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SOME RESULTS DUE TO UNIQUENESS OF MEROMORPHIC FUNCTIONS CONCERNING NON-LINEAR DIFFERENTIAL POLYNOMIALS

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Abstract. In this paper, we deal with the uniqueness problems on meromorphic functions concerning non-linear differential polynomials with regard to multiplicity. Moreover, we greatly generalize and improve some results obtained by V. Husna [J. Anal., Vol. 29, 1191-1206, 2021].

1. Introduction, Definitions and Results

In what follows by a meromorphic function we mean that the function has poles as its singularities only in the complex plane $\mathbb{C}$, we assume that the reader is familier with standard notations such as $T(r,f)$, $m(r,f)$, $N(r,f)$ and the fundamental results of Nevanlinna theory of meromorphic functions (see [3, 9, 14, 16]). The notation $S(r,f)$ denotes any quantity that satisfies the condition $S(r,f) = o\{T(r,f)\}$ as $r \to \infty$, possibly outside of a set of finite linear measure. A meromorphic function $\alpha(z)(\not\equiv 0,\infty)$ defined in $\mathbb{C}$ is called a small function with respect to f if $T(r,\alpha(z)) = S(r,f)$.
Let $a \in \mathbb{C} \cup \{\infty\}$. Set $E(a,f) = \{z : f(z) - a = 0\}$, where a zero with multiplicity k is counted k times. If the zeros are counted only once, then we denote the set by $\overline{E}(a,f)$. Let f and g be two nonconstant meromorphic functions. If $E(a,f) = E(a,g)$, then we say that f and g share the value a CM (Counting Multiplicities). If $\overline{E}(a,f) = \overline{E}(a,g)$, then we say that f and g share the value a IM (Ignoring Multiplicities). We denote by $E_{k)}(a,f)$ the set of all $a-$points of f with multiplicities not exceeding k, where an $a-$point is counted according to its multiplicity. Also we denote by $\overline{E}_{k)}(a,f)$ the set of all distinct $a-$points of f with multiplicities not greater than k. In addition, we need the following definitions.

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Definition 1.1. *A non-linear differential polynomial P[f] of a nonconstant meromorphic function f is defined as*

$$P[f] = \sum_{i=1}^{t} M_i[f]$$

where $M_i[f] = a_i \prod_{j=0}^{l} \left(f^{(j)}\right)^{n_{ij}}$ *with* $n_{i0}, n_{i1}, \ . \ . \ . \ , n_{il}$ *as nonnegative integers and* $a_i (\not\equiv 0)$ *are meromorphic functions satisfying* $T(r, a_i) = o(T(r, f))$ *as* $r \to \infty$.
The numbers $\overline{d}(P) = max_{1 \le i \le t} \sum_{j=0}^{l} n_{ij}$ *and* $\underline{d}(P) = min_{1 \le i \le t} \sum_{j=0}^{l} n_{ij}$ *are respectively called the degree and lower degree of* $P[f]$. *If* $\overline{d}(P) = \underline{d}(P) = d'$ *(say), then we say that* $P[f]$ *is a homogeneous differential polynomial of degree* d'. *Also we define* $Q = \max_{1 \le i \le t} \{n_{i0} + n_{i1} + 2n_{i2} + ... + ln_{il}\}$.

Definition 1.2. [5] *Let* $a \in \mathbb{C} \cup \{\infty\}$. *We denote by* $N(r, a; f)$ *the counting function of simple a-points of* f. *For a positive integer* k *we denote by* $N_{k)}(r, a; f)$ *the counting function of those a-points of* f *(counted with proper multiplicities) whose multiplicities are not greater than* k. *By* $\overline{N}_{k)}(r, a; f)$ *we denote the corresponding reduced counting function. Analogously we can define* $N_{(k}(r, a; f)$ *and* $\overline{N}_{(k}(r, a; f)$.

Definition 1.3. [6, 7] *For a complex number* $a \in \mathbb{C} \cup \{\infty\}$, *we denote by* $E_k(a, f)$ *the set of all a-points of* f *where an a-points with multiplicity* m *is counted* m *times if* $m \le k$ *and* $k + 1$ *times if* $m > k$. *If* $E_k(a, f) = E_k(a, g)$, *then we say that* f *and* g *share the value* a *with weight* k.
The definition implies that if f, g *share a value* a *with weight* k, *then* z_0 *is a zero of* $f - a$ *with multiplicity* $m(\le k)$ *if and only if it is a zero of* $g - a$ *with multiplicity* $m(\le k)$ *and* z_0 *is a zero of* $f - a$ *with multiplicity* $m(> k)$ *if and only if it is a zero of* $g - a$ *with multiplicity* $n(> k)$, *where* m *is not necessarily equal to* n. *We write* f, g *share* (a, k) *to mean that* f, g *share the value* a *with weight* k. *Clearly if* f, g *share* (a, k) *then* f, g *share* (a, p) *for all integers* $p, 0 \le p < k$. *Also we note that* f, g *share a value* a *IM or CM if and only if* f, g *share* $(a, 0)$ *or* (a, ∞) *respectively.*

Definition 1.4. [6] *Let* S *be a set of distinct elements of* $\mathbb{C} \cup \{\infty\}$ *and* k *a nonnegtive integer or* ∞. *We denote by* $E_f(S, K)$ *the set* $\cup_{a \in S} E_k(a, f)$. *Clearly* $E_f(S) = E_f(S, \infty)$ *and* $\overline{E}_f(S) = E_f(S, 0)$.

