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NUMERICAL SOLUTION OF VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER WITH INITIAL CONDITIONS USING COLLOCATION APPROACH

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Abstract. In this paper, we developed and implemented a numerical method for the solution of Volterra integro- differential equations of fractional order using the collocation approach. We obtained the integral form of the problem, which is transformed into a system of algebraic equations using the standard collocation points. We then solve the algebraic equation using matrix inversion. The analysis of the developed method was investigated, and the solution was found to be continuous and convergent. The uniqueness of the solution was also proven. Numerical examples were considered to test the consistency and efficiency of the method.

1. Introduction

The applications of fractional differential and integral equations are widely used in engineering, physics, chemistry, and mathematics. Ordinary and partial differential equations are examples of functional equations where real-world situations are mathematically modeled. In order to investigate population growth processes, Vito Volterra created a novel kind of equation known as Integro-differen- tial Equations (IDEs) in the early 1900s. One or more derivatives of the unknown function can be found under the integral sign in these equations. IDEs are found in many mathematical formulations of physical events and are used in the modeling of various scientific and engineering phenomena.[23]. In the open literature, many numerical approaches for solving integro-differential difference equations and integro-differential equations have been proposed, including the Adomian decomposition method by

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[16], Collocation method by [3, 1, 5], Hybrid multistep method [14, 6, 17], Homotopy analysis method [15], Bernoulli matrix method [11], Differential transform method [13], Shifted Legendre polynomials [18], Bernstein Polynomials Method [19], Differential transformation [12], Chebyshev polynomials[20], Weighted mean-value theorem[8], Optimal Auxiliary Function Method (OAFM) [26], Laplace decomposition method [15], Reproducing kernel method [14], Block pulse functions operational matrices [22] and Spectral Homotopy Analysis Method [9]. [4] used the standard collocation method to solve first-order Volterra integro-differential equations. Assuming an approximation solution, the class of integro-differential equations was restated in terms of the derived polynomial. After solving for the unknown, we collocated the resultant equation at many points within the range [0, 1], yielding a system of linear algebraic equations. [7] introduced a collocation method for the computational solution of the integro-differential equations with Fredholm-Volterra fractional order. They first obtained the problem in linear integral form, which they then converted into a set of linear algebraic equations using standard collocation points.

This research paper consider the numerical solution of Volterra Integro-differential equation fractional order of the form

$$ {}_0^cD_x^\alpha y\left(x\right) = g\left(x\right) + h(x)y(x) + \int_0^x K(x,t)(y(t))^v dt, \qquad 0 \le x, t \le 1, v \ge 1 \quad (1) $$

subject to initial condition

$$ y^{(j)}(0) = q_j, \ j = 0, 1, ..., N \quad (2) $$

where $y(x)$ is the unknown function to be determine, D^α is the Caputo's derivative, $K(x,t)$ is the Volterra integral kernel function. $g(x)$ and $h(x)$ are the known functions, a_j and q_j are known constants.

2. Basic definitions

In this section, we define some basic terms that would be encountered in this research

Definition 2.1. *A metric on a set M is a function $d : M \times M \longrightarrow \mathbb{R}$ with the following properties. For all $x, y \in M$* [7]
(a) $d(x,y) \ge 0$;
(b) $d(x,y) = 0 \Longleftrightarrow x = y$
(c) $d(x,y) = d(y,x)$
(d) $d(x,y) \le d(x,z) + d(x,y)$
If d is a metric on M, then the pair (M,d) is called a metric space.

Definition 2.2. *Metric space* [10] *Let (X,d) be a metric space, A mapping $T : X \longrightarrow X$ is Lipschitzian if $\exists$ a constant $L > 0$ such that $d(Tx,Ty) \le Ld(x,y)\ \forall\ x,y \in X$.*

Definition 2.3. *Left Caputo's derivatives)* [2] *The left Caputo's definition of fractional derivative operator is given by*

$$ {}_0^cD_t^\alpha f\left(x\right) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (x-t)^{m-\alpha-1} f^{(m)}(t) dt \quad (3) $$

where $m - 1 \le \alpha \le m, m \in N, x > 0$.

