Journal of Fractional Calculus and Applications
Vol. 16(1) Jan 2025, No.8.
ISSN: 2090-5858.
ISSN 2090-584X (print)
http://jfca.journals.ekb.eg/

SUM AND PRODUCT RELATED THEOREMS OF ENTIRE FUNCTION IN TERMS OF (α, β, γ)-ORDER

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Abstract. Growth analysis of entire and meromorphic function is a very important part in complex analysis. Order is a classical growth indicator of entire and meromorphic functions. During the past several years, many renowned Mathematicians have made the close investigations on the properties of entire and meromorphic functions in different directions using the concepts of order and some extended definitions of order, like iterated p-order [7, 8], (p, q)-th order [5, 6], (p, q)-φ order [9], φ-order [3] etc. and they have achieved many valuable results. Heittokangas et al. [4] have introduced another the concept of φ-order of entire and meromorphic functions considering φ as subadditive function. Later, Belaïdi et al. [1] have extended the above ideas and have introduced the definitions of (α, β, γ)-order and (α, β, γ)-lower order of entire and meromorphic functions. In this paper, we investigate some basic properties in connection with sum and product of (α, β, γ)-order and (α, β, γ)-lower order of entire function with respect to another entire function where α, β, γ are continuous non-negative functions defined on $(-\infty, +\infty)$ with $\alpha \in L_1$, $\beta \in L_2$, $\gamma \in L_3$.

1. Introduction

Let $\mathbb{C}$ be the set of all finite complex numbers and $f = \sum_{n=0}^{+\infty} a_n z^n$ be an entire function defined on $\mathbb{C}$. The maximum modulus function $M(r, f)$ of f, is defined as $M(r, f) = \max_{|z|=r} |f(z)|$. Clearly, $M(r, f)$ is a real and increasing function of r. The ratio $\frac{M(r,f)}{M(r,g)}$ is also called the growth of the entire function f with respect to the entire function g in terms of maximum modulus. Order and lower order are classical growth indicators of entire functions in complex analysis. Several authors have made the close investigations on the growth properties of entire functions

2010 *Mathematics Subject Classification.* 30D35, 30D20.
Key words and phrases. Growth, Property (A), (α, β, γ)-order.
Submitted Sept. 02, 2024. Revised Jan. 08, 2025.

in different directions using the concepts of order, iterated p-order [7, 8], (p,q)-th order [5, 6], (p,q)-φ order [9] and achieved many valuable results. The standard notations and definitions of the theory of entire functions are available in [10, 11] and therefore we do not explain those in details. To start our paper, we just recall the following definitions:

Definition 1.1. *The order ρ_f and the lower order λ_f of an entire function f are defined as:*

$$\rho_f = \limsup_{r\to+\infty} \frac{\log\log M(r,f)}{\log r} \text{ and } \lambda_f = \liminf_{r\to+\infty} \frac{\log\log M(r,f)}{\log r}.$$

Definition 1.2. [2] *A non-constant entire function f is said to have the Property (A) if for any $\delta > 1$ and for all sufficiently large r, $[M(r,f)]^2 \leq M(r^\delta, f)$ holds.*

First of all, let L be a class of continuous non-negative functions α defined on $(-\infty,+\infty)$ such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x_0 \leq x \to +\infty$. We say that $\alpha \in L_1$, if $\alpha \in L$ and $\alpha(a+b) \leq \alpha(a) + \alpha(b) + c$ for all $a, b \geq R_0$ and fixed $c \in (0,+\infty)$. Further we say that $\alpha \in L_2$, if $\alpha \in L$ and $\alpha(x + O(1)) = (1 + o(1))\alpha(x)$ as $x \to +\infty$. Finally, $\alpha \in L_3$, if $\alpha \in L$ and $\alpha(a+b) \leq \alpha(a) + \alpha(b)$ for all $a, b \geq R_0$, i.e., α is subadditive. Clearly $L_3 \subset L_1$.

