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SOLUTION OF FRACTIONAL ORDER SCHRÖDINGER EQUATION BY USING ABOODH TRANSFORM HOMOTOPY PERTURBATION METHOD

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Abstract. This study presents an analytical approach for solving the fractional-order Schrödinger equation using the Aboodh Transform Homotopy Perturbation Method (ATHPM). The proposed method provides an approximate analytical solution, benchmarked against the exact solution to demonstrate its accuracy. Unlike conventional methods, ATHPM offers a computationally efficient technique without requiring linearization or small-parameter assumptions. The results validate the method's effectiveness, showing excellent agreement with exact solutions. The study includes a comparative analysis, highlighting ATHPM's advantages over existing techniques. Additionally, multiple figures illustrate the behavior of the obtained solutions, reinforcing the accuracy of the method. Several comparison tables showcase ATHPM's performance against the exact solution. Furthermore, a detailed comparative study with the Modified Generalized Mittag-Leffler Function Method (MGMLFM) is provided, demonstrating the robustness and efficiency of the proposed method.

1. Introduction

Many improvements have been made in the field of solving nonlinear differential equations in recent years. However, several nonlinear differential equations do not have analytical solutions. The Aboodh Transform Homotopy Perturbation Method (ATHPM) can be used to obtain an analytical solution for such equations. The Aboodh transform was developed by Khalid Aboodh to solve differential equations in the time domain [1]. J. H. He introduced the Homotopy Perturbation Method (HPM) for the first time [11]. HPM is a semi-analytical method for solving both linear and nonlinear differential equations.

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Quantum mechanical systems are governed by the Schrödinger equation, a fundamental partial differential equation that describes the evolution of wave functions. This equation encapsulates essential quantum properties such as position, momentum, and energy of particles, making it crucial in fields such as quantum optics, condensed matter physics, and superconductivity.
The time-fractional Schrödinger equation governs wave function evolution in complex quantum systems. It is given as:

$$\frac{\partial^\alpha \nu}{\partial t^\alpha} + i\frac{\partial^2 \nu}{\partial \varkappa^2} = 0, \quad \nu(\varkappa, 0) = \nu(\varkappa). \tag{1}$$

The Schrödinger equation in its nonlinear time-fractional form is given by:

$$i\frac{\partial^\alpha \nu}{\partial t^\alpha} + \gamma\frac{\partial^2 \nu}{\partial \varkappa^2} + \beta|\nu|^2\nu = 0, \quad \nu(\varkappa, 0) = \nu(\varkappa). \tag{2}$$

Here, $\nu(\varkappa, t)$ is the wave function, β and γ are constants, and α is the fractional derivative order, modeling memory and hereditary effects in quantum systems. The significance of this formulation lies in its ability to generalize classical quantum mechanics to fractional-order domains.
The aim of this research is to use ATHPM to find accurate solutions for linear and nonlinear Schrödinger wave equations that arise in physics, such as plasma physics, nonlinear optics, quantum mechanics, and superconductivity. Dehghan [9], Bairwa [7], Eladdad [10], Aruna [6], and Wazwaz [19] investigated the Schrödinger equation using the Numerical Method, Sumudu Iterative Method, Picard and Homotopy Perturbation Method, Differential Transform Method, and Variational Iteration Method, respectively.

The ATHPM simplifies both linear and nonlinear problems by continuously transforming a complex equation into a series of solvable subproblems. Unlike other methods, such as the Adomian Decomposition Method (ADM) or the Variational Iteration Method (VIM), ATHPM does not require complicated polynomial expansions or iterative corrections. This makes it computationally efficient while preserving accuracy.

2. ATHPM implementation

Let us consider a fractional order partial differential equation in the Caputo sense that is both nonlinear and nonhomogeneous.

$$\frac{\partial^\alpha \nu}{\partial t^\alpha} + R\nu(\varkappa, t) + N\nu(\varkappa, t) = f(\varkappa, t). \tag{3}$$

where $f(x, y)$ is the source term, N is the nonlinear differential term, and R is the linear differential operator.
Apply Aboodh transform on it,

$$A[\frac{\partial^\alpha \nu}{\partial t^\alpha}] + A[R\nu(\varkappa, t)] + A[N\nu(\varkappa, t)] = A[f(\varkappa, t)],$$

$$K(s) = \frac{1}{s^2}\nu(\varkappa, 0) - \frac{1}{s^\alpha}A[R\nu(\varkappa, t) + N\nu(\varkappa, t) - f(\varkappa, t)].$$

Implement Aboodh inverse transform and HPM for the nonlinear terms [4]

$$\sum_{n=0}^{\infty} p^n \nu_n(\varkappa, t) = \nu(\varkappa, 0) - pA^{-1}[\frac{1}{s^\alpha}A(R\sum_{n=0}^{\infty} p^n H_n(\varkappa, t) - N\sum_{n=0}^{\infty} p^n H_n(\varkappa, t))].$$