It has the following two basic properties:

(i): $D^\alpha I^\alpha f(x) = f(x)$

(ii): $I^\alpha D^\alpha f(x) = f(x) - \sum\limits_{k=0}^{m-1} f^{(k)}(0^+)\frac{x^k}{k!}, x > 0$

Definition 2.4. *(Riemann - Liouvile fractional integral)* [10] *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a continuous function $u : (0, \infty) \to \mathbb{R}$ is defined by*

$$ {}_0I_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} u(s) ds. \tag{4}$$

Definition 2.5. *Integration of nth derivatives* [10] *For $\alpha > 0$, let $u(t)$ be a continuous function, then*

$$ {}_0I_t^\alpha \left({}^cDu\right)(t) = u(t) - \sum_{k=0}^{\alpha-1} c_k t^k \tag{5}$$

Definition 2.6. [4] *Let $(a_m), m \geq 0$ be a sequence of real numbers. The power series in k with coefficients a_m is an expression.*

$$ y(w) = \sum_{m=0}^{M} a_m w^m = \phi(w) \ \mathbf{A} \tag{6}$$

where $\phi(w) = [1 \ w \ \ w^2 \ \cdots \ \ w^M]$, $\mathbf{A} = [a_0 \ a_1 \ \ \cdots \ \ a_M]^T$

Theorem 2.1. *(Banach's fixed point theorem)* [21]
Let $(X, \|.\|)$ be a complete norm space, then each contraction mapping $T : X \to X$ has a unique fixed point x of T in X, such that $T(x) = x$.

3. Material and Method

This section considers the development of our method, which was achieved by developing the integral form of the modeled equation (1) and obtaining the algebraic equations using some lemmas.

3.1. **Method of Solution. Lemma:** Let $y \in C([a,b], \mathbb{R})$ be the solution to equation (1) and equation (2), $K \in C([a,b], \mathbb{R})$ then equation (1) and equation(2) is equivalent to

$$\begin{aligned} y(x) &= P(x) + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1}(h(x)y(x)ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \left(\int_0^x K(x,t)(y(t))^v dt \right) ds \end{aligned}$$

where

$$ P(x) = \sum_{k=0}^{N} \frac{y^{(k)}(0)}{k!} x^k + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} g(x) ds $$

Proof. Multiply equation (1) by ${}_0I_x^\beta(.)$ gives

$$ {}_0I_x^\alpha \left({}_0^cD_x^\alpha y(x)\right) = {}_0I_x^\alpha(g(x)) + {}_0I_x^\alpha(h(x)y(x)) + {}_0I_x^\alpha \left(\int_0^x K(x,t)(y(t))^v dt \right) $$

using equation (5) on equation (1) gives

$$y(x) = \sum_{k=0}^{N} \frac{y^{(k)}(0)}{k!} x^k + \ {}_0I_x^{\alpha} g(x)) +_0 I_x^{\alpha} (h(x)y(x)) +_0 I_x^{\alpha} \int_0^x K(x,t)(y(t))^v dt \quad (7)$$

applying equation (4) to equation (7) gives

$$y(x) = \sum_{k=0}^{N} \frac{y^{(k)}(0)}{k!} x^k + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} (h(x)y(x)ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \left(\int_0^x K(x,t)(y(t))^v dt \right) ds + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} g(x) ds \quad (8)$$

collocating at x_i in equation (8) gives

$$y(x_i) = \sum_{k=0}^{N} \frac{y^{(k)}(0)}{k!} x_i^k + \frac{1}{\Gamma(\alpha)} \int_0^{x_i} (x_i-s)^{\alpha-1} (h(x_i)y(x_i)ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^{x_i} (x_i-s)^{\alpha-1} \left(\int_0^{x_i} K(x_i,t)(y(t))^v dt \right) ds + \frac{1}{\Gamma(\alpha)} \int_0^{x_i} (x_i-s)^{\alpha-1} g(x_i) ds \quad (9)$$

