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HARTMAN-WINTNER-TYPE INEQUALITY FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH k-PRABHAKAR DERIVATIVE

NARAYAN G. ABUJ, DEEPAK B. PACHPATTE

Abstract. In this manuscript, we investigate a non-local fractional boundary value problem of the form:

$$\begin{cases} ({}_k\mathbf{D}^{\gamma}_{\rho,\beta,\omega,a+}y)(t) + q(t)y(t) = 0, \quad a < t < b,\, 2 < \beta \leq 3, \\ y(a) = y'(a) = 0,\, y'(b) = \alpha y(\xi), \end{cases}$$

and establish a Hartman-Wintner-type inequality for this problem within the framework of k-Prabhakar fractional derivatives. By leveraging the properties of Green's function and its analytical characteristics, we derive the corresponding integral equation for the proposed nonlocal fractional boundary value problem. The resulting inequality provides a significant generalization of previous results in the literature [26]. This work broadens the scope of fractional boundary value problems, offering new insights and laying a foundation for future applications across various fields. The generalization highlighted here emphasizes the flexibility and depth of the k-Prabhakar framework, paving the way for further advancements in fractional calculus research. Furthermore, this study contributes to both the theoretical development of fractional calculus and its practical applications in modeling complex real-world phenomena.

1. Introduction

In recent years, fractional calculus has seen growing popularity across diverse fields, thanks to its wide-ranging applications in science and engineering (see [1],[3], [4],[7],[12],[29],[32],[33] and references therein). This growing interest has accelerated the development of various Lyapunov-type inequalities for fractional boundary value problems.

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In 1951, Hartman and Wintner [17] consider the boundary value problem

$$\begin{cases} x''(t)+q(t)x(t)=0, \quad a<t<b, \\ x(a)=x(b)=0, \end{cases} \tag{1}$$

and if (1) has a nontrivial solution then they proved the following inequality

$$\int_a^b (b-s)(s-a)q^+(s)ds > b-a, \tag{2}$$

where $q^+(s)=max\{q(s),0\}$.

In 1907, Lyapunov [20] explore the following remarkable inequality if (1) has nontrivial solution

$$\int_a^b |q(s)|ds > \frac{4}{b-a}. \tag{3}$$

The above Lyapunov inequality (3) can be deduced from (2) using the following fact

$$\max_{s\in[a,b]}(b-s)(s-a)=\frac{(b-a)^2}{4}. \tag{4}$$

Many generalizations and extensions of the Lyapunov inequality (3) are exists in the literature [8, 5, 10, 9, 23, 21, 22, 34] and therein references. Recently, few Lyapunov type inequalities were established in connections with different fractional boundary value problem using differential operators, for details see [15, 16, 18, 19, 27, 30, 24, 2].

In [11], Cabrera et al. investigated the Hartman-Wintner-type inequality

$$\int_a^b (b-s)^{\alpha-2}(s-a)|q(s)|ds \geq \left(1+\frac{\beta(b-a)^{\alpha-1}}{(\alpha-1)(b-a)^{\alpha-2}-\beta(\xi-a)^{\alpha-1}}\right)^{-1}\Gamma(\alpha). \tag{5}$$

for following nonlocal fractional boundary value problem:

$$\begin{cases} D_a^\alpha x(t)+q(t)x(t)=0, \quad a<t<b, \\ x(a)=x'(a)=0, x'(b)=\beta x(\xi), \end{cases} \tag{6}$$

where D_a^α denotes the standard Riemann-Liouville fractional derivative of order α, $a<\xi<b$, $0\leq\beta(\xi-a)^{\alpha-1}<(\alpha-1)(b-a)^{\alpha-2}$ and $q(t)$ is continuous real valued function on $[a,b]$.