W. K. Hayman proposed the following well-known conjecture.
Hayman's Conjecture [3] If an entire function satisfies $f^n f' \neq 1$ for all positive integers $n \in N$, then f is a constant.
In 1997, corresponding to the famous conjecture of Hayman, Yang and Hua [15] studied the unicity of differential monomials and obtained the following theorem.

Theorem A. *Let* $f(z)$ *and* $g(z)$ *be two nonconstant entire functions,* $n \ge 6$ *a positive integer. If* $f^n f'$ *and* $g^n g'$ *share 1 CM, then either* $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$ *where* c_1, c_2, c *are three constants satisfying* $(c_1 c_2)^{n+1} c^2 = -1$ *or* $f = tg$ *for a constant* t *such that* $t^{n+1} = 1$.

In 2018, V. H. An and H. H. Khoai [1] considered the set of roots of unity of degree d and studied the relations of f and g when $E_{(f^n)^{(k)}}(S) = E_{(g^n)^{(k)}}(S)$ and they proved the following result.

Theorem B. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions and let n, d, k be positive integers with $n > 2k + \frac{2k+8}{d}$, $d \geq 2$ and $S = \{a \in \mathbb{C} : a^d = 1\}$. If $E_{(f^n)^{(k)}}(S) = E_{(g^n)^{(k)}}(S)$, then one of the following two cases holds: (i) $f(z) = c_1e^{cz}$ and $g(z) = c_2e^{-cz}$ for three nonzero constants c_1, c_2 and c such that $(-1)^{kd}(c_1c_2)^{nd}(nc)^{2kd} = 1$; (ii) $f = tg$ with $t^{nd} = 1$, $t \in \mathbb{C}$.*

In 2019, C. Meng and X. Li [11] proved the following results.

Theorem C. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions and let n, d, k be positive integers with $n > 2k + \frac{3k+9}{d}$, $d \geq 2$ and $S = \{a \in \mathbb{C} : a^d = 1\}$. If $E_{(f^n)^{(k)}}(S, 1) = E_{(g^n)^{(k)}}(S, 1)$, then one of the following two cases holds: (i) $f(z) = c_1e^{cz}$ and $g(z) = c_2e^{-cz}$ for three nonzero constants c_1, c_2 and c such that $(-1)^{kd}(c_1c_2)^{nd}(nc)^{2kd} = 1$; (ii) $f = tg$ with $t^{nd} = 1$, $t \in \mathbb{C}$.*

Theorem D. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions and let n, d, k be positive integers with $n > 2k + \frac{8k+14}{d}$, $d \geq 2$ and $S = \{a \in \mathbb{C} : a^d = 1\}$. If $E_{(f^n)^{(k)}}(S, 0) = E_{(g^n)^{(k)}}(S, 0)$, then one of the following two cases holds: (i) $f(z) = c_1e^{cz}$ and $g(z) = c_2e^{-cz}$ for three nonzero constants c_1, c_2 and c such that $(-1)^{kd}(c_1c_2)^{nd}(nc)^{2kd} = 1$; (ii) $f = tg$ with $t^{nd} = 1$, $t \in \mathbb{C}$.*

Question 1.1. *What can be said about the relationship between two meromorphic functions $f(z)$ and $g(z)$, if $(f^n(z)(f-1)^s)^{(k)}$ is the differential polynomials, where $n(\geq 1)$, $s(\geq 1)$ are integers?*

Considering Question 1.1, V. Husna [4] proved the following theorems.

Theorem E. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions and let n, d, k, s be positive integers with $n > 2k-s+\frac{3k+9}{d}$, $d \geq 2$ and $S = \{a \in \mathbb{C} : a^d = 1\}$. If $E_{(f^n(f-1)^s)^{(k)}}(S, 1) = E_{(g^n(g-1)^s)^{(k)}}(S, 1)$, then one of the following two cases holds: (i) $f(z) = c_1e^{cz}$ and $g(z) = c_2e^{-cz}$ for three nonzero constants c_1, c_2 and c such that $(-1)^{kd}(c_1c_2)^{(n+s)d}((n+s)c)^{2kd} = 1$; (ii) $f = tg$ with $t^{(n+s)d} = 1$, $t \in \mathbb{C}$.*

Theorem F. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions and let n, d, k, s be positive integers with $n > 2k - s + \frac{8k+14}{d}$, $d \geq 2$ and $S = \{a \in \mathbb{C} : a^d = 1\}$. If $E_{(f^n(f-1)^s)^{(k)}}(S, 0) = E_{(g^n(g-1)^s)^{(k)}}(S, 0)$, then one of the following two cases holds: (i) $f(z) = c_1e^{cz}$ and $g(z) = c_2e^{-cz}$ for three nonzero constants c_1, c_2 and c such that $(-1)^{kd}(c_1c_2)^{(n+s)d}((n+s)c)^{2kd} = 1$; (ii) $f = tg$ with $t^{(n+s)d} = 1$, $t \in \mathbb{C}$.*

According to the results of V. Husna, it is natural to ask the following question which is the motive of the present paper.

Question 1.2. *What will happen if we replace $(f^n(z)(f-1)^s)^{(k)}$ by the nonlinear differential polynomial $(f^n(z)(f-1)^sP[f])^{(k)}$, where $P[f]$ is defined in Definition 1.1 ?*

In this paper, our main concern is to find the possible answer of Question 1.2. We prove the following theorems which generalize and improve Theorems E and F. The following theorems are the main results of the paper.