using equation (10) gives

$$\sum_{n=0}^{N} a_n x_i^n = \frac{1}{\Gamma(\alpha)} \int_0^{x_i} (x_i-s)^{\alpha-1} \left(h(x) \sum_{n=0}^{N} a_n x_i^n \right) ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^{x_i} (x_i-s)^{\alpha-1} \left(\int_0^{x_i} K(x_i,t) \left(\sum_{n=0}^{N} a_n x_i^n \right)^v dt \right) ds + P(x_i) \quad (10)$$

where

$$P(x_i) = + \frac{1}{\Gamma(\alpha)} \int_0^{x_i} (x_i-s)^{\alpha-1} g(x) ds$$

factorise the values of $\mathbf{a}'s$ from equation (10) gives

$$\left[\sum_{n=0}^{N} a_n \begin{array}{c} x_i^n - \frac{1}{\Gamma(\alpha)} \int_0^{x_i} (x_i-s)^{\alpha-1} \left(h(x) x_i^n \right) ds \\ - \frac{1}{\Gamma(\alpha)} \int_0^{x_i} (x_i-s)^{\alpha-1} \left(\int_0^{x_i} K(x_i,t) \left(x_i^n \right)^v dt \right) ds \end{array} \right] = P(x_i) \quad (11)$$

equation (11) can be in the form

$$\sum_{n=0}^{N} a_n L(x_i) = P(x_i) \quad (12)$$

where

$$\begin{aligned} L(x_i) \quad &= \quad x_i^n - \frac{1}{\Gamma(\alpha)} \int_0^{x_i} (x_i-s)^{\alpha-1} \left(h(x) x_i^n \right) ds \\ & \quad - \frac{1}{\Gamma(\alpha)} \int_0^{x_i} (x_i-s)^{\alpha-1} \left(\int_0^{x_i} K(x_i,t) \left(x_i^n \right)^v dt \right) ds \end{aligned}$$

□

3.2. **Consideration of Initial Condition.** using the initial condition in equation (2)

$$y^{(j)}(0) = q_j,\ j = 0, 1, ..., N \tag{13}$$

hence, equation (13) becomes

$$\frac{d^j}{dx^j}\sum_{n=0}^{N} a_n x^n = q_j$$

$$\sum_{n=0}^{N} a_n \left(\frac{d^j}{dx^j} x^n\right) = q_j \tag{14}$$

adding equation (14) to equation (12) gives

$$\sum_{n=0}^{N} a_n L^*(x_i) {=} P^*(x_i) \tag{15}$$

equation (15) is the algebraic equation that we solve to obtain the values of a's and substitute into the approximate solution to give the numerical solution.

3.3. **Uniqueness of the Method.** In this section, we assumed that the solution to equation (1) and equation (2) exist, we then establish the uniqueness of solution and present solutions from the method of solution.
In order to prove the uniqueness theorem, we use the following hypothesis
H_1 : Let $T : X \to X$ be a mapping for $y_1, y_2 \in X$,There exist a constant, $L > 0$. such that

$$|y_1^v(t) - y_2^v(t)| \leq L\,|y_1(t) - y_2(t)|$$

H_2 : There exist a function $K^* \in C\left([0,1] \times [0,1], \mathbb{R}\right)$, the set of all positive functions such that

$$K^* = \max_{x\in[0,1]} \int_0^x |K(x,t)|\, dt < \infty$$

H_3 : The function $h \in \mathbb{R}$ is continuous such that

$$h^* = \max_{x\in[0,1]} |h(x)|$$

Theorem 3.2. *Assume the H_1- H_3 hold. If*

$$\left(\frac{h^* + LK^*}{\Gamma\left(\alpha + 1\right)}\right) < 1 \tag{16}$$

then there exist a unique solution $y(x) \in T$

Proof. Let $y_1(x),\ y_2(x) \in X$,then

$$(Ty_1)(x) = P(x) + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}(h(x)y_1(x)ds+$$
$$\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(\int_0^x K(x,t)(y_1(t))^v dt\right) ds \tag{17}$$