Recently, Pachpatte et al. [26] studied the Hartman-Wintner-type inequality for following nonlocal fractional boundary problem with Prabhakar derivative

$$\begin{cases} \mathbf{D}^\gamma_{\rho,\mu,\omega,a^+}x(t)+q(t)x(t)=0, \quad a<t<b, \quad 2<\mu\leq 3, \\ x(a)=x'(a)=0, x'(b)=\beta x(\xi), \end{cases} \tag{7}$$

In present paper, for $a<\xi<b, 0\leq\alpha(\xi-a)^{\beta-1}<(\beta-1)(b-a)^{\beta-2}$, $q:[a,b]\to\mathbb{R}$, we consider the following nonlocal fractional boundary value problem:

$$\begin{cases} ({}_k\mathbf{D}^\gamma_{\rho,\beta,\omega,a+}y)(t)+q(t)y(t)=0, \quad a<t<b,\ 2<\beta\leq 3, \\ y(a)=y'(a)=0,\ y'(b)=\alpha y(\xi), \end{cases} \tag{8}$$

where ${}_k\mathbf{D}^\gamma_{\rho,\beta,\omega,a+}$ denotes the k-Prabhakar derivative of order β.
and establish the Hartman-Wintner-type inequality for(8).

In Section 1, we provide an overview of the essential concepts in fractional calculus, Lyapunov-type inequalities, and Hartman-Winter-type inequalities. Section 2 covers key preliminaries concerning the Mittag-Leffler function and the k-Prabhakar derivative, along with its Laplace transform. In Section 3, we explore the integral equation for a nonlocal fractional boundary value problem involving the k-Prabhakar fractional derivative to derive a Hartman-Winter-type inequality.

2. Preliminaries

In this section, we give some basic definitions and lemmas that will be necessary to prove main results in the sequel.

Definition 2.1. [12] *The k-Mittag-Leffler function is denoted by $E^{\gamma}_{k,\alpha,\beta}(z)$ and is defined as*

$$E^{\gamma}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}\, z^n}{\Gamma_k(\alpha n + \beta) n!}, \tag{9}$$

where $k \in \mathbb{R}^+, \alpha, \beta, \gamma \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$; $\Gamma_k(x)$ is the k-Gamma function and $(\gamma)_{n,k} = \frac{\Gamma_k(\gamma+nk)}{\Gamma_k(\gamma)}$ is the pochhammer k-symbol.

Definition 2.2. [13] *Let $\alpha, \beta, \omega, \gamma \in \mathbb{C}$, $k \in \mathbb{R}^+$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$ and $\phi \in L^1([0, b])$,$(0 < x < b \leq \infty)$. The k-Prabhakar integral operator involving k-Mittag-Leffler function is defined as*

$$\begin{aligned}({}_k\boldsymbol{P}_{\alpha,\beta,\omega}\phi)(x) &= \int_0^x \frac{(x-t)^{\frac{\beta}{k}-1}}{k} E^{\gamma}_{k,\alpha,\beta}[\omega(x-t)^{\frac{\alpha}{k}}]\phi(t)dt, \quad (x > 0)\\ &= ({}_k\mathcal{E}^{\gamma}_{\alpha,\beta,\omega} * f)(x),\end{aligned} \tag{10}$$

where

$${}_k\mathcal{E}^{\gamma}_{\alpha,\beta,\omega}(t) = \begin{cases} \frac{t^{\frac{\beta}{k}-1}}{k} E^{\gamma}_{k,\alpha,\beta}(\omega t^{\frac{\alpha}{k}}), & t > 0; \\ 0, & t \leq 0. \end{cases} \tag{11}$$

Definition 2.3. [13] *Let $k \in \mathbb{R}^+$, $\rho, \beta, \gamma, \omega \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0$, $m = [\frac{\beta}{k}] + 1$, $f \in L^1([0, b])$. The k-Prabhakar derivative is defined as*

$${}_k\boldsymbol{D}^{\gamma}_{\rho,\beta,\omega,}f(x) = \left(\frac{d}{dx}\right)^m k^m\, {}_k\boldsymbol{P}^{-\gamma}_{\rho,mk-\beta,\omega} f(x). \tag{12}$$

In order to prove our main results we shall need following lemma.