When we equate coefficients of like powers of p on both sides, we obtain

$$p^i : \nu_i(\varkappa, t) = -A^{-1}[\frac{1}{s^\alpha} A(R\nu_{i-1}(\varkappa, t) - N\nu_{i-1}(\varkappa, t))]; i > 0.$$

Ultimately, the approximate analytical solution is given by,

$$\nu(\varkappa, t) = \nu_0 + \nu_1 + \nu_2 + \nu_3 + ...$$

3. Solution of Schrödinger equations by ATHPM

Example 1 Consider the following Schrödinger equation,

$$\frac{\partial^\alpha \nu}{\partial t^\alpha} + i\frac{\partial^2 \nu}{\partial \varkappa^2} = 0,$$

where $\nu(\varkappa, 0) = ce^{id\varkappa}$, c and d are complex constants [10].
Apply Aboodh transform on it,

$$A[\frac{\partial^\alpha \nu}{\partial t^\alpha}] = -A[i\frac{\partial^2 \nu}{\partial \varkappa^2}],$$

$$K(s) = \frac{1}{s^2}\nu(\varkappa, 0) - \frac{1}{s^\alpha} A[i\frac{\partial^2 \nu}{\partial \varkappa^2}].$$

Implement Aboodh inverse transform,

$$\nu(\varkappa, t) = \nu(\varkappa, 0) - A^{-1}[\frac{1}{s^\alpha} A(i\frac{\partial^2 \nu}{\partial \varkappa^2})].$$

By HPM [11],

$$\nu_0 = \nu(\varkappa, 0) = ce^{id\varkappa},$$

$$p^1 : \nu_1 = -A^{-1}[\frac{1}{s^\alpha} A(i\frac{\partial^2 \nu_0}{\partial \varkappa^2})] = icd^2 e^{id\varkappa} \frac{t^\alpha}{(\alpha)!},$$

$$p^2 : \nu_2 = -A^{-1}[\frac{1}{s^\alpha} A(i\frac{\partial^2 \nu_1}{\partial \varkappa^2})] = i^2 cd^4 e^{id\varkappa} \frac{t^{2\alpha}}{(2\alpha)!},$$

$$p^3 : \nu_3 = -A^{-1}[\frac{1}{s^\alpha} A(i\frac{\partial^2 \nu_2}{\partial \varkappa^2}) = i^3 cd^6 e^{id\varkappa} \frac{t^{3\alpha}}{(3\alpha)!}].$$

Approximate solution,

$$\nu(\varkappa, t) = \nu_0 + \nu_1 + \nu_2 + \nu_3 + ...$$

$$\nu(\varkappa, t) = ce^{id\varkappa} + icd^2 e^{id\varkappa} \frac{t^\alpha}{(\alpha)!} + i^2 cd^4 e^{id\varkappa} \frac{t^{2\alpha}}{(2\alpha)!} + i^3 cd^6 e^{id\varkappa} \frac{t^{3\alpha}}{(3\alpha)!} + ...$$

Classical Schrödinger equation for $\alpha = 1$ has the solution,

$$\nu(\varkappa, t) = ce^{id\varkappa} + icd^2 e^{id\varkappa} t + i^2 cd^4 e^{id\varkappa} \frac{t^2}{(2)!} + i^3 cd^6 e^{id\varkappa} \frac{t^3}{(3)!} + ...$$

Applying convergence analysis, we have,

$$\gamma_0 = \frac{\|\nu_1\|}{\|\nu_0\|} = 0.3162 < 1,$$

$$\gamma_1 = \frac{\|\nu_2\|}{\|\nu_1\|} = 0.2236 < 1,$$

$$\gamma_2 = \frac{\|\nu_3\|}{\|\nu_2\|} = 0.1290 < 1.$$

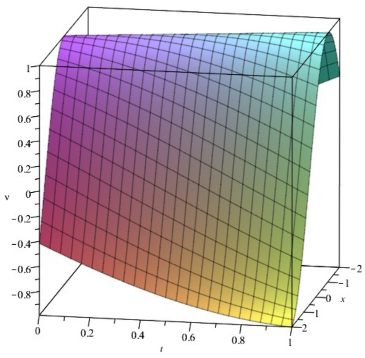

FIGURE 1. ATHPM solution for real part of Example 3.1 with the values $d = c = 1$ and $\alpha = 1$.

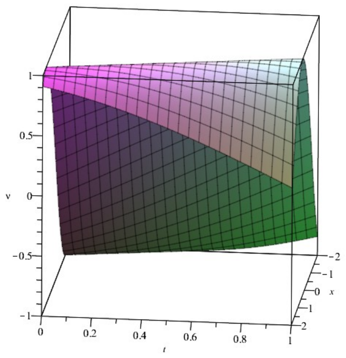

FIGURE 2. ATHPM solution for imaginary part of Example 3.1 with the values $d = c = 1$ and $\alpha = 1$.

