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SOME FAMILIES OF ANALYTIC FUNCTIONS RELATED TO THE ERDELY-KOBER INTEGRAL OPERATOR

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Abstract. The Erdelyi-Kober integral operator is a specific integral transform used in mathematical analysis, particularly in connection with solving certain differential equations and studying properties of functions. The study of analytic functions in connection with the Erdelyi-Kober integral operator involves analyzing how the operator affects the analytic properties of functions, ensuring convergence, and understanding the behavior near singularities. These properties are crucial for applications in various branches of mathematics, including differential equations, harmonic analysis, and integral transforms. This paper aims to explore a novel category of regular mapping characterized by negative coefficients in connection with the Erdely-Kober integral operator within the unit disk. We will establish fundamental properties such as coefficient inequalities, extreme points, integral means inequalities and subordination results for this class.

1. Introduction

The Erdelyi-Kober integral operator, named after mathematicians Arthur Erdelyi and Hans Kober, finds applications in various areas of mathematics, physics, engineering, and other fields. Some of the key applications include: Integral Equations, Differential Equations, Potential Theory, Fractional Calculus, Special Functions, Probability Theory and Analytic Number Theory. It serves as a bridge between different mathematical concepts and provides a common framework for tackling complex problems. Furthermore, the Erdelyi-Kober operator continues to inspire research and innovation, as mathematicians and scientists explore new applications, extensions, and connections with other areas of mathematics and physics.

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Its theoretical foundations, coupled with its practical implications, make it an indispensable tool in the toolkit of researchers and practitioners alike. The Erdelyi-Kober integral operator represents not only a mathematical concept but also a gateway to deeper understanding and broader applications across disciplines. Its continued study and exploration promise further advancements and insights into the intricate workings of the mathematical universe.
The Erdelyi-Kober integral is a versatile mathematical tool with applications across different domains, including pure mathematics, applied mathematics, physics, and engineering. Its importance lies in providing solutions to various types of differential and integral equations, as well as facilitating the study of special functions and their properties.
Let $\mathcal{A}$ represent the set encompassing all mapping $v(z)$ in the following type:

$$v(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$, we define a subclass, signified as S, within the larger class $\mathcal{A}$, which consists of univalent mapping. These mappings adhere to the common normalization condition of $v(0) = v'(0) - 1 = 0$.

Specifically, the subclass S is a subset of $\mathcal{A}$ comprising mapping $v(z)$ that are schlicht in the unit disk U.

A function $v \in \mathcal{A}$ is classified as a star shape mapping of order ξ, where $0 \leq \xi < 1$, if it fulfils the following criteria:

$$\Re\left\{\frac{zv'(z)}{v(z)}\right\} > \xi, \ z \in U. \tag{2}$$

We represent this class as $S^*(\xi)$. A mapping u belonging to the broader class $\mathcal{A}$ is considered a convex function of order ξ, where $0 \leq \xi < 1$, when it meets the subsequent conditions:

$$\Re\left\{1 + \frac{zv''(z)}{v'(z)}\right\} > \xi, \ z \in U. \tag{3}$$

We represent this class using the notation $K(\xi)$. It's worth noting that $S^*(0) = S^*$ and $K(0) = K$ correspond to the conventional classes of star-shape and convex mapping within the unit disk U, respectively.

For a mapping v in $\mathcal{A}$ defined by equation (1) and as a mapping $g(z) \in \mathcal{A}$ provided by:

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{4}$$

their convolution , signified by $(v * g)$, is specified as

$$(v * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * v)(z), \ z \in U. \tag{5}$$

Note that $v * g \in \mathcal{A}$.
We'll designate the class of mapping that is regular within the unit disk U as T, and these mapping can be expressed sequentially:

$$v(z) = z - \sum_{n=2}^{\infty} a_n z^n, \ (a_n \geq 0, \ z \in U) \tag{6}$$

and let $T(\xi) = S(\xi) \cap T$ and $C(\xi) = K(\xi) \cap T$. The category of $T(\xi)$ and related groups exhibits noteworthy characteristics and has been thoroughly explored by Silverman [27].