and

$$(Ty_2)(x) = P(x) + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}(h(x)y_2(x)ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(\int_0^x K(x,t)(y_2(t))^v dt\right) ds \tag{18}$$

substract equation (18) from equation (17) gives

$$(Ty_1)(x) - (Ty_2)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1}(h(x)\left[y_1(x) - y_2(x)\right] ds +$$
$$\frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \left(\int_0^x K(x,t)\left[y_1^v(t) - y_2^v(t)\right] dt \right) ds$$

taking the absolute value gives

$$\left|(Ty_1)(x) - (Ty_2)(x)\right| \leq \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} |h(x)| \, |y_1(x) - y_2(x)| \, ds +$$
$$\frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \left(\int_0^x |K(x,t)| \, |y_1^v(t) - y_2^v(t)| \, dt \right) ds$$

taking maximum of both sides and using H_1 *and* H_2

$$d\left(Ty_1(x), Ty_2(x)\right) \leq \left[\frac{h^* + LK^*}{\Gamma(\alpha+1)}\right] d(y_1, y_2)$$

based on the inequality (16) we have

$$d\left(Ty_1(x), Ty_2(x)\right) \leq d(y_1, y_2)$$

By the Banach contraction principle, we can conclude that T has a unique fixed point. □

4. Convergence of the method

Theorem 4.3. *(Convergence of method) Let (X, d) be a metric space and $T : X \longrightarrow X$ be a continuous mapping and $y_N(x), y_{N-1}(x)\epsilon X$ are approximate solutions of equation (7). Let $\Delta_N(x) = |y_N(x) - y_{N-1}(x)|$, if $\lim_{N\to 0}(\Delta_N(x)) \to 0$, then the method converges to exact solution.*

Proof. Let $y_1(x),\ y_2(x)$ be approximated by $y_N(x) = \sum_{n=0}^{M} a_n x^n = \ \phi(x)\ \mathbf{A}$ *and* $y_{N-1} = \sum_{n=0}^{M} b_n x^n = \ \phi(x)\ \mathbf{B}$ respectively. Substitute the approximate solution into equation (8) gives

$$(Ty_N)(x) = P(x) + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1}(h(x)\left(\ \phi(x)\right)\mathbf{A} ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \left(\int_0^x K(x,t)\left(\ \phi(t)^v\right) dt \right) ds\mathbf{A}$$

Similarly

$$(Ty_{N-1})(x) = P(x) + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1}(h(x)\left(\ \phi(x)\right)\mathbf{A} ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \left(\int_0^x K(x,t)\left(\ \phi(t)^v\right) dt \right) ds\mathbf{B}$$

$$|Ty_N(x) - Ty_{N-1}(x)| = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \left(h(x) + \int_0^x K(x,t)\left(\phi(t)^v\ \right) dt \right) |\mathbf{B} - \mathbf{A}| \, ds$$

Since $x\epsilon\,[0,1]$ *and* $|\mathbf{B} - \mathbf{A}| \neq 0$, hence $\lim_{N\to 0} \Delta_N(x) \to 0$

Therefore the method of solution converges. □

4.1. **Numerical Examples. Example 1:** Considering linear fracrional Volterra integro-differential equation

$$D^{0.75}y\left(x\right)+\frac{x^{2}e^{x}}{5}y(x)-\int_{0}^{x}e^{x}ty(t)dt=f(x) \tag{19}$$

subject to initial condition

$$y(0)=0$$

where

$$f\left(x\right)=\frac{6x^{9/4}}{\Gamma(3.25)}$$

Exact solution: $y\left(x\right)=x^{3}$

The approximate solution of equation (19) at $N=3$ gives

$$y_{3}=-7.941025615000\times10^{-11}x+1.422222340000\times10^{-10}x^{2}+0.999999999999x^{3}$$

