 = 1$, we conclude

$$\lambda \cdot c_f = \frac{1}{6} \cdot 1 = \frac{1}{6} < 1.$$

Thus, the assumptions of the Theorem (2.2) are valid. Then, the solution to the problem (3.3)–(3.4) is unique.

4. Conclusion

The cornerstone of this research is the employing of Caratheodory's theorem to establish the existence of a solution for non-local two-point boundary value problems related to functional integro-fractional differential equations. By converting the problem into a clearer integral form of the required solution. We use suitable conditions provided that the solution exists to a class of continuous solutions on $[0, 1]$.

Moreover, we introduced the sufficient conditions to deduce this solution is unique by embedding the Lipschitz continuity condition on the nonlinear term in the equation. As a result, to exhibit the reliability of our unique solution, the Hyers-Ulam stability and the continuous dependence on certain parameters are addressed.

In the final phase, we demonstrated the applicability of our results through various special cases and examples to illustrate the flexibility of our framework in mentioning distinct different types of boundary value problems. To conclude, this work widens the domain of equations that can be solved, studied for the scholars the solvability of functional equations.

In future work, we could generalize and extend these results to other ideas in the same direction and also find some practical problems and verify our results with numerical methods [10].

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