Particularly, when $\alpha \in L_3$, then one can easily verify that $\alpha(mr) \leq m\alpha(r)$, $m \geq 2$ is an integer. Up to a normalization, subadditivity is implied by concavity. Indeed, if $\alpha(r)$ is concave on $[0,+\infty)$ and satisfies $\alpha(0) \geq 0$, then for $t \in [0,1]$,

$$\begin{aligned} \alpha(tx) &= \alpha(tx + (1-t)\cdot 0) \\ &\geq t\alpha(x) + (1-t)\alpha(0) \geq t\alpha(x), \end{aligned}$$

so that by choosing $t = \frac{a}{a+b}$ or $t = \frac{b}{a+b}$,

$$\begin{aligned} \alpha(a+b) &= \frac{a}{a+b}\alpha(a+b) + \frac{b}{a+b}\alpha(a+b) \\ &\leq \alpha\left(\frac{a}{a+b}(a+b)\right) + \alpha\left(\frac{b}{a+b}(a+b)\right) \\ &= \alpha(a) + \alpha(b), \ a, b \geq 0. \end{aligned}$$

As a non-decreasing, subadditive and unbounded function, $\alpha(r)$ satisfies

$$\alpha(r) \leq \alpha(r + R_0) \leq \alpha(r) + \alpha(R_0)$$

for any $R_0 \geq 0$. This yields that $\alpha(r) \sim \alpha(r + R_0)$ as $r \to +\infty$. Throughout the present paper we take $\alpha \in L_1$, $\beta \in L_2$, $\gamma \in L_3$.

Heittokangas et al. [4] introduced a new concept of φ-order of entire function considering φ as subadditive function. For details, one may see [4]. Recently, Belaïdi et al. [1] have extended this idea and have introduced the definitions of (α,β,γ)-order and (α,β,γ)-lower order of an entire function f in terms of maximum moduli in the following way:

Definition 1.3. [1] *The (α,β,γ)-order denoted by $\rho_{(\alpha,\beta,\gamma)}[f]$ and (α,β,γ)-lower order denoted by $\lambda_{(\alpha,\beta,\gamma)}[f]$, of an entire function f, are defined as:*

$$\begin{aligned}\rho_{(\alpha,\beta,\gamma)}[f] &= \limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}(M(r,f)))}{\beta\left(\log(\gamma(r))\right)}\\ \text{and } \lambda_{(\alpha,\beta,\gamma)}[f] &= \liminf_{r\to+\infty}\frac{\alpha(\log^{[2]}(M(r,f)))}{\beta\left(\log(\gamma(r))\right)}.\end{aligned}$$

Remark 1. *An entire function f is said to have regular (α,β,γ)-order if $\rho_{(\alpha,\beta,\gamma)}[f] = \lambda_{(\alpha,\beta,\gamma)}[f]$.*

In this paper, we have studied some growth properties relating to sum and product of entire functions on the basis of (α,β,γ)-order and (α,β,γ)-lower order in terms of maximum modulus. Throughout this paper, we assume that all the growth indicators are all nonzero finite.

2. Lemma

In this section, we present some lemmas which will be needed in the sequel.

Lemma 2.1. [2] *Suppose that f be an entire function, $a>1, 0<b<a$. Then*

$$M(ar,f) > bM(r,f).$$

3. Main results

In this section, we present the main results of the paper.

Theorem 3.1. *Let f and g be two non-constant entire functions with finite (α,β,γ)-order, then*

$$\rho_{(\alpha,\beta,\gamma)}[f\pm g] \le \max\{\rho_{(\alpha,\beta,\gamma)}[f],\ \rho_{(\alpha,\beta,\gamma)}[g]\}.$$

The equality holds when $\rho_{(\alpha,\beta,\gamma)}[f] \neq \rho_{(\alpha,\beta,\gamma)}[g]$.