Lemma 2.1. [13] *Let $\alpha, \beta, \omega, \gamma \in \mathbb{C}, k \in \mathbb{R}^+, \Re(\alpha) > 0$; $\Re(\beta) > 0$, $\phi \in L^1(\mathbb{R}^+_0)$ and $|\omega k(ks)^{\frac{-\rho}{k}}| < 1$ then*

$$\begin{aligned}L\{({}_k\boldsymbol{P}^{\gamma}_{\rho,\beta,\omega,}\phi)(x)\}(s) &= L\{{}_k\varepsilon^{\gamma}_{\rho,\beta,\omega}(t)\}(s)L\{\phi\}(s)\\ &= (ks)^{\frac{-\beta}{k}}(1 - \omega k(ks)^{\frac{-\rho}{k}})^{\frac{-\gamma}{k}} L\{\phi\}(s).\end{aligned} \tag{13}$$

Lemma 2.2. [28] *The Laplace transform of k-Prabhakar derivative* (12) *is*

$$\begin{aligned}L\{{}_k\boldsymbol{D}^{\gamma}_{\rho,\beta,\omega}f(x)\} &= (ks)^{\frac{\beta}{k}}(1 - \omega * k(ks)^{\frac{-\rho}{k}})^{\frac{\gamma}{k}} F(s)\\ &= -\sum_{n=0}^{m-1} k^{n+1}s^n \left({}_k\boldsymbol{D}^{\gamma}_{\rho,\beta-(n+1)k,\omega} f(0^+)\right).\end{aligned} \tag{14}$$

For the case $[\frac{\beta}{k}] + 1 = m = 1$,

$$L\{{}_k\boldsymbol{D}^{\gamma}_{\rho,\beta,\omega,}y(x)\} = (ks)^{\frac{-\beta}{k}}(1 - \omega k(ks)^{\frac{-\rho}{k}})^{\frac{\gamma}{k}} L\{y(x)\}(s) - k({}_k\boldsymbol{P}^{-\gamma}_{\rho,k-\beta,\omega,}y)(0)$$

with $|\omega k(ks)^{\frac{-\rho}{k}}| < 1$.

Lemma 2.3. [26] *If* $f(x) \in C(a,b) \cap L(a,b)$; *then* ${}_k\boldsymbol{D}^{\gamma}_{\rho,\beta,\omega,a+}{}_k\boldsymbol{P}^{\gamma}_{\rho,\beta,\omega,a+}f(x) = f(x)$ *and if* $f(x), {}_k\boldsymbol{D}^{\gamma}_{\rho,\beta,\omega,a+}f(x) \in C(a,b) \cap L(a,b)$, *then for* $c_j \in \mathbb{R}$ *and* $m-1 < \beta \leq m$, *we have*

$$\begin{aligned}{}_k\boldsymbol{P}^{\gamma}_{\rho,\beta,\omega,a+}{}_k\boldsymbol{D}^{\gamma}_{\rho,\beta,\omega,a+}f(x) =& f(x) + c_0(x-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(x-a)^{\frac{\rho}{k}}) \\ &+ c_1(x-a)^{\frac{\beta}{k}-2}E^{\gamma}_{k,\rho,\beta-k}(\omega(x-a)^{\frac{\rho}{k}}) \\ &+ c_2(x-a)^{\frac{\beta}{k}-3}E^{\gamma}_{k,\rho,\beta-2k}(\omega(x-a)^{\frac{\rho}{k}}) + ... \\ &+ c_{m-1}(x-a)^{\frac{\beta}{k}-m}E^{\gamma}_{k,\rho,\beta-(m-1)k}(\omega(x-a)^{\frac{\rho}{k}}) \qquad (15)\end{aligned}$$

Lemma 2.4. [26]*Let* $k \in \mathbb{R}^+$,$\rho, \beta, \gamma, \omega \in \mathbb{C}$, $\Re(\alpha) > 0$; $\Re(\beta) > 0$ *then for any* $j \in \mathbb{N}$ *we have*

$$\frac{d^j}{dx^j}[x^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega x^{\frac{\rho}{k}})] = \frac{x^{\frac{\beta}{k}-(j+1)}}{k^j}E^{\gamma}_{k,\rho,\beta-jk}(\omega x^{\frac{\rho}{k}}) \qquad (16)$$

Inspired by above mentioned work and stimulated specially by work of Pachpatte et al. [26]. Here we propose to establish the Hartman-Wintner-type inequality in framework of k-Prabhakar fractional order derivative.

3. Main Results

In this section, we investigate integral equation for nonlocal fractional boundary value problem with the k-Prabhakar fractional derivative to establish a Hartman-Wintner-type inequality.