Theorem 1.1. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions whose zeros and poles are of multiplicities atleast m, where m is a positive integer and let n, d, k, s, l, Q, $\overline{d}(P)$ be positive integers with $n > 2k - s - \overline{d}(P) + \frac{6k+5Q+18}{2md}$, $d \geq 2$ and $S = \{a \in \mathbb{C} : a^d = 1\}$. If $E_{(f^n(f-1)^sP[f])^{(k)}}(S, 1) = E_{(g^n(g-1)^sP[g])^{(k)}}(S, 1)$, then* *either* $f = tg$ *for a constant* t *such that* $t^{\sigma d} = 1$, *where* $\sigma = gcd(n+s+\overline{d}(P), ..., n+s-i+\overline{d}(P), ..., n+\overline{d}(P))$ *or* f *and* g *satisfy the algebric relation equation* $R(f,g) = 0$, *where* $R(f,g) = (f^n(f-1)^s P[f])^{(k)} - (g^n(g-1)^s P[g])^{(k)}$.

Theorem 1.2. *Let* $f(z)$ *and* $g(z)$ *be two nonconstant meromorphic functions whose zeros and poles are of multiplicities atleast* m, *where* m *is a positive integer and let* n, d, k, s, l, Q, $\overline{d}(P)$ *be positive integers with* $n > 2k - s - \overline{d}(P) + \frac{8k+5Q+14}{md}$, $d \geq 2$ *and* $S = \{a \in \mathbb{C} : a^d = 1\}$. *If* $E_{(f^n(f-1)^s P[f])^{(k)}}(S, 0) = E_{(g^n(g-1)^s P[g])^{(k)}}(S, 0)$, *then the conclusion of Theorem 1.1 holds.*

Remark 1. *Since Theorems E-F are the special cases of Theorems 1.1 and 1.2 respectively for* $Q = 0$ *and* $\overline{d}(P) = 0$ *then Theorems 1.1 and 1.2 improve and extend Theorems E-F respectively.*

2. **Lemmas**

In this section, we state some Lemmas which will be needed in the sequel. We denote by H the following function:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right),$$

where F and G are nonconstant meromorphic functions defined in the complex plane $\mathbb{C}$.

Lemma 2.1. [13] *Let* $f(z)$ *be a non-constant meromorphic function and* $P[f]$ *be a differential polynomial of* $f(z)$. *Then*

$$T(r, P[f]) \quad \leq \quad \overline{d}(P)T(r,f) + Q\overline{N}(r,\infty;f) + S(r,f)$$

and

$$N(r,\infty;P[f]) \quad \leq \quad \overline{d}(P)N(r,\infty;f) + Q\overline{N}(r,\infty;f) + S(r,f)$$

where $Q = \max_{1\leq i\leq t}\{n_{i0} + n_{i1} + 2n_{i2} + ... + ln_{il}\}$.

Lemma 2.2. [2] *Let* H *be defined as above. If* F *and* G *share* $(1,1)$ *and* $H \not\equiv 0$, *then*

$$\begin{aligned} T(r,F) \quad \leq \quad & N_2(r,0;F) + N_2(r,\infty;F) + N_2(r,0;G) + N_2(r,\infty;G) + \frac{1}{2}\overline{N}(r,0;F) \\ & + \frac{1}{2}\overline{N}(r,\infty;F) + S(r,F) + S(r,G) \end{aligned}$$

and the same inequality holds for $T(r,G)$.

Lemma 2.3. [8] *Let* $f(z)$ *be a nonconstant meromorphic function,* k *be a positive integer, then*

$$N_p(r,0;f^{(k)}) \leq N_{p+k}(r,0;f) + k\overline{N}(r,\infty;f) + S(r,f).$$

Lemma 2.4. *Let* $f(z)$ *be a non-constant meromorphic function and* n, s, $\overline{d}(P)$, Q, k *be positive integers with* $n > k$. *Then*

$$N\left(r,\infty;\frac{(f^n(f-1)^s P[f])^{(k)}}{f^{n+s+\overline{d}(P)-k}}\right) \leq kN(r,\infty;f) + Q\overline{N}(r,\infty;f) + k\overline{N}(r,\infty;f) + S(r,f).$$

Proof. Let z_0 be a pole of $f(z)$ with multiplicity p, then in some neighbourhood of z_0

$$\frac{(f^n(f-1)^sP[f])^{(k)}}{f^{n+s+\overline{d}(P)-k}} = \frac{h_k(z)}{(z-z_0)^{(p+1)k+Q_1}}, \tag{2.1}$$

where $h_k(z)$ is regular at z_0 and $h_k(z_0) \neq 0$ $Q_1 = \max\limits_{1\leq i\leq t}\{n_{i1} + 2n_{i2} + ... + ln_{il}\}$.
Now from (2.1), we have

$$\begin{aligned} N\left(r,\infty;\frac{(f^n(f-1)^sP[f])^{(k)}}{f^{n+s+\overline{d}(P)-k}}\right) &\leq kN(r,\infty;f) + Q_1\overline{N}(r,\infty;f) + k\overline{N}(r,\infty;f) \\ &\quad +S(r,f). \\ &\leq kN(r,\infty;f) + Q\overline{N}(r,\infty;f) + k\overline{N}(r,\infty;f) \\ &\quad +S(r,f). \end{aligned}$$

□

This completes the proof of Lemma.