Let's now revisit the definition of the Erdély-Kober type integral operator ([14], Ch. 5) that will be consistently employed in this paper, as obeys:

Definition 1 Given $\vartheta > 0$, and complex plane values for $\mathcal{A}$ and c with $\Re(c-a) \geq 0$, we express an Erdély-Kober type integral operator indicated as $I_{\vartheta}^{a,c} : \mathcal{A} \to \mathcal{A}$. The operator is specified in the domain where $\Re(c-a) > 0$ and $\Re(a) > -\vartheta$, and its definition is as obeys:

$$I_{\vartheta}^{a,c} v(z) = \frac{\Gamma(c+\vartheta)}{\Gamma(a+\vartheta)} \frac{1}{\Gamma(c-a)} \int_0^1 (1-t)^{c-a-1} v(zt^{\vartheta}) dt, \ \vartheta > 0. \tag{7}$$

When $\vartheta > 0$, $\Re(c-a) \geq 0$, $\Re(a) > -\vartheta$, and $v \in \mathcal{A}$ of the type (1), it obeys that:

$$I_{\vartheta}^{a,c} v(z) = z + \sum_{n=2}^{\infty} \aleph_{\vartheta}^{a,c}(n) a_n z^n, \tag{8}$$

where

$$\aleph_{\vartheta}^{a,c}(n) = \frac{\Gamma(c+\vartheta)\Gamma(a+n\vartheta)}{\Gamma(a+\vartheta)\Gamma(c+n\vartheta)} \text{ and } \aleph_{\vartheta}^{a,c}(2) = \frac{\Gamma(c+\vartheta)\Gamma(a+2\vartheta)}{\Gamma(a+\vartheta)\Gamma(c+2\vartheta)}. \tag{9}$$

Note that $I_{\vartheta}^{a,a} v(z) = v(z)$. The operator $I_{\vartheta}^{a,c} v(z)$ extends to encompass various well-known operators that have been previously explored. Several notable specific cases include:

(i). When $a = \kappa$, $c = \varsigma + \kappa$, and $\vartheta = 1$, we acquire the operator $Q_{\kappa}^{\varsigma} v(z)$ for $\varsigma \geq 0$ and $\kappa > 1$, as explored by Jung et al. [13].

(ii). When $a = \varsigma - 1, c = \kappa - 1$, and $\vartheta = 1$, we acquire the operator $L_{\varsigma,\kappa} v(z)$, where ς and κ belong to $\mathbb{C} \in \mathbb{Z}_0$ ($\mathbb{Z}_0 = \{0, -1, -2, \cdots\}$), as examined by Carlson and Shafer [9].

(iii). If $a = \varsigma - 1$, $c = \ell$, and $\vartheta = 1$, the operator $I_{\varsigma,\ell}$ is acquireed, where $\varsigma > 0$ and $\ell > 0$, as studied by Choi et al. [11].

(iv). When $a = \varsigma$, $c = 0$, and $\vartheta = 1$, the operator D^{ς} is acquired, with $\varsigma > -1$, as explored by Ruschweyh [24].

(v). For $a = 1$, $c = n$, and $\vartheta = 1$, the operator I_n is acquired, where $n > \mathbb{N}_0$, as deliberated by Noor [19], Noor and Noor [20].

(vi). In the case where $a = \kappa$, $c = \kappa + 1$, and $\vartheta = 1$, the integral operator $I_{\kappa,1}$ is acquired, as deliberated by Bernardi [8].

(vii). When $a = 1$, $c = 2$, and $\vartheta = 1$, the integral operator $I_{1,1} = I$ is acquired, as examined by Libera [15] and Livingston [17].

Inspired by the contributions of several researchers [1, 2, 5, 6, 18, 23, 31] , we propose a novel subclass of mapping within the broader class $\mathcal{A}$.

Definition 2 For $\hbar \geq 0, 0 \leq \ell < 1$, we set $S_{\vartheta}^{a,c}(\hbar, \ell)$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1) and satisfy

$$Re\left(\frac{I_{\vartheta}^{a,c} v(z)}{z}\right) \geq \hbar \left|(I_{\vartheta}^{a,c} v(z))' - \frac{I_{\vartheta}^{a,c} v(z)}{z}\right| + \ell where$$

$\mathrm{I}_{\vartheta}^{a,c} v(z)$ is given by (7).

We further let $TS_{\vartheta}^{a,c}(\hbar, \ell) = S_{\vartheta}^{a,c}(\hbar, \ell) \cap T$.

In this paper, we obtain coefficient inequalities, extreme points, integral means inequalities for the functions in the class $TS_{\vartheta}^{a,c}(\hbar,\ell)$ and also subordination results for the class of function $v \in S_{\vartheta}^{a,c}(\hbar,\ell)$.