Theorem 3.1. *Assume that* $2 < \beta \leq 3$ *and* $y \in C[a,b]$. *If the nonlocal fractional boundary value problem* (8) *has unique nontrivial solution, then it satisfies*

$$\begin{aligned}y(t) =& \int_a^b G(t,s)q(s)y(s)ds \\ &+ \frac{\alpha(t-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(t-a)^{\frac{\rho}{k}})}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}}) - \alpha(\xi-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(\xi-a)^{\frac{\rho}{k}})}\int_a^b G(\xi,s)q(s)y(s)ds,\end{aligned}$$

where the Green's function $G(t,s)$ *is defined as follows*

$$G(t,s) = \begin{cases} \dfrac{(t-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(t-a)^{\frac{\rho}{k}})\frac{(b-s)^{\frac{\beta}{k}-2}}{k^2}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-s)^{\frac{\rho}{k}})}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}})} \\ -\frac{(t-s)^{\frac{\beta}{k}-1}}{k}E^{\gamma}_{k,\rho,\beta}(\omega(t-s)^{\frac{\rho}{k}}), & a \leq s \leq t \leq b, \\ \dfrac{(t-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(t-a)^{\frac{\rho}{k}})\frac{(b-s)^{\frac{\beta}{k}-2}}{k^2}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-s)^{\frac{\rho}{k}})}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}})}, & a \leq t \leq s \leq b. \end{cases} \qquad (17)$$

Proof. From lemma 2.3, the general solution to (8) in $C[a,b]$ can be written as follows

$$y(t) = c_0(t-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(t-a)^{\frac{\rho}{k}}) + c_1(t-a)^{\frac{\beta}{k}-2}E^{\gamma}_{k,\rho,\beta-k}(\omega(t-a)^{\frac{\rho}{k}}) + c_2(t-a)^{\frac{\beta}{k}-3}E^{\gamma}_{k,\rho,\beta-2k}(\omega(t-a)^{\frac{\rho}{k}}) - \int_a^t \frac{(t-s)^{\frac{\beta}{k}-1}}{k}E^{\gamma}_{k,\rho,\beta}(\omega(t-s)^{\frac{\rho}{k}})q(s)y(s)ds. \tag{18}$$

Employing the first boundary condition $y(a) = y'(a) = 0$ we obtain $c_1 = c_2 = 0$. Therefore the solution (18) becomes

$$y(t) = c_0(t-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(t-a)^{\frac{\rho}{k}}) - \int_a^t \frac{(t-s)^{\frac{\beta}{k}-1}}{k}E^{\gamma}_{k,\rho,\beta}(\omega(t-s)^{\frac{\rho}{k}})q(s)y(s)ds. \tag{19}$$

For second boundary condition we find

$$y'(t) = c_0\frac{(t-a)^{\frac{\beta}{k}-2}}{k}E^{\gamma}_{k,\rho,\beta-k}(\omega(t-a)^{\frac{\rho}{k}}) - \int_a^t \frac{(t-s)^{\frac{\beta}{k}-2}}{k^2}E^{\gamma}_{k,\rho,\beta-k}(\omega(t-s)^{\frac{\rho}{k}})q(s)y(s)ds.$$

Employing the second boundary condition $y'(b) = \alpha y(\xi)$ we get

$$\Rightarrow c_0\frac{(b-a)^{\frac{\beta}{k}-2}}{k}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}}) - \int_a^b \frac{(b-s)^{\frac{\beta}{k}-2}}{k^2}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-s)^{\frac{\rho}{k}})q(s)y(s)ds =$$

$$\alpha c_0(\xi-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(\xi-a)^{\frac{\rho}{k}}) - \alpha\int_a^{\xi} \frac{(\xi-s)^{\frac{\beta}{k}-1}}{k}E^{\gamma}_{k,\rho,\beta}(\omega(\xi-s)^{\frac{\rho}{k}})q(s)y(s)ds,$$

$$\Rightarrow c_0\left[\frac{(b-a)^{\frac{\beta}{k}-2}}{k}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}}) - \alpha(\xi-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(\xi-a)^{\frac{\rho}{k}})\right] =$$

$$\int_a^b \frac{(b-s)^{\frac{\beta}{k}-2}}{k^2}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-s)^{\frac{\rho}{k}})q(s)y(s)ds - \alpha\int_a^{\xi} \frac{(\xi-s)^{\frac{\beta}{k}-1}}{k}E^{\gamma}_{k,\rho,\beta}(\omega(\xi-s)^{\frac{\rho}{k}})q(s)y(s)ds,$$