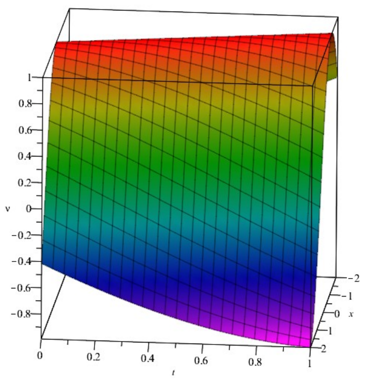

FIGURE 3. Exact solution for real part of Example 3.1 with the values $d = c = 1$ and $\alpha = 1$.

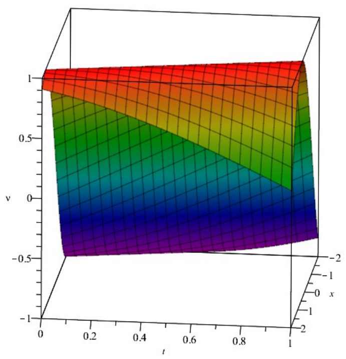

FIGURE 4. Exact solution for imaginary part of Example 3.1 with the values $d = c = 1$ and $\alpha = 1$.

Hence, for $\alpha = 1$ above series solution $\nu(\varkappa, t)$ can be converge to,

$$\nu(\varkappa, t) = ce^{id(\varkappa + d \cdot t)}.$$

Which is the exact solution.

t	x	ATHPM	Exact	Absolute error	
				Real part	Imaginary part
0.1	0.5	0.8227917992+0.5641270686i	0.8253356149+0.5646424734i	0.0025438157	0.0005154048
	0.75	0.6473037184+0.7480471618i	0.6599831459+0.7512804051i	0.0126794275	0.0032332433
	1	0.4143075688+0.8790868464i	0.4535961214+0.8912073601i	0.0392885526	0.0121205137
0.3	0.5	0.6943159955+0.7163455820i	0.6967067093+0.7173560909i	0.0023907138	0.0010105089
	0.75	0.4857867110+0.8617354189i	0.4975710479+0.8674232256i	0.0117843369	0.0056878067
	1	0.2314014056+0.9438738448i	0.2674988286+0.9635581854i	0.0360974230	0.0196843406
0.5	0.5	0.5381600045+0.8400056573i	0.5403023059+0.8414709848i	0.0021423014	0.0014653275
	0.75	0.3049029205+0.9410690031i	0.3153223624+0.9489846194i	0.0104194419	0.0079156163
	1	0.03926999875+0.9710315707i	0.7073720167+0.9974949866i	0.03146720292	0.0264634159

TABLE 1. $\nu(\varkappa, t)$ in case of $\alpha = 1$ in Example 3.1.

Example 2 Consider the followig Schrödinger equation,

$$\frac{\partial^\alpha \nu}{\partial t^\alpha} + i\frac{\partial^2 \nu}{\partial \varkappa^2} = 0.$$

Where $\nu(\varkappa, 0) = 1 + \cosh(2x)$ [10].

Apply Aboodh transform and inverese Aboodh transform on it,

$$\nu(\varkappa, t) = \nu(\varkappa, 0) - A^{-1}[\frac{1}{s^\alpha} A(i\frac{\partial^2 \nu}{\partial \varkappa^2})].$$

By HPM [11],

$$\nu_0 = \nu(\varkappa, 0) = 1 + \cosh(2x),$$

$$p^1 : \nu_1 = -A^{-1}[\frac{1}{s^\alpha} A(i\frac{\partial^2 \nu_0}{\partial \varkappa^2})] = -4i\cosh(2\varkappa)\frac{t^\alpha}{(\alpha)!},$$

$$p^2 : \nu_2 = -A^{-1}[\frac{1}{s^\alpha} A(i\frac{\partial^2 \nu_1}{\partial \varkappa^2})] = 16i^2\cosh(2x)\frac{t^{2\alpha}}{(2\alpha)!},$$

$$p^3 : \nu_3 = -A^{-1}[\frac{1}{s^\alpha} A(i\frac{\partial^2 \nu_2}{\partial \varkappa^2}) = -64i^3\cosh(2x)\frac{t^{3\alpha}}{(3\alpha)!}.$$

Approximate solution,

$$\nu(\varkappa, t) = \nu_0 + \nu_1 + \nu_2 + \nu_3 + ...$$

$$\nu(\varkappa, t) = 1+\cosh(2x)-4i\cosh(2\varkappa)\frac{t^\alpha}{(\alpha)!}+16i^2\cosh(2x)\frac{t^{2\alpha}}{(2\alpha)!}-64i^3\cosh(2x)\frac{t^{3\alpha}}{(3\alpha)!}+...$$