Proof. If $\rho_{(\alpha,\beta,\gamma)}[f\pm g] = 0$ then the result is obvious. So we suppose that $\rho_{(\alpha,\beta,\gamma)}[f\pm g] > 0$. Let us assume that $\max\{\rho_{(\alpha,\beta,\gamma)}[f],\ \rho_{(\alpha,\beta,\gamma)}[g]\} = \Delta$. Now for any arbitrary $\varepsilon>0$, from the definitions of $\rho_{(\alpha,\beta,\gamma)}[f]$ and $\rho_{(\alpha,\beta,\gamma)}[g]$, we have for all sufficiently large values of r that

$$\begin{aligned}M(r,f) &\le \exp^{[2]}(\alpha^{-1}[(\rho_{(\alpha,\beta,\gamma)}[f]+\varepsilon)\cdot\beta\left(\log(\gamma(r))\right)]),\\ i.e.,\ M(r,f) &\le \exp^{[2]}(\alpha^{-1}[(\Delta+\varepsilon)\cdot\beta\left(\log(\gamma(r))\right)]),\end{aligned}\tag{1}$$

and

$$\begin{aligned}M(r,g) &\le \exp^{[2]}(\alpha^{-1}[(\rho_{(\alpha,\beta,\gamma)}[g]+\varepsilon)\cdot\beta\left(\log(\gamma(r))\right)]),\\ i.e.,\ M(r,g) &\le \exp^{[2]}(\alpha^{-1}[(\Delta+\varepsilon)\cdot\beta\left(\log(\gamma(r))\right)]),\end{aligned}\tag{2}$$

In view of (1) and (2), we obtain for all sufficiently large values of r that

$$\begin{aligned}M(r,f\pm g) &\le M(r,f)+M(r,g),\\ i.e.,\ M(r,f\pm g) &\le 2\exp^{[2]}(\alpha^{-1}[(\Delta+\varepsilon)\cdot\beta\left(\log(\gamma(r))\right)]),\end{aligned}$$

$$i.e.,\ \frac{1}{2}M(r,f\pm g) \le \exp^{[2]}(\alpha^{-1}[(\Delta+\varepsilon)\cdot\beta\left(\log(\gamma(r))\right)]).\tag{3}$$

Therefore, in view of Lemma 2.1, we obtain from (3) for all sufficiently large values of r that

$$i.e.,\ M(\frac{r}{3},f\pm g) \le \exp^{[2]}(\alpha^{-1}[(\Delta+\varepsilon)\cdot\beta\left(\log(\gamma(r))\right)]),$$

$$i.e.,\ \log^{[2]} M(\frac{r}{3}, f \pm g) \leq \alpha^{-1}[(\Delta + \varepsilon) \cdot \beta\left(\log(\gamma(r))\right)],$$

$$i.e.,\ \alpha\left(\log^{[2]} M(\frac{r}{3}, f \pm g)\right) \leq (\Delta + \varepsilon) \cdot \beta\left(\log(\gamma(r))\right).$$

Thus for all sufficiently large values of r, we get

$$\frac{\alpha\left(\log^{[2]} M(\frac{r}{3}, f \pm g)\right)}{\beta\left(\log(\gamma(r))\right)} \leq \Delta + \varepsilon.$$

Hence,

$$\limsup_{r\to+\infty}\left[\frac{\alpha\left(\log^{[2]} M(\frac{r}{3}, f \pm g)\right)}{\beta\left(\log(\gamma(\frac{r}{3}))\right)} \cdot \frac{\beta\left(\log(\gamma(\frac{r}{3}))\right)}{\beta\left(\log(\gamma(r))\right)}\right] \leq \Delta + \varepsilon,$$

$$\limsup_{r\to+\infty}\frac{\alpha\left(\log^{[2]} M(\frac{r}{3}, f \pm g)\right)}{\beta\left(\log(\gamma(\frac{r}{3}))\right)} \cdot \lim_{r\to+\infty}\frac{\beta\left(\log(\gamma(\frac{r}{3}))\right)}{\beta\left(\log(\gamma(r))\right)} \leq \Delta + \varepsilon,$$

$$i.e.,\ \rho_{(\alpha,\beta,\gamma)}[f \pm g] \leq \Delta + \varepsilon.$$

As $\varepsilon > 0$ is arbitrary,

$$\rho_{(\alpha,\beta,\gamma)}[f \pm g] \leq \Delta.$$

Hence,

$$\rho_{(\alpha,\beta,\gamma)}[f \pm g] \leq \max\{\rho_{(\alpha,\beta,\gamma)}[f], \rho_{(\alpha,\beta,\gamma)}[g]\}.$$

This completes the proof of first part of the theorem.