$$\Rightarrow c_0 = \frac{1}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}}) - \alpha(\xi-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(\xi-a)^{\frac{\rho}{k}})} \times \int_a^b \frac{(b-s)^{\frac{\beta}{k}-2}}{k^2}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-s)^{\frac{\rho}{k}})q(s)y(s)ds$$

$$- \frac{\alpha}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}}) - \alpha(\xi-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(\xi-a)^{\frac{\rho}{k}})} \times \int_a^{\xi} \frac{(\xi-s)^{\frac{\beta}{k}-1}}{k}E^{\gamma}_{k,\rho,\beta}(\omega(\xi-s)^{\frac{\rho}{k}})q(s)y(s)ds.$$

Thus the solution $y(t)$ becomes

$$\begin{aligned}
y(t) =& \frac{(t-a)^{\frac{\beta}{k}-1} E^{\gamma}_{k,\rho,\beta}(\omega(t-a)^{\frac{\rho}{k}})}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k} E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}}) - \alpha(\xi-a)^{\frac{\beta}{k}-1} E^{\gamma}_{k,\rho,\beta}(\omega(\xi-a)^{\frac{\rho}{k}})} \\
&\quad \times \int_a^b \frac{(b-s)^{\frac{\beta}{k}-2}}{k^2} E^{\gamma}_{k,\rho,\beta-k}(\omega(b-s)^{\frac{\rho}{k}}) q(s) y(s) ds \\
&- \frac{\alpha(t-a)^{\frac{\beta}{k}-1} E^{\gamma}_{k,\rho,\beta}(\omega(t-a)^{\frac{\rho}{k}})}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k} E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}}) - \alpha(\xi-a)^{\frac{\beta}{k}-1} E^{\gamma}_{k,\rho,\beta}(\omega(\xi-a)^{\frac{\rho}{k}})} \\
&\quad \times \int_a^{\xi} \frac{(\xi-s)^{\frac{\beta}{k}-1}}{k} E^{\gamma}_{k,\rho,\beta}(\omega(\xi-s)^{\frac{\rho}{k}}) q(s) y(s) ds \\
&- \int_a^t \frac{(t-s)^{\frac{\beta}{k}-1}}{k} E^{\gamma}_{k,\rho,\beta}(\omega(t-s)^{\frac{\rho}{k}}) q(s) y(s) ds.
\end{aligned}$$

Taking into account that

$$\begin{aligned}
&\frac{E^{\gamma}_{k,\rho,\beta}(\omega(t-a)^{\frac{\rho}{k}})}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k} E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}}) - \alpha(\xi-a)^{\frac{\beta}{k}-1} E^{\gamma}_{k,\rho,\beta}(\omega(\xi-a)^{\frac{\rho}{k}})} \\
&= \left(\frac{E^{\gamma}_{k,\rho,\beta}(\omega(t-a)^{\frac{\rho}{k}})}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k} E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}})} \right) \\
&\quad \times \left[\frac{\frac{(b-a)^{\frac{\beta}{k}-2}}{k} E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}})}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k} E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}}) - \alpha(\xi-a)^{\frac{\beta}{k}-1} E^{\gamma}_{k,\rho,\beta}(\omega(\xi-a)^{\frac{\rho}{k}})} \right] \\
&= \left(\frac{E^{\gamma}_{k,\rho,\beta}(\omega(t-a)^{\frac{\rho}{k}})}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k} E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}})} \right) \\
&\quad \times \left[1 + \frac{\alpha(\xi-a)^{\frac{\beta}{k}-1} E^{\gamma}_{k,\rho,\beta}(\omega(\xi-a)^{\frac{\rho}{k}})}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k} E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}}) - \alpha(\xi-a)^{\frac{\beta}{k}-1} E^{\gamma}_{k,\rho,\beta}(\omega(\xi-a)^{\frac{\rho}{k}})} \right],
\end{aligned}$$