Classical Schrödinger equation for $\alpha = 1$ has the solution,

$$\nu(\varkappa, t) = 1 + \cosh(2x) - 4i\cosh(2\varkappa)t + 16i^2\cosh(2x)\frac{t^2}{(2)!} - 64i^3\cosh(2x)\frac{t^3}{(3)!} + ...$$

Applying convergence analysis, we have

$$\gamma_0 = \frac{\|\nu_1\|}{\|\nu_0\|} = 0.4494 < 1,$$

$$\gamma_1 = \frac{\|\nu_2\|}{\|\nu_1\|} = 0.4472 < 1,$$

$$\gamma_2 = \frac{\|\nu_3\|}{\|\nu_2\|} = 0.3651 < 1.$$

Hence, for $\alpha = 1$ above series solution $\nu(\varkappa, t)$ can be converge to,

$$\nu(\varkappa, t) = 1 + 4e^{-4it}\cosh(2x).$$

Which is the exact solution.

t	x	ATHPM	Exact	Absolute error	
				Real part	Imaginary part
0.1	0.5	2.42121-0.601046i	2.42127-0.600904i	0.00006	0.00014
	0.75	3.16662-0.916288i	3.16671-0.916071i	0.0009	0.00021
	1	4.46507-1.165410i	4.46521-1.465070i	0.00014	0.00035
0.3	0.5	1.55872-1.43838i	1.55915-1.43821i	0.00043	0.00017
	0.75	1.85176-2.19279i	1.85241-2.19254i	0.00066	0.00025
	1	2.36221-3.50692i	2.36326-3.50651i	0.00105	0.00040
0.5	0.5	0.3571550-1.40280i	0.3578520-1.40312i	0.00070	0.00032
	0.75	0.0199828-2.13855i	0.0210522-2.13904i	0.00107	0.00049
	1	-0.567336-3.42017i	-0.565626-3.42095i	0.00171	0.00078

TABLE 2. $\nu(\varkappa, t)$ in case of $\alpha = 1$ in Example 3.2.

Example 3 Consider the followig nonlinear Schrödinger equation with $\gamma = 1, \beta = -2$

$$i\frac{\partial^\alpha \nu}{\partial t^\alpha} + \frac{\partial^2 \nu}{\partial \varkappa^2} - 2|\nu|^2\nu = 0.$$

Where $\nu(\varkappa, 0) = e^{i\varkappa}$ [10].
Multiply (-i) in above nonlinear Schrödinger equation then it can be rewritten as,

$$\frac{\partial^\alpha \nu}{\partial t^\alpha} - i\frac{\partial^2 \nu}{\partial \varkappa^2} + 2i\nu^2\bar{\nu} = 0.$$

Try Aboodh transform and inverese Aboodh transform on it,

$$\nu(\varkappa, t) = \nu(\varkappa, 0) - A^{-1}[\frac{1}{s^\alpha}A(-i\frac{\partial^2 \nu}{\partial \varkappa^2} + 2i\nu^2\bar{\nu})].$$

By HPM [11]

$$\nu_0 = \nu(\varkappa, 0) = e^{i\varkappa},$$

$$p^1 : \nu_1 = A^{-1}[\frac{1}{s^\alpha}A(i\frac{\partial^2 \nu_0}{\partial \varkappa^2} - 2i\nu_0^2\bar{\nu_0})] = -3ie^{i\varkappa}\frac{t^\alpha}{(\alpha)!},$$

$$p^2 : \nu_2 = A^{-1}[\frac{1}{s^\alpha}A(i\frac{\partial^2 \nu_1}{\partial \varkappa^2} - 2i\nu_0^2\bar{\nu_1} - 4i\nu_0\nu_1\bar{\nu_0})] = 9i^2e^{i\varkappa}\frac{t^{2\alpha}}{(2\alpha)!},$$

similarly,

$$p^3 : \nu_3 = -27i^3e^{i\varkappa}\frac{t^{3\alpha}}{(3\alpha)!}.$$

Approximate solution,

$$\nu(\varkappa, t) = \nu_0 + \nu_1 + \nu_2 + \nu_3 + ...$$

$$\nu(\varkappa, t) = e^{i\varkappa} - 3ie^{i\varkappa}\frac{t^\alpha}{(\alpha)!} + 9i^2e^{i\varkappa}\frac{t^{2\alpha}}{(2\alpha)!} - 27i^3e^{i\varkappa}\frac{t^{3\alpha}}{(3\alpha)!} + ...$$

Classical Schrödinger equation for $\alpha = 1$ has the solution,

$$\nu(\varkappa, t) = e^{i\varkappa} - 3ie^{i\varkappa}t + 9i^2e^{i\varkappa}\frac{t^2}{(2)!} - 27i^3e^{i\varkappa}\frac{t^3}{(3)!} + ...$$

Applying convergence analysis, we have

$$\gamma_0 = \frac{\|\nu_1\|}{\|\nu_0\|} = 0.3 < 1,$$

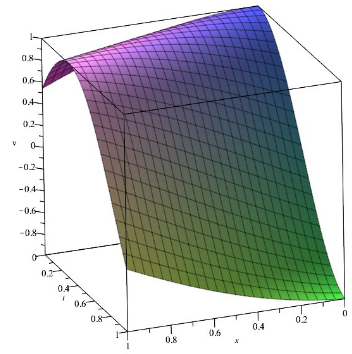

Figure 5. ATHPM solution for real part of Example 3.3 with $\alpha = 1$.