Lemma 2.5. *Let $g(z)$ be a non-constant meromorphic function and n, s, $\overline{d}(P)$, Q, k be positive integers with $n > k$. Then*

$$\overline{N}\left(r,0;\frac{(g^n(g-1)^sP[g])^{(k)}}{g^{n+\overline{d}(P)-k}}\right) \leq k\overline{N}(r,0;g) + Q\overline{N}(r,\infty;g) + k\overline{N}(r,\infty;g) + S(r,g).$$

Proof. Let z_0 be a zero of $g(z)$ with multiplicity p, then in some neighbourhood of z_0

$$\frac{(g^n(g-1)^sP[g])^{(k)}}{g^{n+\overline{d}(P)-k}} = \frac{h_k(z)(z-z_0)^{pk}}{(z-z_0)^{Q_1+k}}, \tag{2.2}$$

where $h_k(z)$ is regular at z_0 and $h_k(z_0) \neq 0$, $Q_1 = \max\limits_{1\leq i\leq t}\{n_{i1} + 2n_{i2} + ... + ln_{il}\}$.

Now from (2.2), we have

$$\begin{aligned} \overline{N}\left(r,0;\frac{(g^n(g-1)^sP[g])^{(k)}}{g^{n+\overline{d}(P)-k}}\right) &\leq k\overline{N}(r,0;g) + Q_1\overline{N}(r,\infty;g) + k\overline{N}(r,\infty;g) \\ &\quad +S(r,g). \\ &\leq k\overline{N}(r,0;g) + Q\overline{N}(r,\infty;g) + k\overline{N}(r,\infty;g) \\ &\quad +S(r,g). \end{aligned}$$

□

This completes the proof of Lemma.

Lemma 2.6. [2] *Let H be defined as above. If F and G share $(1,0)$ and $H \not\equiv 0$, then*

$$\begin{aligned} T(r,F) &\leq N_2(r,0;F) + N_2(r,\infty;F) + N_2(r,0;G) + N_2(r,\infty;G) + 2\overline{N}(r,0;F) \\ &\quad +2\overline{N}(r,\infty;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + S(r,F) + S(r,G) \end{aligned}$$

and the same inequality holds for $T(r,G)$.

Lemma 2.7. *Let* $f(z)$ *be a nonconstant meromorphic function and* n, k, s, Q, $\overline{d}(P)$ *be positive integers,* $n>2k$. *Then*

$$(n+s+\overline{d}(P)-2k)T(r,f)+kN(r,\infty;f)+N\left(r,\infty;\frac{f^{n+s+\overline{d}(P)-k}}{(f^n(f-1)^sP[f])^{(k)}}\right) \leq T(r,(f^n(f-1)^sP[f])^{(k)})+S(r,f).$$

Proof. Let $F_2=(F_1)^{(k)}$, where $F_1=f^n(f-1)^sP[f]$.
By Lemma 2.1, we have

$$\begin{aligned} N(r,\infty;F_2) &= N(r,\infty;F_1)+k\overline{N}(r,\infty;F_1) \\ &\leq (n+s+\overline{d}(P))N(r,\infty;f)+(Q+k)\overline{N}(r,\infty;f). \end{aligned} \tag{2.3}$$

From this and noting that $S(r,f)=S(r,F_1)$ and $m\left(r,\frac{(f)^{(k)}}{f}\right)=S(r,f)$, we obtain

$$\begin{aligned} (n+s+\overline{d}(P)-k)m(r,\infty;f) &= m(r,\infty;f^{n+s+\overline{d}(P)-k}) \\ &\leq m(r,\infty;F_2)+m\left(r,\infty;\frac{f^{n+s+\overline{d}(P)-k}}{F_2}\right)+S(r,f) \\ &= m(r,\infty;F_2)+T\left(r,\frac{F_2}{f^{n+s+\overline{d}(P)-k}}\right) \\ &\quad -N\left(r,\infty;\frac{f^{n+s+\overline{d}(P)-k}}{F_2}\right)+S(r,f) \\ &\leq m(r,\infty;F_2)+m\left(r,\infty;\frac{F_2}{f^{n+s+\overline{d}(P)-k}}\right) \\ &\quad +N\left(r,\infty;\frac{F_2}{f^{n+s+\overline{d}(P)-k}}\right)-N\left(r,\infty;\frac{f^{n+s+\overline{d}(P)-k}}{F_2}\right) \\ &\quad +S(r,f) \\ &\leq m(r,\infty;F_2)+km(r,\infty;f) \\ &\quad +N\left(r,\infty;\frac{F_2}{f^{n+s+\overline{d}(P)-k}}\right)-N\left(r,\infty;\frac{f^{n+s+\overline{d}(P)-k}}{F_2}\right) \\ &\quad +S(r,f) \end{aligned}$$

By Lemma 2.4, we get

$$\begin{aligned} (n+s+\overline{d}(P)-k)m(r,\infty;f) &\leq m(r,\infty;F_2)+km(r,\infty;f)+kN(r,\infty;f)+Q\overline{N}(r,\infty;f) \\ &\quad +k\overline{N}(r,\infty;f)-N\left(r,\infty;\frac{f^{n+s+\overline{d}(P)-k}}{F_2}\right)+S(r,f) \\ &\leq m(r,\infty;F_2)+kT(r,f)+(k+Q)\overline{N}(r,\infty;f)) \\ &\quad -N\left(r,\infty;\frac{f^{n+s+\overline{d}(P)-k}}{F_2}\right)+S(r,f). \end{aligned} \tag{2.4}$$