Table 1: Exact, approximate and absolute error values for example 1

x	Exact	Our method$_{N=3}$	error$_3$
0.2	0.008000000000	0.008000000000	0.00
0.4	0.064000000000	0.064000000000	0.00
0.6	0.216000000000	0.600000000000	0.00
0.8	0.800000000000	0.216000000000	0.00
1.0	1.000000000000	1.000000000000	0.00

Example 2: Considering nonlinear fractional Volterra integro-differential equation

$$D^{1.2}y\left(x\right)-\int_{0}^{x}\left(x-t\right)^{2}\left(y(t)\right)^{3}dt=f(x) \tag{20}$$

subject to initial condition

$$y(0)=0,y^{'}(0)=0$$

where

$$f\left(x\right)=\frac{5}{2\Gamma(0.8)}x^{\frac{4}{5}}-\frac{x^{9}}{252}$$

Exact solution: $y\left(x\right)=x^{2}$

The approximate solution of equation (20) at $N=5$ gives

$$y_{5}=\left(\begin{array}{c}4.163336342344\times10^{-16}+8.326672684689\times10^{-17}x+1.000000000000x^{2}\\ +4.547473508865\times10^{-13}x^{3}-4.547473508865\times10^{-13}x^{4}+6.821210263297\times10^{-13}x^{5}\end{array}\right)$$

Table 2: Exact and approximate values for example 2

x	Exact	$N=5$	error$_5$
0.2	0.040000000000	0.040000000000	0.00
0.4	0.160000000000	0.160000000000	0.00
0.6	0.360000000000	0.360000000000	0.00
0.8	0.640000000000	0.640000000000	0.00
1.0	1.000000000000	1.000000000000	0.00

Example 3: Considering nonlinear fractional Volterra integro-differential equation

$$D^{1.6}y\left(x\right)-\int_{0}^{x}\left(x+t\right)^{2}\left(y(t)\right)^{2}dt=f(x \tag{21}$$

subject to initial condition

$$y(0)=0,y^{'}(0)=0$$

where

$$f\left(x\right)=\frac{6}{\Gamma(0.33)}x^{\frac{1}{3}}-\frac{x^{2}}{7}-\frac{x}{4}-\frac{1}{9}$$

Exact solution: $y\left(x\right)=x^{2}$

The approximate solution of equation (21) at $N=5$ *and* 10 gives

$$y_5=\begin{pmatrix} -0.79545696e-4-0.148975638e-3x+0.926538952030x^2 \\ +0.200221882790x^3-0.512476554541x^4+0.287008050336x^5 \end{pmatrix}$$

$$2.751660086000\times10^{-16}+1.000000000000x-1.097149038000\times10^{-9}x^2+4.323889395000\times10^{-9}x^3-5.598697802000\times10^{-9}x^4+2.344192277000\times10^{-9}x^5$$

$$y_{10}=\begin{pmatrix} -2.907896146098\times10^{-12}-3.314681862321\times10^{-12}x+1.051124559715x^2 \\ -0.344825714827x^3+0.579640269279x^4-1.121343612671x^5 \\ +1.666065216064x^6-1.773351669312x^7+1.265789985657x^8 \\ -0.511327743530x^9+0.95399975777e-1\times x^{10} \end{pmatrix}$$

Table 3: Exact and approximate values for example 3

x	Exact	Our method$_{N=5}$	Our method$_{N=10}$	error$_5$	error $_{N=10}$
0.2	0.040000000000	0.037825872410	0.039941888810	2.174127590e-3	5.811119e-5
0.4	0.160000000000	0.152740859400	0.159091569800	9.2591406e-3	5.9084302e-5
0.6	0.360000000000	0.352533802800	0.396621880900	2.74661972e-3	2.33781191e-5
0.8	0.640000000000	0.569436208300	0.584976498200	6.05637917e-3	5.50235018e-4
1.0	1.000000000000	0.981063809800	0.997171266300	6.0563791700e-3	4.28287337e-4

5. Discussion of Results

In this section, we discussed the results obtained from the solved problems using our developed method. We also established the uniqueness and convergence of the solution.