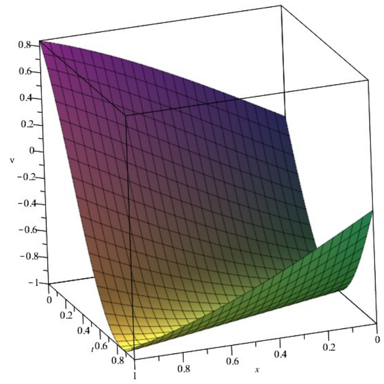

Figure 6. ATHPM solution for imaginary part of Example 3.3 with $\alpha = 1$.

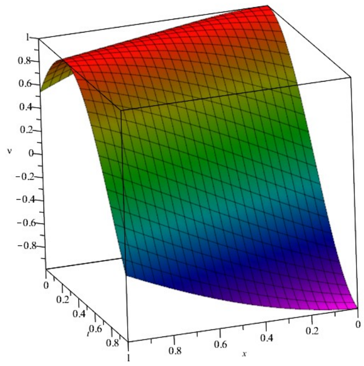

Figure 7. Exact solution for real part of Example 3.3 with $\alpha = 1$.

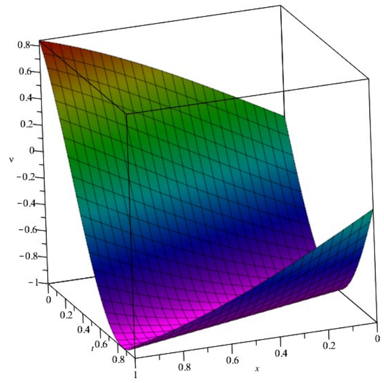

Figure 8. Exact solution for imaginary part of Example 3.3 with $\alpha = 1$.

$$\gamma_1 = \frac{\|\nu_2\|}{\|\nu_1\|} = 0.15 < 1,$$

$$\gamma_2 = \frac{\|\nu_3\|}{\|\nu_2\|} = 0.0001 < 1.$$

Hence, for $\alpha = 1$ above series solution $\nu(\varkappa, t)$ can be converge to,

$$\nu(\varkappa, t) = e^{i(x-3t)}.$$

Which is the exact solution.

t	x	ATHPM	Exact	Absolute error	
				Real part	Imaginary part
0.1	0.5	0.980086+0.198571i	0.980067+0.198669i	0.0000020	0.000098
	0.75	0.900491+0.434875i	0.900447+0.434966i	0.000043	0.000090
	1	0.764907+0.644141i	0.764842+0.644218i	0.000064	0.000076
0.3	0.5	0.920944-0.389695i	0.921061-0.3894180i	0.00012	0.000276
	0.75	0.988726-0.149735i	0.988771-0.149438i	0.000045	0.000297
	1	0.995034+0.0995349i	0.995004+0.099833i	0.00003	0.000299
0.5	0.5	0.539882-0.841741i	0.540302-0.841471i	0.000421	0.00027
	0.75	0.731348-0.682005i	0.731689-0.681639i	0.000341	0.000366
	1	0.877343-0.479864i	0.877583-0.479426i	0.00024	0.000439

TABLE 3. $\nu(\varkappa, t)$ in case of $\alpha = 1$ in Example 3.3.

Example 4 Consider the followig nonlinear Schrödinger equation $\gamma = \frac{1}{2}$, $\beta = -1$ and non-homogeneous term $-\nu\cos^2(x)$ [10],

$$i\frac{\partial^\alpha \nu}{\partial t^\alpha} + \frac{1}{2}\frac{\partial^2 \nu}{\partial \varkappa^2} + \nu\cos^2(x) + |\nu|^2\nu = 0.$$

Where $\nu(\varkappa, 0) = \sin(x)$ [6].