From (2.3) and (2.4) it implies that

$$\begin{aligned}(n+s+\overline{d}(P)-k)T(r,f)+kN(r,\infty;f) &= (n+s+\overline{d}(P)-k)m(r,\infty;f)\\ &\quad +(n+s+\overline{d}(P))N(r,\infty;f)\\ &\leq m(r,\infty;F_2)+kT(r,f)+(k+Q)\overline{N}(r,\infty;f))\\ &\quad -N\left(r,\infty;\frac{f^{n+s+\overline{d}(P)-k}}{F_2}\right)+N(r,\infty;F_2)\\ &\quad -(Q+k)\overline{N}(r,\infty;f)+S(r,f). \qquad (2.5)\end{aligned}$$

Thus from (2.5), we have

$$\begin{aligned}(n+s+\overline{d}(P)-2k)T(r,f)+kN(r,\infty;f)+N\left(r,\infty;\frac{f^{n+s+\overline{d}(P)-k}}{F_2}\right)\\ \leq T(r,F_2)+S(r,f).\end{aligned}$$

□

Lemma 2.8. [12] *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions, whose zeros and poles are of multiplicities at least m, where m is a positive integer. Let $P[f]$ be defined as in Definition 1.1, and let n, s, k, Q and $\overline{d}(P)$ are positive integers. Then*

$$(f^n(f-1)^sP[f])^{(k)}(g^n(g-1)^sP[g])^{(k)}\not\equiv 1.$$

Lemma 2.9. *Let $f(z)$ and $g(z)$ be two non constant meromorphic functions, whose zeros and poles are of multiplicities atleast m, where m is a positive integer. Let $P[f]$ be defined as in Definition 1.1. Let n, s, k, l, $\overline{d}(P)$ and Q be positive integers. Let $F=(F_2)^d$ and $G=(G_2)^d$, where $d\geq 2$, $F_2=(f^n(f-1)^sP[f])^{(k)}$ and $G_2=(g^n(g-1)^sP[g])^{(k)}$. If there exists two non zero constants b_1 and b_2 such that $\overline{N}(r,0;F)=\overline{N}(r,b_1;G)$ and $\overline{N}(r,0;G)=\overline{N}(r,b_2;F)$, then $n<2k-s-\overline{d}(P)+\frac{k+Q+3}{md}$.*

Proof. By Nevanlinna second fundamental theorem we get

$$\begin{aligned}T(r,F) &\leq \overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}\left(r,b_2;F\right)+S(r,F)\\ &\leq \overline{N}(r,\infty;F)+N_2(r,0;F)+\overline{N}(r,0;G)+S(r,f). \qquad (2.6)\end{aligned}$$

By Lemma 2.3 and Lemma 2.5, we have

$$\begin{aligned}\overline{N}(r,0;G) &= \overline{N}\left(r,0;((g^n(g-1)^sP[g])^{(k)})^d\right)\\ &= \overline{N}(r,0;(g^n(g-1)^sP[g])^{(k)})\\ &\leq \overline{N}\left(r,0;\frac{(g^n(g-1)^sP[g])^{(k)}}{g^{n+\overline{d}(P)-k}}\right)+\overline{N}(r,0;g^{n+\overline{d}(P)-k})+S(r,g)\\ &\leq k\overline{N}(r,0;g)+Q\overline{N}(r,\infty;g)+k\overline{N}(r,\infty;g)+\overline{N}(r,0;g)+S(r,g)\\ &\leq (k+1)\overline{N}(r,0;g)+(Q+k)\overline{N}(r,\infty;g)+S(r,g) \qquad (2.7)\end{aligned}$$

and similarly the same argument can be obtained for $\overline{N}(r,0;F)$.

$$\begin{aligned} N_2(r,0;F) &= 2\overline{N}(r,0;(f^n(f-1)^sP[f])^{(k)}) \\ &\leq 2\left(\overline{N}(r,0;f^{n+s+\overline{d}(P)-k}) + N\left(r,\infty;\frac{f^{n+s+\overline{d}(P)-k}}{(f^n(f-1)^sP[f])^{(k)}}\right)\right) + S(r,f) \\ &\leq 2\left(\overline{N}(r,0;f) + N\left(r,\infty;\frac{f^{n+s+\overline{d}(P)-k}}{(f^n(f-1)^sP[f])^{(k)}}\right)\right) + S(r,f) \qquad (2.8) \end{aligned}$$

and similarly the same argument can be obtained for $N_2(r,0;G)$.
Also

$$\overline{N}(r,\infty;F) = \overline{N}(r,\infty;f). \qquad (2.9)$$

Using (2.7), (2.8) and (2.9) in (2.6), we obtain

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,\infty;f) + 2\left(\overline{N}(r,0;f) + N\left(r,\infty;\frac{f^{n+s+\overline{d}(P)-k}}{(f^n(f-1)^sP[f])^{(k)}}\right)\right) \\ &\quad +(k+1)\overline{N}(r,0;g) + (Q+k)\overline{N}(r,\infty;g) + S(r,f) \qquad (2.10) \end{aligned}$$

and similarly

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,\infty;g) + 2\left(\overline{N}(r,0;g) + N\left(r,\infty;\frac{g^{n+s+\overline{d}(P)-k}}{(g^n(g-1)^sP[g])^{(k)}}\right)\right) \\ &\quad +(k+1)\overline{N}(r,0;f) + (Q+k)\overline{N}(r,\infty;f) + S(r,g). \qquad (2.11) \end{aligned}$$