we have

$$
\begin{aligned}
y(t) = & \left[1 + \frac{\alpha(\xi-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(\xi-a)^{\frac{\rho}{k}})}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}}) - \alpha(\xi-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(\xi-a)^{\frac{\rho}{k}})}\right] \\
& \times \frac{(t-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(t-a)^{\frac{\rho}{k}})}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}})}\int_a^b \frac{(b-s)^{\frac{\beta}{k}-2}}{k^2}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-s)^{\frac{\rho}{k}})q(s)y(s)ds \\
& - \frac{\alpha(t-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(t-a)^{\frac{\rho}{k}})}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}}) - \alpha(\xi-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(\xi-a)^{\frac{\rho}{k}})} \\
& \times \int_a^{\xi} \frac{(\xi-s)^{\frac{\beta}{k}-1}}{k}E^{\gamma}_{k,\rho,\beta}(\omega(\xi-s)^{\frac{\rho}{k}})q(s)y(s)ds \\
& - \int_a^t \frac{(t-s)^{\frac{\beta}{k}-1}}{k}E^{\gamma}_{k,\rho,\beta}(\omega(t-s)^{\frac{\rho}{k}})q(s)y(s)ds.
\end{aligned}
$$

On simplifying,

$$
\begin{aligned}
y(t) = & \frac{(t-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(t-a)^{\frac{\rho}{k}})}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}})}\int_a^t \frac{(b-s)^{\frac{\beta}{k}-2}}{k^2}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-s)^{\frac{\rho}{k}})q(s)y(s)ds \\
& + \frac{(t-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(t-a)^{\frac{\rho}{k}})}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}})}\int_t^b \frac{(b-s)^{\frac{\beta}{k}-2}}{k^2}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-s)^{\frac{\rho}{k}})q(s)y(s)ds \\
& + \left[\frac{\alpha(\xi-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(\xi-a)^{\frac{\rho}{k}})}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}}) - \alpha(\xi-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(\xi-a)^{\frac{\rho}{k}})}\right] \\
& \times \left(\frac{(t-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(t-a)^{\frac{\rho}{k}})}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}})}\right)\int_a^{\xi} \frac{(b-s)^{\frac{\beta}{k}-2}}{k^2}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-s)^{\frac{\rho}{k}})q(s)y(s)ds \\
& + \left[\frac{\alpha(\xi-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(\xi-a)^{\frac{\rho}{k}})}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}}) - \alpha(\xi-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(\xi-a)^{\frac{\rho}{k}})}\right] \\
& \times \left(\frac{(t-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(t-a)^{\frac{\rho}{k}})}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}})}\right)\int_{\xi}^{b} \frac{(b-s)^{\frac{\beta}{k}-2}}{k^2}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-s)^{\frac{\rho}{k}})q(s)y(s)ds \\
& - \frac{\alpha(t-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(t-a)^{\frac{\rho}{k}})}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}}) - \alpha(\xi-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(\xi-a)^{\frac{\rho}{k}})} \\
& \times \int_a^{\xi} \frac{(\xi-s)^{\frac{\beta}{k}-1}}{k}E^{\gamma}_{k,\rho,\beta}(\omega(\xi-s)^{\frac{\rho}{k}})q(s)y(s)ds \\
& - \int_a^t \frac{(t-s)^{\frac{\beta}{k}-1}}{k}E^{\gamma}_{k,\rho,\beta}(\omega(t-s)^{\frac{\rho}{k}})q(s)y(s)ds.
\end{aligned}
$$