Next let, $\rho_{(\alpha,\beta,\gamma)}[f] > \rho_{(\alpha,\beta,\gamma)}[g]$. Taking $f \pm g = h$, we get

$$\rho_{(\alpha,\beta,\gamma)}[h] \leq \rho_{(\alpha,\beta,\gamma)}[f]. \tag{4}$$

Also, we have $f = h \mp g$. So,

$$\begin{aligned} \rho_{(\alpha,\beta,\gamma)}[f] &\leq \max\{\rho_{(\alpha,\beta,\gamma)}[h], \rho_{(\alpha,\beta,\gamma)}[g]\}, \\ i.e.,\ i.e.,\ \rho_{(\alpha,\beta,\gamma)}[f] &\leq \rho_{(\alpha,\beta,\gamma)}[h]. \end{aligned} \tag{5}$$

From (4) and (5), we have $\rho_{(\alpha,\beta,\gamma)}[h] = \rho_{(\alpha,\beta,\gamma)}[f]$,

$$i.e.,\ \rho_{(\alpha,\beta,\gamma)}[f \pm g] = \max\{\rho_{(\alpha,\beta,\gamma)}[f], \rho_{(\alpha,\beta,\gamma)}[g]\}.$$

This completes the proof. □

Corollary 3.0. *Let f and g be two non-constant entire functions with finite (α, β, γ)-lower order with f or g has regular (α, β, γ)-order, then*

$$\lambda_{(\alpha,\beta,\gamma)}[f \pm g] \leq \max\{\lambda_{(\alpha,\beta,\gamma)}[f],\ \lambda_{(\alpha,\beta,\gamma)}[g]\}.$$

The equality holds when either (i) $\lambda_{(\alpha,\beta,\gamma)}[f] > \lambda_{(\alpha,\beta,\gamma)}[g]$ with g has regular (α, β, γ)-order or (ii) $\lambda_{(\alpha,\beta,\gamma)}[g] > \lambda_{(\alpha,\beta,\gamma)}[f]$ with f has regular (α, β, γ)-order.

Theorem 3.2. *Let f and g be two non-constant entire functions with finite (α, β, γ)-lower order with f or g has regular (α, β, γ)-order and both of f and g satisfy the Property (A), then*

$$\lambda_{(\alpha,\beta,\gamma)}[f \cdot g] \leq \max\{\lambda_{(\alpha,\beta,\gamma)}[f],\ \lambda_{(\alpha,\beta,\gamma)}[g]\}.$$

Proof. If $\lambda_{(\alpha,\beta,\gamma)}[f\cdot g]=0$ then the result is obvious. So we suppose that $\lambda_{(\alpha,\beta,\gamma)}[f\cdot g]>0$. Without loss of generality, we assume that $\lambda_{(\alpha,\beta,\gamma)}[f]\geq\lambda_{(\alpha,\beta,\gamma)}[g]$ and f has regular (α,β,γ)-order. Now for any arbitrary $\varepsilon>0$, we have from the definition of $\lambda_{(\alpha,\beta,\gamma)}[f]$ $(=\rho_{(\alpha,\beta,\gamma)}[f])$, for all sufficiently large values of r that

$$M(r,f)\leq\exp^{[2]}(\alpha^{-1}[(\lambda_{(\alpha,\beta,\gamma)}[f]+\varepsilon)\cdot\beta\left(\log(\gamma(r))\right)]). \tag{6}$$