Again since $d \geq 2$,

$$kdN(r,\infty;g) \geq (k+1)\overline{N}(r,\infty;g), \qquad (2.12)$$

$$kdN(r,\infty;f) \geq (k+1)\overline{N}(r,\infty;f). \qquad (2.13)$$

By Lemma 2.7, we have

$$\begin{aligned} (n+s+\overline{d}(P)-2k)dT(r,f) + kdN(r,\infty;f) + dN\left(r,\infty;\frac{f^{n+s+\overline{d}(P)-k}}{(f^n(f-1)^sP[f])^{(k)}}\right) \\ \leq T(r,F) + S(r,f) \qquad (2.14) \end{aligned}$$

and similarly the same argument can be obtained for $T(r,G)$.
Hence from (2.10)-(2.14), we obtain

$$\left(nd + sd + \overline{d}(P)d - 2kd - \frac{k+Q+3}{m}\right)(T(r,f) + T(r,g)) \leq (S(r,f) + S(r,g)),$$

which gives $n < 2k - s - \overline{d}(P) + \frac{k+Q+3}{md}$.
This proves the lemma. □

3. Proof of Theorems

Proof of Theorem 1.1. Let $F = (F_2)^d$, $G = (G_2)^d$ and $F_2 = (F_1)^{(k)}$, $G_2 = (G_1)^{(k)}$, where $F_1 = f^n(f-1)^s P[f]$ and $G_1 = g^n(g-1)^s P[g]$.
Since $E_{F_2}(S,1) = E_{G_2}(S,1)$, we see that F and G share $(1,1)$. If $H \not\equiv 0$, then by Lemma 2.2

$$\begin{aligned} T(r,F) \leq\ & N_2(r,0;F) + N_2(r,\infty;F) + N_2(r,0;G) + N_2(r,\infty;G) + \frac{1}{2}\overline{N}(r,0;F) \\ & + \frac{1}{2}\overline{N}(r,\infty;F) + S(r,F) + S(r,G). \end{aligned} \tag{3.1}$$

By Lemma 2.7, we obtain

$$\begin{aligned} (n+s+\overline{d}(P)-2k)T(r,f) &\leq T(r,F_2) + S(r,f) \\ &\leq (k+1)(n+s+\overline{d}(P)+Q)T(r,f) \\ &\quad + S(r,f) \end{aligned} \tag{3.2}$$

and similarly the same argument can be obtained for T(r,g).
Since $T(r,F) = dT(r,F_2) + S(r,F_2)$ and $T(r,G) = dT(r,G_2) + S(r,G_2)$, it is easy to see that $S(r,F) = S(r,F_2) = S(r,f)$ and $S(r,G) = S(r,G_2) = S(r,g)$.
On the other hand

$$N_2(r,\infty;F) = 2\overline{N}(r,\infty;f), \tag{3.3}$$

$$N_2(r,\infty;G) = 2\overline{N}(r,\infty;g), \tag{3.4}$$

$$\frac{1}{2}\overline{N}(r,\infty;F) = \frac{1}{2}\overline{N}(r,\infty;f). \tag{3.5}$$

By Lemma 2.3 and Lemma 2.5, we have

$$\begin{aligned} N_2(r,0;F) &= N_2\left(r,0;((f^n(f-1)^sP[f])^{(k)})^d\right) \\ &= 2\overline{N}(r,0;(f^n(f-1)^sP[f])^{(k)}) \\ &\leq 2\overline{N}\left(r,0;\frac{(f^n(f-1)^sP[f])^{(k)}}{f^{n+\overline{d}(P)-k}}\right) + 2\overline{N}(r,0;f^{n+\overline{d}(P)-k}) + S(r,f) \\ &\leq 2k\overline{N}(r,0;f) + 2Q\overline{N}(r,\infty;f) + 2k\overline{N}(r,\infty;f) + 2\overline{N}(r,0;f) + S(r,f) \\ &\leq 2(k+1)\overline{N}(r,0;f) + 2(Q+k)\overline{N}(r,\infty;f) + S(r,f) \end{aligned} \tag{3.6}$$

and similarly the same argument can be obtained for $N_2(r,0;G)$.
Similarly, we have

$$\frac{1}{2}\overline{N}(r,0;F) \leq \frac{1}{2}(k+1)\overline{N}(r,0;f) + \frac{1}{2}(Q+k)\overline{N}(r,\infty;f) + S(r,f). \tag{3.7}$$

Again

$$\begin{aligned} N_2(r,0;G) &= 2\overline{N}(r,0;(g^n(g-1)^sP[g])^{(k)}) \\ &\leq 2\left(\overline{N}(r,0;g^{n+s+\overline{d}(P)-k}) + N\left(r,\infty;\frac{g^{n+s+\overline{d}(P)-k}}{(g^n(g-1)^sP[g])^{(k)}}\right)\right) + S(r,g) \\ &\leq 2\left(\overline{N}(r,0;g) + N\left(r,\infty;\frac{g^{n+s+\overline{d}(P)-k}}{(g^n(g-1)^sP[g])^{(k)}}\right)\right) + S(r,g) \end{aligned} \tag{3.8}$$

and similarly the same argument can be obtained for $N_2(r,0;F)$.
Combining (3.1) and (3.3)-(3.8), we deduce