2. Coefficient Estimates

Theorem 1 The function v defined by (1) is in the class $S_{\vartheta}^{a,c}(\hbar,\ell)$ if

$$\sum_{n=2}^{\infty}[1+\hbar(n-1)]\aleph_{\vartheta}^{a,c}(n)|a_n| \le 1-\ell, \tag{11}$$

where $\hbar \ge 0, 0 \le \ell < 1$ and $\aleph_{\vartheta}^{a,c}(n)$ is given by (7).

Proof. It suffices to show that

$$\hbar\left|(I_{\vartheta}^{a,c}v(z))' - \frac{I_{\vartheta}^{a,c}v(z)}{z}\right| - Re\left\{\frac{I_{\vartheta}^{a,c}v(z)}{z} - 1\right\} \le 1-\ell.$$

We have the next inequality

$$\hbar\left|(I_{\vartheta}^{a,c}v(z))' - \frac{I_{\vartheta}^{a,c}v(z)}{z}\right| - Re\left\{\frac{I_{\vartheta}^{a,c}v(z)}{z} - 1\right\}$$

$$\le \hbar\left|\frac{\sum_{n=2}^{\infty}(n-1)\aleph_{\vartheta}^{a,c}(n)a_n z^n}{z}\right| + \left|\frac{\sum_{n=2}^{\infty}\aleph_{\vartheta}^{a,c}(n)a_n z^n}{z}\right|$$

$$\le \hbar\sum_{n=2}^{\infty}(n-1)\aleph_{\vartheta}^{a,c}(n)|a_n| + \sum_{n=2}^{\infty}\aleph_{\vartheta}^{a,c}(n)|a_n|$$

$$= \sum_{n=2}^{\infty}[1+\hbar(n-1)]\,\aleph_{\vartheta}^{a,c}(n)|a_n|.$$

The last expression is bounded above by $(1-\ell)$ if

$$\sum_{n=2}^{\infty}[1+\hbar(n-1)]\aleph_{\vartheta}^{a,c}(n)|a_n| \le 1-\ell$$

and the proof of theorem is completed.

In the following theorem, we obtain necessary and sufficient conditions for functions in $TS_{\vartheta}^{a,c}(\hbar,\ell)$.

Theorem 2 For $\hbar \ge 0, 0 \le \ell < 1$, a function v of the form (2) to be in the class $TS_{\vartheta}^{a,c}(\hbar,\ell)$ if and only if

$$\sum_{n=2}^{\infty}[1+\hbar(n-1)]\aleph_{\vartheta}^{a,c}(n)|a_n| \le 1-\ell.$$

Proof. Suppose $v(z)$ of the form (2) is in the class $TS_{\vartheta}^{a,c}(\hbar,\ell)$. Then

$$Re\left\{\frac{I_{\vartheta}^{a,c}v(z)}{z}\right\} - \hbar\left|(I_{\vartheta}^{a,c}v(z))' - \frac{I_{\vartheta}^{a,c}v(z)}{z}\right| \ge \ell.$$

Equivalently,

$$Re\left[1-\sum_{n=2}^{\infty}\aleph_{\vartheta}^{a,c}(n)|a_n|z^{n-1}\right] - \hbar\left[\sum_{n=2}^{\infty}(n-1)\aleph_{\vartheta}^{a,c}(n)a_n z^{n-1}\right] \ge \ell.$$

Letting z to be real values and as $|z| \to 1$, we have

$$1-\sum_{n=2}^{\infty}\aleph_{\vartheta}^{a,c}(n)|a_n|-\hbar\sum_{n=2}^{\infty}(n-1)\aleph_{\vartheta}^{a,c}(n)|a_n|\geq\ell$$

which implies

$$\sum_{n=2}^{\infty}[1+\hbar(n-1)]\aleph_{\vartheta}^{a,c}(n)|a_n|\leq 1-\ell,$$

where $\hbar\geq 0, 0\leq\ell<1, \aleph_{\vartheta}^{a,c}(n)$ is given by (7) and the sufficiency follows from Theorem 2.

Corollary 1 If $v\in TS_{\vartheta}^{a,c}(\hbar,\ell)$ then

$$a_n\leq\frac{1-\ell}{[1+\hbar(n-1)]\aleph_{\vartheta}^{a,c}(n)}.$$

Equality holds for the function

$$v(z)=z-\frac{1-\ell}{[1+\hbar(n-1)]\aleph_{\vartheta}^{a,c}(n)}z^n,$$

$\hbar\geq 0, 0\leq\ell<1, \aleph_{\vartheta}^{a,c}(n)$ is given by (7).