From the result obtained for example 1, as shown in Table 1, the approximate solution at N = 3 gives $y_3=-7.941025615000\times10^{-11}x+1.422222340000\times10^{-10}x^2+0.999999999999x^3$.

The numerical result converged to an exact solution.

The approximate solution obtained in example 2 at N = 5 gives $y_5=4.163336342344\times10^{-16}+8.326672684689\times10^{-17}x+1.000000000000x^2+4.547473508865\times10^{-13}x^3-4.547473508865\times10^{-13}x^4+6.821210263297\times10^{-13}x^5$.The numerical result converged to an exact solution as shown in Tables 2.

The numeical results of example 3 shows that the approximate solution at N = 5 gives $y_5=0.79545696e-4-0.148975638e-3x+0.926538952030x^2+0.200221882790x^3$
$-0.512476554541x^4+0.287008050336x^5$. Solving for the value of N = 10, the numerical results converge to an exact solution as the value of N increases as shown in Table 3.

The numerical method is observed to be consistent and converges faster to the exact solution, as shown in problems 1, 2, and 3. It is also observed that as N increases, the solution gets better. Hence the stability of the method.

6. Conclusion

This work investigates the collocation approach for solving fractional Volterra integro-differential equations. This method is easy to compute, reliable, and efficient. All of the computations in this paper were performed using Maple 18.
Conflicts of Interest: There are no competing interests.

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References

[1] A. O. Agbolade and T. A. Anake, Solution of first order volterra linear integro differential equations by collocation method, J. Appl. Math. Article, 5, 2017. ID. 1510267 , doi:10.1155/2017/1510267.

[2] A. H. Ahmed, P. G. Kirtiwant and M. A. Shakir, The approximate solutions of fractional integro-differential equations by using modified adomian decomposition method. Khayyan J. Math. 5 (1), 21-39, 2019.

[3] G. Ajileye and F. A. Aminu, A Numerical Method using Collocation approach for the solution of Volterra-Fredholm Integro-differential Equations, African Scientific Reports 1, 205–211, 2022.

[4] G. Ajileye and F. A. Aminu, Approximate Solution to First-Order Integro-differential Equations Using Polynomial Collocation Approach, J. Appl. Computat Math., 11, 486, 2022.

[5] G. Ajileye, and S. A. Amoo, Numerical solution to Volterra integro-differential equations usingcollocation approximation, Mathematics and Computational Sciences, 4(1), 1-8, 2023.

[6] G. Ajileye, S. A. Amoo and O. D. Ogwumu, Hybrid Block Method Algorithms for Solution of First Order Initial Value Problems in Ordinary Differential Equations, J Appl Computat Math. **7**,390, (2018. doi:10.4172/2168-9679.1000390.

[7] G. Ajileye, A. A. James, A. M. Ayinde and T. Oyedepo, Collocation Approach for the Computational Solution Of Fredholm-Volterra Fractional Order of Integro-Differential Equations, J. Nig. Soc. Phys. Sci., 4, 834, 2022.

[8] A. Altürk, Numerical solution of linear and nonlinear Fredholm integral equations by using weighted mean-value theorem, Springer Plus, **5,** 2016. DOI 10.1186/s40064-016-3645-8.

[9] Z. P. Atabakan, A. K. Nasab, A. Kiliçman and Z. K. Eshkuvatov, Numerical Solution of Nonlinear Fredholm Integro-Differential Equations Using Spectral Homotopy Analysis Method, Mathematical Problems in Engineering, 9 (7), 2013. http://dx.doi.org/10.1155/2013/674364.