$$\begin{aligned} T(r,F) \quad \leq \quad & \frac{6k+5Q+10}{2m}T(r,f)+\frac{4}{m}T(r,g)+2k\overline{N}(r,\infty;f) \\ & +2N\left(r,\infty;\frac{g^{n+s+\overline{d}(P)-k}}{(g^n(g-1)^sP[g])^{(k)}}\right)+S(r,f)+S(r,g) \end{aligned} \tag{3.9}$$

and similarly the same argument can be obtained for $T(r,G)$.
By Lemma 2.7, we have

$$\begin{aligned} (n+s+\overline{d}(P)-2k)dT(r,f)+kdN(r,\infty;f)+dN\left(r,\infty;\frac{f^{n+s+\overline{d}(P)-k}}{(f^n(f-1)^sP[f])^{(k)}}\right) \\ \leq T(r,F)+S(r,f) \end{aligned} \tag{3.10}$$

and similarly the same argument can be obtained for $T(r,G)$.
Again since $d \geq 2$

$$dN\left(r,\infty;\frac{f^{n+s+\overline{d}(P)-k}}{(f^n(f-1)^sP[f])^{(k)}}\right) \geq 2N\left(r,\infty;\frac{f^{n+s+\overline{d}(P)-k}}{(f^n(f-1)^sP[f])^{(k)}}\right), \tag{3.11}$$

$$kdN(r,\infty;f) \geq 2k\overline{N}(r,\infty;f) \tag{3.12}$$

and similarly the same argument can be obtained for g.
From (3.9) - (3.12), we have

$$\left(nd+sd+\overline{d}(P)d-2kd-\frac{6k+5Q+18}{2m}\right)(T(r,f)+T(r,g)) \leq (S(r,f)+S(r,g)),$$

which contradicts $n > 2k-s-\overline{d}(P)+\frac{6k+5Q+18}{2md}$.

Therefore we must have $H \equiv 0$. Then

$$\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right)=0. \tag{3.13}$$

Integrating both side twice, we get from (3.13)

$$\frac{1}{G-1}=\frac{A}{F-1}+B, \tag{3.14}$$

where $A \neq 0$ and B are constants. Thus

$$G=\frac{(B+1)F+(A-B-1)}{BF+(A-B)} \tag{3.15}$$

and

$$F=\frac{(B-A)G+(A-B-1)}{BG-(B+1)}. \tag{3.16}$$

Next we consider the following three cases:
Case 1. $B \neq 0,-1$. Then from (3.16), we have

$$\overline{N}\left(r,\frac{B+1}{B};G\right)=\overline{N}(r,\infty;F). \tag{3.17}$$

By Nevanlinna second fundamental theorem and (3.8), we get

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}\left(r,\frac{B+1}{B};G\right)+S(r,G) \\ &\leq \overline{N}(r,\infty;G)+N_2(r,0;G)+\overline{N}(r,\infty;F)+S(r,G) \\ &\leq \overline{N}(r,\infty;g)+2\left(\overline{N}(r,0;g)+N\left(r,\infty;\frac{g^{n+s+\overline{d}(P)-k}}{(g^n(g-1)^sP[g])^{(k)}}\right)\right) \\ &\quad +\overline{N}(r,\infty;f)+S(r,g). \end{aligned} \tag{3.18}$$

If $A-B-1\neq 0$, then it follows from (3.15) that

$$\overline{N}\left(r,\frac{B+1-A}{B+1};F\right)=\overline{N}(r,0;G). \tag{3.19}$$

Again by Nevanlinna second fundamental theorem and (3.7), (3.8), we get

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}\left(r,\frac{B+1-A}{B+1};F\right)+S(r,F) \\ &\leq \overline{N}(r,\infty;F)+N_2(r,0;F)+\overline{N}(r,0;G)+S(r,f) \\ &\leq \overline{N}(r,\infty;f)+2\left(\overline{N}(r,0;f)+N\left(r,\infty;\frac{f^{n+s+\overline{d}(P)-k}}{(f^n(f-1)^sP[f])^{(k)}}\right)\right) \\ &\quad +(k+1)\overline{N}(r,0;g)+(Q+k)\overline{N}(r,\infty;g)+S(r,f). \end{aligned} \tag{3.20}$$

Again since $d\geq 2$,

$$kdN(r,\infty;g)\geq (k+1)\overline{N}(r,\infty;g), \tag{3.21}$$

$$kdN(r,\infty;f)\geq 2\overline{N}(r,\infty;f). \tag{3.22}$$

Hence combining (3.10), (3.11), (3.12), (3.18), (3.20), (3.21) and (3.22), we obtain

$$\left(nd+sd+\overline{d}(P)d-2kd-\frac{2}{m}\right)T(r,f)+\left(nd+sd+\overline{d}(P)d-2kd-\frac{k+3+Q}{m}\right)T(r,g) \leq S(r,f)+S(r,g).$$

which contradicts with $n>2k-s-\overline{d}(P)+\frac{6k+5Q+18}{2md}$. Hence $A-B-1=0$.
Then by (3.15), we have

$$\overline{N}\left(r,-\frac{1}{B};F\right)=N(r,\infty;G). \tag{3.23}$$

Again by Nevanlinna second fundamental theorem

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}\left(r,-\frac{1}{B};F\right)+S(r,F) \\ &\leq \overline{N}(r,\infty;F)+N_2(r,0;F)+\overline{N}(r,\infty;G)+S(r,f) \\ &\leq \overline{N}(r,\infty;f)+2\left(\overline{N}(r,0;f)+N\left(r,\infty;\frac{f^{n+s+\overline{d}(P)-k}}{(f^n(f-1)^sP[f])^{(k)}}\right)\right) \\ &\quad +\overline{N}(r,\infty;g)+S(r,f). \end{aligned} \tag{3.24}$$