3. Extreme Points

Theorem 3 Let $v_1(z)=z$ and $v_n(z)=z-\frac{1-\ell}{[1+\hbar(n-1)]\aleph_{\vartheta}^{a,c}(n)}z^n, n\geq 2$ for $\hbar\geq 0, 0\leq\ell<1, \aleph_{\vartheta}^{a,c}(n)$ is given by (7). Then $v(z)$ is in the class $\aleph_{\vartheta}^{a,c}(n)$ if and only if it can be expressed in the form $v(z)=\sum\limits_{n=1}^{\infty}\lambda_n v_n(z)$, where λ_n and $\sum\limits_{n=1}^{\infty}\lambda_n=1$.

Proof. If $v(z)=\sum\limits_{n=1}^{\infty}\lambda_n v_n(z)$ with $\lambda_n\geq 0$ and $\sum\limits_{n=1}^{\infty}\lambda_n=1$. Then

$$\begin{aligned}
v(z)&=\sum_{n=1}^{\infty}\lambda_n v_n(z)\\
&=\lambda_1 v_1(z)+\sum_{n=2}^{\infty}\lambda_n v_n(z)\\
&=\left(1-\sum_{n=2}^{\infty}\lambda_n\right)z+\sum_{n=2}^{\infty}\left[\lambda_n\left(z-\frac{1-\ell}{[1+\hbar(n-1)]\aleph_{\vartheta}^{a,c}(n)}z^n\right)\right]\\
&=z-\sum_{n=2}^{\infty}\frac{1-\ell}{[1+\hbar(n-1)]\aleph_{\vartheta}^{a,c}(n)}z^n.
\end{aligned}$$

$$\begin{aligned}
\text{Now }&\sum_{n=2}^{\infty}\frac{[1+\hbar(n-1)]\aleph_{\vartheta}^{a,c}(n)}{1-\ell}\frac{1-\ell}{[1+\hbar(n-1)]\aleph_{\vartheta}^{a,c}(n)}\lambda^n\\
&=\sum_{n=2}^{\infty}\lambda_n=1-\lambda_1\leq 1.
\end{aligned}$$

Then $v \in TS_{\vartheta}^{a,c}(\hbar, \ell)$. Conversely suppose that $v \in TS_{\vartheta}^{a,c}(\hbar, \ell)$. Then Corollary gives

$$a_n \leq \frac{1-\ell}{[1+\hbar(n-1)]\aleph_{\vartheta}^{a,c}(n)}, \ n \geq 2$$

$$\text{set } \lambda_n = \frac{[1+\hbar(n-1)]\aleph_{\vartheta}^{a,c}(n)}{1-\ell} a_n, \ n \geq 2$$

$$\text{where } \lambda_n = 1 - \sum_{n=2}^{\infty} \lambda_n.$$

Then

$$\begin{aligned} v(z) &= z - \sum_{n=2}^{\infty} a_n z^n \\ &= z - \sum_{n=2}^{\infty} \lambda_n \frac{1-\ell}{[1+\hbar(n-1)]\aleph_{\vartheta}^{a,c}(n)} \\ &= z - \left[1 - \sum_{n=2}^{\infty} \lambda_n\right] + \sum_{n=2}^{\infty} \lambda_n v_n(z) \\ &= \lambda_1 v_1(z) + \sum_{n=2}^{\infty} \lambda_n v_n(z) \\ &= \sum_{n=1}^{\infty} \lambda_n v_n(z). \end{aligned}$$

The poof of theorem is completed

4. Integral Means Inequalities

Definition 3 (Subordination principle) for analytic function g and h with $g(0) = h(0)$, g is said to be subordinate to h, denoted by $g \prec h$ if there exists an analytic function ω such that $\omega(0) = 0, |\omega(z)| < 1$ and $g(z) = h(\omega(z))$, for all $z \in U$.
Lemma 1[16] If the function $v(z)$ and $g(z)$ are analytic in U with $g(z) \prec v(z)$ then

$$\int_0^{2\pi} \left|g(re^{i\theta})\right|^p d\theta \leq \int_0^{2\pi} \left|v(re^{i\theta})\right|^p d\theta \ (0 \leq r < 1, p > 0 \ and \ z = re^{i\theta}).$$