Further, on rearranging the terms, we have

$$
\begin{aligned}
y(t) = &\int_a^t \Bigg[\frac{(t-a)^{\frac{\beta}{k}-1} E^{\gamma}_{k,\rho,\beta}(\omega(t-a)^{\frac{\rho}{k}}) \frac{(b-s)^{\frac{\beta}{k}-2}}{k^2} E^{\gamma}_{k,\rho,\beta-k}(\omega(b-s)^{\frac{\rho}{k}})}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k} E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}})} \\
&- \frac{(t-s)^{\frac{\beta}{k}-1}}{k} E^{\gamma}_{k,\rho,\beta}(\omega(t-s)^{\frac{\rho}{k}}) \Bigg] q(s)y(s)ds \\
&+ \int_t^b \Bigg[\frac{(t-a)^{\frac{\beta}{k}-1} E^{\gamma}_{k,\rho,\beta}(\omega(t-a)^{\frac{\rho}{k}}) \frac{(b-s)^{\frac{\beta}{k}-2}}{k^2} E^{\gamma}_{k,\rho,\beta-k}(\omega(b-s)^{\frac{\rho}{k}})}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k} E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}})} \Bigg] q(s)y(s)ds \\
&+ \frac{\alpha(t-a)^{\frac{\beta}{k}-1} E^{\gamma}_{k,\rho,\beta}(\omega(t-a)^{\frac{\rho}{k}})}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k} E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}}) - \alpha(\xi-a)^{\frac{\beta}{k}-1} E^{\gamma}_{k,\rho,\beta}(\omega(\xi-a)^{\frac{\rho}{k}})} \\
&\times \int_a^{\xi} \Bigg[\frac{(\xi-a)^{\frac{\beta}{k}-1} E^{\gamma}_{k,\rho,\beta}(\omega(\xi-a)^{\frac{\rho}{k}}) \frac{(b-s)^{\frac{\beta}{k}-2}}{k^2} E^{\gamma}_{k,\rho,\beta-k}(\omega(b-s)^{\frac{\rho}{k}})}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k} E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}})} \\
&- \frac{(\xi-s)^{\frac{\beta}{k}-1}}{k} E^{\gamma}_{k,\rho,\beta}(\omega(\xi-s)^{\frac{\rho}{k}}) \Bigg] q(s)y(s)ds \\
&+ \Bigg[\frac{\alpha(\xi-a)^{\frac{\beta}{k}-1} E^{\gamma}_{k,\rho,\beta}(\omega(\xi-a)^{\frac{\rho}{k}})}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k} E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}}) - \alpha(\xi-a)^{\frac{\beta}{k}-1} E^{\gamma}_{k,\rho,\beta}(\omega(\xi-a)^{\frac{\rho}{k}})} \Bigg] \\
&\times \left(\frac{(t-a)^{\frac{\beta}{k}-1} E^{\gamma}_{k,\rho,\beta}(\omega(t-a)^{\frac{\rho}{k}})}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k} E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}})} \right) \int_{\xi}^{b} \frac{(b-s)^{\frac{\beta}{k}-2}}{k^2} E^{\gamma}_{k,\rho,\beta-k}(\omega(b-s)^{\frac{\rho}{k}}) q(s)y(s)ds,
\end{aligned}
$$

therefore the solution $y(t)$ becomes

$$
\begin{aligned}
y(t) = &\int_a^b G(t,s)q(s)y(s)ds \\
&+ \frac{\alpha(t-a)^{\frac{\beta}{k}-1} E^{\gamma}_{k,\rho,\beta}(\omega(t-a)^{\frac{\rho}{k}})}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k} E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}}) - \alpha(\xi-a)^{\frac{\beta}{k}-1} E^{\gamma}_{k,\rho,\beta}(\omega(\xi-a)^{\frac{\rho}{k}})} \int_a^b G(\xi,s)q(s)y(s)ds,
\end{aligned}
$$

where the Green's function $G(t,s)$ is given by (17).

Theorem 3.2. *The Green's function* (17) *satisfies the following properties:*
(*a*) $G(t,s) \geq 0$, *for all* $(t,s) \in [a,b] \times [a,b]$*;*
(*b*) $G(t,s)$ *is nondecreasing function with respect to the first variable;*
(*c*) $0 \leq G(a,s) \leq G(t,s) \leq G(b,s), \qquad (t,s) \in [a,b] \times [a,b]$.

proof (a). For proof see,(Theorem 3.2, in [26])

proof (b). Proof of this is similar to (Theorem 2, in [14])

Proof (c). Proof of this follows from (b).

Theorem 3.3. *Suppose that problem* (8) *has a nontrivial continuous solution, then*

$$\int_a^b \Bigg(\frac{(b-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(b-a)^{\frac{\rho}{k}})\frac{(b-s)^{\frac{\beta}{k}-2}}{k^2}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-s)^{\frac{\rho}{k}})}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}})} - \frac{b-s)^{\frac{\beta}{k}-1}}{k}E^{\gamma}_{k,\rho,\beta}(\omega(b-s)^{\frac{\rho}{k}})\Bigg)|q(s)|ds$$
$$\geq \left(1+\frac{\alpha(b-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(b-a)^{\frac{\rho}{k}})}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}})-\alpha(\xi-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(\xi-a)^{\frac{\rho}{k}})}\right)^{-1}$$