Multiply (-i) in above nonlinear Schrödinger equation then it can be rewritten as,

$$\frac{\partial^\alpha \nu}{\partial t^\alpha} = \frac{1}{2}i\frac{\partial^2 \nu}{\partial \varkappa^2} - i\nu\cos^2(x) - i\nu^2\bar{\nu}.$$

Applycboodh transform and inverese Aboodh transform on it,

$$\nu(\varkappa, t) = \nu(\varkappa, 0) + A^{-1}[\frac{1}{s^\alpha}A(\frac{1}{2}i\frac{\partial^2 \nu}{\partial \varkappa^2} - i\nu\cos^2(x) - i\nu^2\bar{\nu})].$$

By HPM [11],

$$\nu_0 = \nu(\varkappa, 0) = \sin(x),$$

$$p^1 : \nu_1 = A^{-1}[\frac{1}{s^\alpha}A(\frac{1}{2}i\frac{\partial^2 \nu_0}{\partial \varkappa^2} - i\nu_0\cos^2(x) - i\nu_0^2\bar{\nu_0})] = -\frac{3i}{2}\sin(x)\frac{t^\alpha}{(\alpha)!},$$

$$p^2 : \nu_2 = A^{-1}[\frac{1}{s^\alpha}A(\frac{1}{2}i\frac{\partial^2 \nu_1}{\partial \varkappa^2} - i\nu_1\cos^2(x) - i\nu_0^2\bar{\nu_1} - 2i\nu_0\nu_1\bar{\nu_0})] = \frac{9i^2}{4}\sin(x)\frac{t^{2\alpha}}{(2\alpha)!},$$

$$p^3 : \nu_3 = A^{-1}[\frac{1}{s^\alpha}A(\frac{1}{2}i\frac{\partial^2 \nu_2}{\partial \varkappa^2} - i\nu_2\cos^2(x) - i\nu_0^2\bar{\nu_2} - i\bar{\nu_0}(u_1^2 + 2\nu_0\nu_2) - 2i\nu_0\nu_1\bar{\nu_1})] = -\frac{27i^3}{8}\sin x\frac{t^{3\alpha}}{(3\alpha)!}.$$

Approximate solution,

$$\nu(\varkappa, t) = \nu_0 + \nu_1 + \nu_2 + \nu_3 + ...$$

$$\nu(\varkappa, t) = \sin(x) - \frac{3i}{2}\sin(x)\frac{t^\alpha}{(\alpha)!} + \frac{9i^2}{4}\sin(x)\frac{t^{2\alpha}}{(2\alpha)!} - \frac{27i^3}{8}\sin x\frac{t^{3\alpha}}{(3\alpha)!}...$$

Classical Schrödinger equation for $\alpha = 1$ has the solution,

$$\nu(\varkappa, t) = \sin(x) - \frac{3i}{2}\sin(x)t + \frac{9i^2}{4}\sin(x)\frac{t^2}{(2)!} - \frac{27i^3}{8}\sin x\frac{t^3}{(3)!}...$$

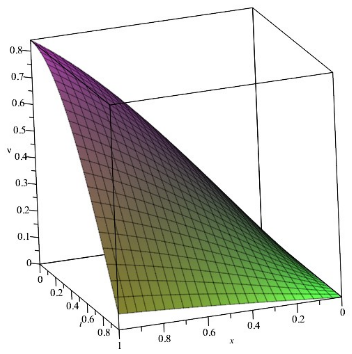

Figure 9. ATHPM solution for real part of Example 3.4 with $\alpha = 1$.

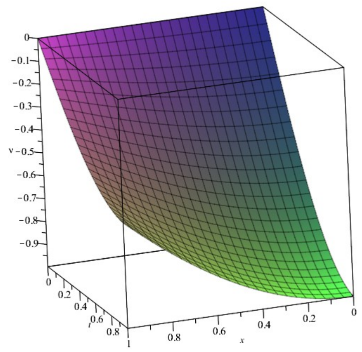

Figure 10. ATHPM solution for imaginary part of Example 3.4 with $\alpha = 1$.

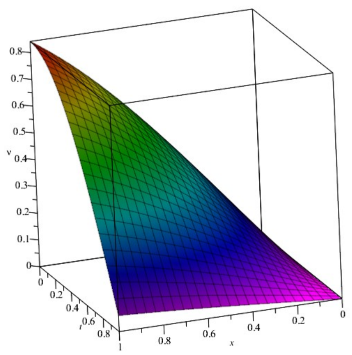

Figure 11. Exact solution for real part of Example 3.4 with $\alpha = 1$.

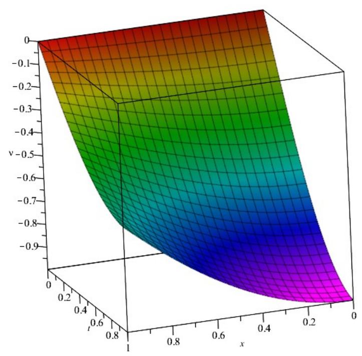

Figure 12. Exact solution for imaginary part of Example 3.4 with $\alpha = 1$.