Again since $d\geq 2$,

$$kdN(r,\infty;f)\geq 2\overline{N}(r,\infty;f), \tag{3.25}$$

$$kdN(r,\infty;g) \geq 2\overline{N}(r,\infty;g). \tag{3.26}$$

Now combining (3.11), (3.18), (3.24), (3.25) and (3.26), we obtain

$$\left(nd + sd + \overline{d}(P)d - 2kd - \frac{2}{m}\right)[T(r,f) + T(r,g)] \leq S(r,f) + S(r,g).$$

which contradicts with $n > 2k - s - \overline{d}(P) + \frac{6k+5Q+18}{2md}$.

Case 2. $B = -1$. Then $G = \frac{A}{A+1-F}$ and $F = \frac{(1+A)G-A}{G}$
If $A+1 \neq 0$. We obtain $\overline{N}(r, A+1; F) = \overline{N}(r,\infty;G)$ and $\overline{N}(r, \frac{A}{A+1}; G) = \overline{N}(r,0;F)$. By similar arguments as in case 1 we arrive at a contradiction. Therefore $A+1 = 0$, then $FG \equiv 1$. That is $((f^n(f-1)^sP[f])^{(k)})^d((g^n(g-1)^sP[g])^{(k)})^d \equiv 1$. Thus we have $(f^n(f-1)^sP[f])^{(k)}(g^n(g-1)^sP[g])^{(k)} = h$, where $h^d = 1$. Then by Lemma 2.8 we arrive at a contradiction.

Case 3. $B = 0$. Then (3.15) and (3.16) gives $G = \frac{F+A-1}{A}$ and $F = AG + 1 - A$. If $A - 1 \neq 0$, then $\overline{N}(r, 1-A; F) = \overline{N}(r,0;G)$ and $\overline{N}(r, \frac{A-1}{A}; G) = \overline{N}(r,0;F)$. By Lemma 2.9 we arrive at a contradiction. Hence $A - 1 = 0$, therefore $F \equiv G$, that is $((f^n(f-1)^sP[f])^{(k)})^d \equiv ((g^n(g-1)^sP[g])^{(k)})^d$. We have

$$(f^n(f-1)^sP[f])^{(k)} = h(g^n(g-1)^sP[g])^{(k)} \tag{3.27}$$

with $h^d = 1$. By integration, we get

$$(f^n(f-1)^sP[f])^{(k-1)} = h(g^n(g-1)^sP[g])^{(k-1)} + c_{k-1},$$

where c_{k-1} is a constant. If $c_{k-1} \neq 0$, using Lemma 2.9, we arrive at a contradiction. Hence $c_{k-1} = 0$.
Repeating the process $(k-1)$-times, we deduce that

$$f^n(f-1)^sP[f] = hg^n(g-1)^sP[g]. \tag{3.28}$$

Let $t = \frac{f}{g}$. If t is a constant, by putting $f = tg$ in (3.28), we get

$$\begin{aligned} g^{n+s}(t^{n+s+\overline{d}(P)} - h) + \ldots.. + (-1)^i s_{C_{s-i}} g^{n+s-i}(t^{n+s-i+\overline{d}(P)} - h) \\ + \ldots\ldots + (-1)^s g^n(h^{n+\overline{d}(P)} - h) = 0, \end{aligned}$$

which implies that $t^\sigma = h$, where $\sigma = gcd(n+s+\overline{d}(P), n+s-1+\overline{d}(P), \ldots., n+\overline{d}(P))$. Thus $f(z) = tg(z)$ for a constant t such that $t^{\sigma d} = 1$.

If t is not constant then from (3.28), we find that f and g satisfying the algebraic equation $R(f,g) = 0$, where $R(f,g) = f^n(f-1)^sP[f] - hg^n(g-1)^sP[g]$ with $h^d = 1$.
This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2.

Proof. Let $F = (F_2)^d$, $G = (G_2)^d$, and $F_2 = (F_1)^{(k)}$, $G_2 = (G_1)^{(k)}$ where $F_1 = f^n(f-1)^sP[f]$ and $G_1 = g^n(g-1)^sP[g]$.

Since $E_{F_2}(S,0)=E_{G_2}(S,0)$, we see that F and G share $(1,0)$. If $H\not\equiv 0$, then by Lemma 2.6

$$\begin{aligned} T(r,F) \quad \leq \quad & N_2(r,0;F)+N_2(r,\infty;F)+N_2(r,0;G)+N_2(r,\infty;G)+2\overline{N}(r,0;F) \\ & +2\overline{N}(r,\infty;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G)+S(r,F)+S(r,G). \quad (3.29) \end{aligned}$$

Now using (3.3)-(3.8) and (3.10)-(3.12) in (3.29), we obtain

$$\left(nd+sd+\overline{d}(P)d-2kd-\frac{8k+5Q+14}{m}\right)(T(r,f)+T(r,g))\leq (S(r,f)+S(r,g)),$$

which contradicts $n>2k-s-\overline{d}(P)+\frac{8k+5Q+14}{md}$.
Thus we must have $H\equiv 0$. Then the result follows from the proof of Theorem 1.1. This completes the proof of Theorem 1.2.

□

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