Theorem 4 Suppose $v \in TS_{\vartheta}^{a,c}(\hbar, \ell), p > 0, \hbar \geq 0, 0 \leq \ell < 1$ and $v(z)$ is defined by

$$v_2(z) = z - \frac{1-\ell}{(1+\hbar)\aleph_{\vartheta}^{a,c}(2)} z^2.$$

Then for $z = re^{i\theta}, \ 0 \leq r < 1$,

$$\int_0^{2\pi} |v(z)|^p d\theta \leq \int_0^{2\pi} |v_2(z)|^p d\theta \tag{12}$$

Proof. For $v(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$, (12) is equivalent to proving that

$$\int_0^{2\pi} \left| 1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} \right|^p d\theta \leq \int_0^{2\pi} \left| 1 - \frac{1-\ell}{(1+\hbar)\aleph_{\vartheta}^{a,c}(2)} z \right|^p d\theta, \ (p > 0).$$

By applying Littlewood's subordination theorem (Lemma 4),
it would be sufficient to show that

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} \prec 1 - \frac{1-\ell}{[1+\hbar(n-1)]\aleph_{\vartheta}^{a,c}(2)} z \tag{13}$$

Setting

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} \prec 1 - \frac{1-\ell}{[1+\hbar(n-1)]\aleph_{\vartheta}^{a,c}(2)} \omega(z).$$

We have $\omega(z) = \frac{[1+\hbar(n-1)]\aleph_{\vartheta}^{a,c}(n)}{1-\ell} \sum_{n=2}^{\infty} a_n z^{n-1}$ and and $\omega(z)$ is analytic in U with $\omega(0) = 0$. Moreover it suffices to prove that $\omega(z)$ satisfies $|\omega(z)| < 1, z \in U$. Now

$$\begin{aligned} |\omega(z)| &= \left| \sum_{n=2}^{\infty} \frac{[(1+\hbar(n-1)\aleph_{\vartheta}^{a,c}(n)}{1-\ell} a_n z^{n-1} \right| \\ &\leq |z| \sum_{n=2}^{\infty} \frac{[1+\hbar(n-1)]\aleph_{\vartheta}^{a,c}(n)}{1-\ell} |a_n| \qquad (14) \\ &\leq |z| < 1. \end{aligned}$$

Thus is view of the inequality (14) the subordination (13) follows, which proves the Theorem.

5. Subordination Results

Definition 4 (Subordination factor sequence) A sequence $\left\{ b_n \right\}_{n=2}^{\infty}$ of complex numbers is said to be a subordinating sequence if, whenever $v(z) = \sum_{n=2}^{\infty} a_n z^n, a_1 = 1$ is regular, univalent and convex in U, we have $\sum_{n=1}^{\infty} b_n a_n z^n \prec v(z), \ z \in U$.

Theorem 5[32] The sequence $\left\{ b_n \right\}_{n=2}^{\infty}$ is a subordinating factor sequence if and only if

$$Re\left\{ 1 + 2\sum_{n=1}^{\infty} b_n z^n \right\} > 0, z \in U.$$

Theorem 6 Let $v \in I_{\vartheta}^{a,c} v(z)$ and $g(z)$ any function in the usual class of convex function $\mathbb{C}$. Then

$$\frac{(1+\hbar)\aleph_{\vartheta}^{a,c}(n)}{2(1-\ell)+(1+\hbar)\aleph_{\vartheta}^{a,c}(n)} (v * g)(z) \prec g(z) \tag{15}$$

where $\hbar \geq 0, 0 \leq \ell < 1$ with $\aleph_{\vartheta}^{a,c}(n)$ is given by (7)

$$Re\{v(z)\} > -\frac{(1-\ell)+(1+\hbar)\aleph_{\vartheta}^{a,c}(n)}{(1+\hbar)\aleph_{\vartheta}^{a,c}(n)}, \ z \in E. \tag{16}$$

The constant $\frac{(1+\hbar)\aleph_{\vartheta}^{a,c}(n)}{2(1-\ell)+(1+\hbar)\aleph_{\vartheta}^{a,c}(n)}$ is the best estimate.