Applying convergence analysis, we have

$$\gamma_0 = \frac{\|\nu_1\|}{\|\nu_0\|} = 0.15 < 1,$$

$$\gamma_1 = \frac{\|\nu_2\|}{\|\nu_1\|} = 0.075 < 1,$$

$$\gamma_2 = \frac{\|\nu_3\|}{\|\nu_2\|} = 0.00151 < 1.$$

Hence, for $\alpha = 1$ above series solution $\nu(\varkappa, t)$ can be converge to ,

$$\nu(\varkappa, t) = e^{\frac{-3it}{2}} \sin(x).$$

Which is the exact solution.

t	x	ATHPM	Exact	Absolute error	
				Real part	Imaginary part
0.1	0.5	0.07175526822-0.06785552537i	0.07304699971-0.06805032633i	0.00129173149	0.00019480096
	0.75	0.03665758267-0.08862165596i	0.04304582485-1-0.09007645665i	0.00638824218	0.00145480069
	1	-0.01247917705-0.09359382813i	0.007061936527-0.09958333260i	0.01954111358	0.0059895044
0.3	0.5	0.2124051486-0.2008613906i	0.2162288458-0.2014380272i	0.0038236972	0.0005766366
	0.75	0.1085113259-0.2623319022i	0.1274213734-0.2666383059i	0.0189100475	0.0043064037
	1	-0.0369400258-0.2770501937i	0.02090427246-0.2947799246i	0.05784429826	0.0177297309
0.5	0.5	0.3445871059-0.3258595458i	0.3507903301-0.3267950296i	0.0062032242	0.0009354838
	0.75	0.1760390649-0.42558380341	0.2067170338-0.4325701273i	0.0306779689	0.0069863239
	1	-0.0599281923-0.4494614424i	0.03391322101-0.4782245712i	0.09384141331	0.0287631288

TABLE 4. $\nu(\varkappa, t)$ in case of $\alpha = 1$ in Example 3.4

4. Comparison of ATHPM solution with MGMLFM

To compare ATHPM solution with MGMLFM [5], take $\gamma = \frac{1}{2}$, $\beta = -1$ and non-homogeneous term $-\nu \cos^2(x)$ in equation (1.2) then we get,

$$i\frac{\partial^\alpha \nu}{\partial t^\alpha} + \frac{1}{2}\frac{\partial^2 \nu}{\partial \varkappa^2} + \nu \cos^2(x) + |\nu|^2 \nu = 0. \tag{4}$$

Where $\nu(\varkappa, 0) = \sin(x)$ [5].
Now, using ATHPM to solve above equation, we get the following solution,

$$\nu(\varkappa, t) = \sin(x) - \frac{3i}{2}\sin(x)\frac{t^\alpha}{(\alpha)!} + \frac{9i^2}{4}\sin(x)\frac{t^{2\alpha}}{(2\alpha)!} - \frac{27i^3}{8}\sin x\frac{t^{3\alpha}}{(3\alpha)!} + ...$$

Similarly by MGMLFM [5], solution of the equation (4) is given by

$$\nu(\varkappa, t) = \sin(x)\left[1 - \frac{3i}{2}\frac{t^\alpha}{(\alpha)!} + \frac{9i^2}{4}\frac{t^{2\alpha}}{(2\alpha)!} - \frac{27i^3}{8}\frac{t^{3\alpha}}{(3\alpha)!} + ...\right].$$

numerical results validate the effectiveness of ATHPM, showing excellent agreement with the exact solutions and MGMFFM. A comparative analysis (see Tables 5,6,7) highlights that the absolute error remains minimal across different fractional orders. This demonstrates that ATHPM is a robust technique for solving fractional Schrödinger equations with high precision.
To further emphasize the novelty and contribution of the work, a detailed comparative study with the Modified Generalized Mittag-Leffler Function Method (MGMLFM) has been performed. Tables 5–7 present side-by-side comparisons of the real and imaginary parts of the solution as well as the absolute errors for different fractional orders ($\alpha = 0.5, 0.8$, and 1). The results demonstrate that ATHPM not only matches but, in several cases, outperforms MGMLFM in terms of computational efficiency and accuracy.

x	t	ATHPM at $\alpha = 0.5$		MGMLFM at $\alpha = 0.5$	
		Re	Im	Re	Im
-5	0.01	0.937348	- 0.15797	0.937348	- 0.15797
	0.03	0.894197	- 0.258595	0.894197	- 0.258595
	0.06	0.829469	- 0.333855	0.829469	- 0.333855
-1	0.01	-0.822538	0.139004	-0.822538	0.139004
	0.03	-0.784672	0.228914	-0.784672	0.228914
	0.06	-0.727872	0.298599	-0.727872	0.298599
1	0.01	0.822538	- 0.139004	0.822538	- 0.139004
	0.03	0.784672	- 0.228914	0.784672	- 0.228914
	0.06	0.727872	- 0.298599	0.727872	- 0.298599
5	0.01	-0.937348	0.15797	-0.937348	0.15797
	0.03	-0.894197	0.258595	-0.894197	0.258595
	0.06	-0.829469	0.333855	-0.829469	0.333855