[10] V. Berinde, Iterative approximation of fixed points. Romania. Editura Efemeride, Baia Mare, 2002.

[11] C. Ercan and T. Kharerah, T. Solving a class of Volterra integral system by the differential transform method, Int. J. Nonlinear Sci. 16, 87-91, 2013.

[12] P. Darania and A. Ebadian, A method for the numerical solution of the integro-differential equations, Applied Mathematics and Computation, 188, 657–668, 2007.

[13] C. Ercan and T. Kharerah, T. Solving a class of Volterra integral system by the differential transform method, Int. J. Nonlinear Sci. 16, 87-91, 2013.

[14] A. A. James, G. Ajileye, A. M. Ayinde and W. Dunama, Hybrid-Block Method for the Solution of Second Order Non-linear Differential Equations, Journal of Advances in Mathematics and Computer Science. 37(12), 156-169, 2022. Article no.JAMCS.94828 ISSN: 2456-9968.

[15] S. B. G. Karakoc, A. Eryilmaz and M. Basbuk, The approximate solutions of Fredholm integro differential difference equations with variable coefficients via homotopy analysis method, Mathematical Problems in Engineering. 2013. Article ID: 261645.

[16] R. H. Khan and B. O. Bakodah, Adomian decomposition method and its modification for nonlinear Abel's integral equations, Computers and Mathematics with Applications, **7**, 2349-2358, 2013.

[17] G. Mehdiyeva, V. Ibrahimov and M. Imanova, On the Construction of the Multistep Methods to Solving the Initial-Value Problem for ODE and the Volterra Integro-Differential Equations, IAPE, Oxford, United Kingdom, 2019. ISBN: 978-1-912532-05-6.

[18] T. Oyedepo, A. A. Ayoade, G. Ajileye and N. J. Ikechukwu, Legendre computational algorithm for linear integro-differential equations, Cumhuriyet Science Journal 44(3), 561-566, 2023.

[19] T. Oyedepo, M. A. Ayinde, A. E. Adenipekun and G. Ajileye, Least-Squares Collocation Bernstein Method for Solving System of Linear Fractional Integro-diffrential Equations, International Journal of Computer Applications, 183(22), 0975 – 8887, 2021.

[20] T. Oyedepo, C. Y. Ishola, A. A. Ayoade and G. Ajileye, Collocation Computational Algorithm For Volterra-Fredholm Integro-Differential Equations, Electronic Journal of Mathematical Analysis and Applications, 11(2), 2023. ISSN. 8ISSN: 2090-729X (online) http://ejmaa.journals.ekb.eg/

[21] R. S. Palais. A simple proof of the Banach contraction principle, Journal of the Fixed Point Theory and Applications, 2, 221-223, 2007. doi:10.1007/s11784-007-0041-6.

[22] L. Rahmani, B. Rahimi, and M. Mordad, Numerical Solution of Volterra-Fredholm Integro-Differential Equation by Block Pulse Functions and Operational Matrices, Gen. Math. Notes, 4 (2), 7-48, 2011.

[23] V. Volterra, Theory of Functionals and of Integral and Integro-differential Equations, Dover Publications, 2005.

[24] J. Wei and T. Tian, Numerical solution of nonlinear Volterra integro-dooferential equation of fractional order by the reproducing kernel method. Applied Mathematical Modelling, 39, 4871-4876, 2015.

[25] C. Yang, and J. Hou, Numerical solution of Volterra integro-differential equations of fractional order by Laplace decomposition method. International Journal of Mathematical and Computational Sciences, 7(5), 863-867, 2013.

[26] L. Zada, M. Al-Hamami, R. Nawaz, S. Jehanzeb, A. Morsy, A. Abdel-Aty and K. S. Nisar, A New Approach for Solving Fredholm Integro-Differential Equations. Information Sciences Letters, 10 (3), 3, 2021.

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