Also, for $\varepsilon>0$ and from the definition of $\lambda_{(\alpha,\beta,\gamma)}[g]$, we have for a sequence of values of r tending to infinity that

$$M(r,g)\leq\exp^{[2]}(\alpha^{-1}[(\lambda_{(\alpha,\beta,\gamma)}[g]+\varepsilon)\cdot\beta\left(\log(\gamma(r))\right)]). \tag{7}$$

Since, $\lambda_{(\alpha,\beta,\gamma)}[f]\geq\lambda_{(\alpha,\beta,\gamma)}[g]$, we can write from (6) and (7), for a sequence of values of r that

$$M(r,f.g)\leq\left[M(r,f)\right]^2. \tag{8}$$

Again, as f satisfies the Property (A), we obtain from (8) for a sequence of values of r and for some $\delta>1$ that

$$M(r,f.g)<M(r^{\delta},f).$$

Using (6) we obtain for a sequence of values of r that

$$M(r,f.g)<\exp^{[2]}(\alpha^{-1}[(\lambda_{(\alpha,\beta,\gamma)}[f]+\varepsilon)\cdot\beta\left(\log(\gamma(r^{\delta}))\right)]),$$

$$i.e.,\ \alpha\left(\log^{[2]}M(r,f.g)\right)<(\lambda_{(\alpha,\beta,\gamma)}[f]+\varepsilon)\cdot\beta\left(\log(\gamma(r^{\delta}))\right),$$

$$i.e.,\ \frac{\alpha\left(\log^{[2]}M(r,f.g)\right)}{\beta\left(\log(\gamma(r^{\delta}))\right)}<(\lambda_{(\alpha,\beta,\gamma)}[f]+\varepsilon),$$

$$i.e.,\ \liminf_{r\to+\infty}\left[\frac{\alpha\left(\log^{[2]}M(r,f.g)\right)}{\beta\left(\log(\gamma(r))\right)}\cdot\frac{\beta\left(\log(\gamma(r))\right)}{\beta\left(\log(\gamma(r^{\delta}))\right)}\right]\leq\lambda_{(\alpha,\beta,\gamma)}[f]+\varepsilon,$$

$$i.e.,\ \liminf_{r\to+\infty}\frac{\alpha\left(\log^{[2]}M(r,f.g)\right)}{\beta\left(\log(\gamma(r))\right)}\cdot\lim_{r\to+\infty}\frac{\beta\left(\log(\gamma(r))\right)}{\beta\left(\log(\gamma(r^{\delta}))\right)}\leq\lambda_{(\alpha,\beta,\gamma)}[f]+\varepsilon,$$

$$i.e.,\ \lambda_{(\alpha,\beta,\gamma)}[f\cdot g]\leq\lambda_{(\alpha,\beta,\gamma)}[f]+\varepsilon.$$

As $\varepsilon>0$ is arbitrary,

$$\lambda_{(\alpha,\beta,\gamma)}[f\cdot g]\leq\lambda_{(\alpha,\beta,\gamma)}[f].$$

Hence,

$$\lambda_{(\alpha,\beta,\gamma)}[f\cdot g]\leq\max\{\lambda_{(\alpha,\beta,\gamma)}[f],\lambda_{(\alpha,\beta,\gamma)}[g]\}.$$

This completes the proof of the theorem. □

Corollary 3.0. *Let f and g be two non-constant entire functions with finite (α,β,γ)-lower order and both satisfy the Property (A), then*

$$\rho_{(\alpha,\beta,\gamma)}[f\cdot g]\leq\max\{\rho_{(\alpha,\beta,\gamma)}[f],\ \rho_{(\alpha,\beta,\gamma)}[g]\}.$$

4. Conclusion

Belaïdi et al. [1] have introduced the concepts of (α,β,γ)-order and (α,β,γ)-lower order of an entire function. The main aim of this paper is to develop some sum and product related results with (α,β,γ)-order and (α,β,γ)-lower order of an entire function. It is no doubt that the interested readers and the involved researchers will be benefited from this idea and we hope that the study will provide a scope for their further research.

Acknowledgement. The authors are very much thankful to the reviewer for his/her valuable suggestions to bring the paper in its present form.

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