TABLE 5. Comparision of ATHPM with MGMLFM in case of $\alpha = 0.5$.

x	t	ATHPM at $\alpha = 0.8$		MGMLFM at $\alpha = 0.8$	
		Re	Im	Re	Im
-5	0.01	0.957976	- 0.0387581	0.957976	- 0.0387581
	0.03	0.953531	- 0.092958	0.953531	- 0.092958
	0.06	0.943374	- 0.1604281	0.943374	- 0.1604281
-1	0.01	-0.840638	0.0340142	-0.840638	0.0340142
	0.03	-0.836709	0.0816136	-0.836709	0.0816136
	0.06	-0.827552	0.140918	-0.827552	0.140918
1	0.01	0.840638	- 0.0340142	0.840638	- 0.0340142
	0.03	0.836709	- 0.0816136	0.836709	- 0.0816136
	0.06	0.827552	- 0.140918	0.827552	- 0.140918
5	0.01	-0.957976	0.0387581	-0.957976	0.0387581
	0.03	-0.953531	0.092958	-0.953531	0.092958
	0.06	-0.943374	0.160428	-0.943374	0.160428

TABLE 6. Comparision of ATHPM with MGMLFM $\nu(\varkappa, t)$ in case of $\alpha = 0.8$.

5. CONCLUSION

This study demonstrates that the Aboodh Transform Homotopy Perturbation Method (ATHPM) is an efficient and robust tool for solving fractional-order Schrödinger equations in both linear and nonlinear forms. By transforming the fractional derivatives into a more tractable algebraic form, ATHPM circumvents the need for linearization or small-parameter assumptions, leading to a solution process that is both straightforward and computationally efficient. The numerical experiments, along with comprehensive comparisons with exact solutions and alternative methods such as the Modified Generalized Mittag-Leffler Function Method (MGMLFM), confirm the rapid convergence and high accuracy of the ATHPM approach. Importantly, the results of this study have broader implications for the fields of applied mathematics and quantum mechanics. The simplicity and effectiveness of ATHPM not

x	t	ATHPM at 1		MGMLFM at 1		Exact		Absolute Error MGMLFM	Absolute Error for ATHPM
		Re	Im	Re	Im	Re	Im		
-5	0.01	0.958816	- 0.0143827	0.958816	- 0.0143827	0.958816	- 0.0143833	6.61738×10^{-7}	6.61738×10^{-7}
	0.03	0.957957	0.0431194	0.957957	0.0431194	0.957954	- 0.043137	1.79569×10^{-5}	1.79569×10^{-5}
	0.06	0.955094	- 0.0860496	0.955094	- 0.0860496	0.955043	- 0.0861867	1.46139×10^{-4}	1.46139×10^{-4}
-1	0.01	-0.841376	0.0126211	-0.841376	0.0126211	-0.841376	0.0126216	4.47172×10^{-7}	4.47172×10^{-7}
	0.03	-0.840621	0.0378414	-0.840621	0.0378414	-0.840619	0.0378534	1.21395×10^{-5}	1.21395×10^{-5}
	0.06	-0.838097	0.0755366	-0.838097	0.0755366	-0.838065	0.0756302	9.89065×10^{-5}	9.89065×10^{-5}
1	0.01	0.841376	- 0.0126211	0.841376	- 0.0126211	0.841376	- 0.0126216	4.47172×10^{-7}	4.47172×10^{-7}
	0.03	0.840621	- 0.0378414	0.840621	- 0.0378414	0.840619	- 0.0378534	1.1395×10^{-5}	1.1395×10^{-5}
	0.06	0.838097	- 0.0755366	0.838097	- 0.0755366	0.838065	- 0.0756302	9.89065×10^{-5}	9.89065×10^{-5}
5	0.01	-0.958816	0.0143827	-0.958816	0.0143827	-0.958816	0.0143833	6.61738×10^{-7}	6.61738×10^{-7}
	0.03	-0.957957	0.0431194	-0.957957	0.0431194	-0.957954	0.043137	1.79569×10^{-5}	1.79569×10^{-5}
	0.06	-0.955094	0.0860496	-0.955094	0.0860496	-0.955043	0.0861867	1.46139×10^{-4}	1.46139×10^{-4}

TABLE 7. Comparision of ATHPM with MGMLFM $\nu(\varkappa, t)$ and absolute erroes with different time t and x at $\alpha = 1$.

only make it a promising alternative for solving complex differential equations but also open up new avenues for its application in areas such as plasma physics, nonlinear optics, and beyond. Looking ahead, future research could explore extending the ATHPM framework to multi-dimensional and more complex fractional differential equations. There is also potential for integrating ATHPM with other numerical or analytical methods to create hybrid approaches that further enhance computational performance. The insights gained from this work provide a solid foundation for the development of advanced techniques that can address increasingly intricate problems across various scientific and engineering disciplines.

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