Proof. Consider the Banach space
$C[a,b]=\{u:[a,b]\longrightarrow\mathbb{R}\setminus u \text{ is continuous}\}$
equipped with norm $\|u\|_\infty = max\{|u(t)| : a\leq t\leq b\}, u\in C[a,b]$.
By theorem 3.1, a solution $y\in C[a,b]$ of (8) has the expression for $a\leq t\leq b$,

$$y(t)=\int_a^b G(t,s)q(s)y(s)ds + \frac{\alpha(t-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(t-a)^{\frac{\rho}{k}})}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}})-\alpha(\xi-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(\xi-a)^{\frac{\rho}{k}})}\int_a^b G(\xi,s)q(s)y(s)ds.$$

From this, for any $t\in[a,b]$, we have

$$|y(t)|\leq \|y\|_\infty\int_a^b |G(t,s)||q(s)|ds + \frac{\alpha(t-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(t-a)^{\frac{\rho}{k}})}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}})-\alpha(\xi-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(\xi-a)^{\frac{\rho}{k}})}\int_a^b |G(\xi,s)||q(s)|ds,$$

therefore,

$$|y(t)|\leq \|y\|_\infty\int_a^b |G(b,s)||q(s)|ds + \frac{\alpha(b-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(b-a)^{\frac{\rho}{k}})\|y\|_\infty}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}})-\alpha(\xi-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(\xi-a)^{\frac{\rho}{k}})}\int_a^b |G(b,s)||q(s)|ds,$$

which yields

$$\|y\|_\infty\leq \|y\|_\infty\left(1+\frac{\alpha(b-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(b-a)^{\frac{\rho}{k}})}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}})-\alpha(\xi-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(\xi-a)^{\frac{\rho}{k}})}\right)\times\int_a^b G(b,s)|q(s)|ds,$$

As y is a nontrivial solution, we have

$$1 \leq \left(1+\frac{\alpha(b-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(b-a)^{\frac{\rho}{k}})}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}})-\alpha(\xi-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(\xi-a)^{\frac{\rho}{k}})}\right) \times \int_a^b |G(b,s)||q(s)|ds.$$

$$\int_a^b |G(b,s)||q(s)|ds \geq \left(1+\frac{\alpha(b-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(b-a)^{\frac{\rho}{k}})}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}})-\alpha(\xi-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(\xi-a)^{\frac{\rho}{k}})}\right)^{-1},$$

therefore

$$\int_a^b \left[\frac{(b-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(b-a)^{\frac{\rho}{k}})\frac{(b-s)^{\frac{\beta}{k}-2}}{k^2}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-s)^{\frac{\rho}{k}})}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}})} - \frac{(t-s)^{\frac{\beta}{k}-1}}{k}E^{\gamma}_{k,\rho,\beta}(\omega(t-s)^{\frac{\rho}{k}})\right]|q(s)|ds$$
$$\geq \left(1+\frac{\alpha(b-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(b-a)^{\frac{\rho}{k}})}{\frac{(b-a)^{\frac{\beta}{k}-2}}{k}E^{\gamma}_{k,\rho,\beta-k}(\omega(b-a)^{\frac{\rho}{k}})-\alpha(\xi-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(\xi-a)^{\frac{\rho}{k}})}\right)^{-1}.$$

Hence the result.

4. Conclusion

This paper explores certain Hartman–Wintner type inequality for nonlocal fractional boundary value problem.The approach used in this paper is based on the calculation of the Green's function associated to the considered problem and obtained more general results than existing results [26]. The results [26] can be obtained for particular values of k and β as $k = 1$ and $\beta = \mu$. The work can also be extended to investigate Hartman–Wintner type inequality for newly generalized weighted fractional derivatives [31]. Mathematically, future reserch can be carried out with the results of newly generalized weighted fractional derivatives [31] with respect to another function are derived in the sense of Caputo and Riemann–Liouville involving a new modified version of a generalized Mittag–Leffler function with three parameters.

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Narayan G. Abuj
School of Basic and Applied Sciences, JSPM University Pune, Wagholi- 412207 (M.S.) India.
Email address: abujng@gmail.com

Deepak B. Pachpatte
Department of mathematics, Dr. Babasaheb Ambedkar Marathwada University, Chhatrapati Sambhajinagar-431004 (M.S.) India.
Email address: pachpatte@gmail.com