Proof Let $v \in I_{\vartheta}^{a,c}v(z)$ and $g(z) = z + \sum\limits_{n=2}^{\infty} c_n z^n \in \mathbb{C}$.
Then

$$\frac{(1+\hbar)\aleph_{\vartheta}^{a,c}(n)}{2(1-\ell)+(1+\hbar)\aleph_{\vartheta}^{a,c}(n)}(v*g)(z) = \frac{(1+\hbar)\aleph_{\vartheta}^{a,c}(n)2(1-\ell)+(1+\hbar)\aleph_{\vartheta}^{a,c}(n)\left(z+\sum\limits_{n=2}^{\infty} c_n a_n z^n\right)}{.}$$

Then by Definition 5.1, the subordination result holds true if $\left\{\frac{(1+\hbar)\aleph_{\vartheta}^{a,c}(n)}{2(1-\ell)+(1+\hbar)\aleph_{\vartheta}^{a,c}(n)}\right\}_{n=1}^{\infty}$ is a subordinating factor sequence with $a_1 = 1$.
In view of Theorem 5, this is equivalent to the following inequality.

$$Re\left\{1+\sum_{n=1}^{\infty}\frac{(1+\hbar)\aleph_{\vartheta}^{a,c}(n)}{(1-\ell)+(1+\hbar)\aleph_{\vartheta}^{a,c}(n)}a_n z^n\right\} > 0,\ \ z \in U. \tag{17}$$

Now for $|z| = r < 1$, we have

$$\begin{aligned}
&Re\left\{1+\sum_{n=1}^{\infty}\frac{(1+\hbar)\aleph_{\vartheta}^{a,c}(n)}{2(1-\ell)+(1+\hbar)\aleph_{\vartheta}^{a,c}(n)}a_n z^n\right\}\\
&= Re\left\{1+\frac{(1+\hbar)\aleph_{\vartheta}^{a,c}(n)}{(1-\ell)+(1+\hbar)\aleph_{\vartheta}^{a,c}(n)}z+\frac{\sum\limits_{n=2}^{\infty}(1+\hbar)\aleph_{\vartheta}^{a,c}(n)a_n z^n}{(1-\ell)+(1+\hbar)\aleph_{\vartheta}^{a,c}(n)}\right\}\\
&\geq 1-\frac{(1+\hbar)\aleph_{\vartheta}^{a,c}(n)}{(1-\ell)+(1+\hbar)\aleph_{\vartheta}^{a,c}(n)}r-\frac{\sum\limits_{n=2}^{\infty}(1+\hbar)\aleph_{\vartheta}^{a,c}(n)a_n r^n}{(1-\ell)+(1+\hbar)\aleph_{\vartheta}^{a,c}(n)}\\
&\geq 1-\frac{(1+\hbar)\aleph_{\vartheta}^{a,c}(n)}{(1-\ell)+(1+\hbar)\aleph_{\vartheta}^{a,c}(n)}r-\frac{1-\ell}{(1-\ell)+(1+\hbar)\aleph_{\vartheta}^{a,c}(n)}r\\
&> 0.
\end{aligned}$$

Using (11) and the fact that $[1+\hbar(n-1)]\aleph_{\vartheta}^{a,c}(n)$ is increasing function for $n \geq 2$.
This proves the inequality (17) and hence also the subordination result (15) asserted by Theorem 5.
The inequality (16) follows from (15) by taking

$$g(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n \in \mathbb{C}.$$

Now we consider the function $v(z) = z - \frac{1-\ell}{(1+\hbar)\aleph_{\vartheta}^{a,c}(n)}z^2$, where $\hbar \geq 0, 0 \leq \ell < 1$.
Clearly $F \in I_{\vartheta}^{a,c}v(z)$. For the function (15) becomes

$$\frac{(1+\hbar)\aleph_{\vartheta}^{a,c}(n)}{2(1-\ell)+(1+\hbar)\aleph_{\vartheta}^{a,c}(n)}v(z) \prec \frac{z}{1-z}.$$

It is easily verified that

$$\min Re\left\{\frac{(1+\hbar)\aleph_{\vartheta}^{a,c}(n)}{2(1-\ell)+(1+\hbar)\aleph_{\vartheta}^{a,c}(n)}v(z)\right\} = \frac{-1}{2}, z \in U.$$

This shows that the constant $\frac{(1+\hbar)\aleph_{\vartheta}^{a,c}(n)}{2(1-\ell)+(1+\hbar)\aleph_{\vartheta}^{a,c}(n)}v(z) \prec \frac{z}{1-z}$ is best possible.

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