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# RELATIONSHIP BETWEEN B-FUNCTION, K-B FUNCTION, AND EXTENDED K-B FUNCTION OF MATRIX ARGUMENTS

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Abstract. This paper focuses on the interrelationship between beta, k-beta, and extended k-beta function of the matrix argument. The study highlights these function's mathematical properties, functional relations, integral formula, integral representation, and interrelationships highlighting their applications and importance in the context of matrix argument. Additionally, we aim to create a stronger relation that enables the derivation of further results applicable in fractional calculus. Here in this paper, we applied some basic properties of beta and gamma functions to obtain new identities and relationships. We also used Laplace transform and the Legendre-duplication formulas to get new relations of beta, k-beta, and extended k-beta functions of matrix arguments. Of particular interest is the use of this transformation to produce identities that offer improved solutions to challenging issues about the beta function. We hope that this article will be helpful to the new researchers for their further research work and applications of fractional calculus.

## 1. Introduction

In the field of mathematics fractional calculus and its applications have gained excellent popularity over time. Special functions like beta, gamma, k-beta, and k-gamma and their extended form are being used to investigate and resolve challenging issues in the field of engineering, research, and statistics. These special functions are strong mathematical assets to deal with a variety of problems in different domains. Applications of k-beta and k-gamma functions and their extended forms are the main subject or research in pure statistics and mathematical analysis. Researchers are looking into the characteristics and possible potential applications

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of these special functions to improve our understanding and maximize their usage in resolving mathematical issues. Researchers aim to push the limits of mathematical study and open the door of opportunity for practical applications by the use of these special functions to insights into and solve complicated problems of science, engineering, and statistics. In fact, the special function of matrix argument is crucial in many branches of mathematics and theoretical physics. With the help of these functions various applications can be made in the area of statistics, group theory, number theory, and theoretical physics (see [8],[11],[20], [21] ). In the study of spherical function on specific symmetrical spaces and multivariate analysis in statistics, James A.T. [14] examined the special functions of matrix argument. Here in this paper, we investigated and applied certain properties of the beta and gamma functions of matrix argument (see [3],[4],[7],[11],[15],[18]). We introduced, investigated, and applied the applications of special functions like k-beta, extended k-beta k-gamma, and extended gamma function of matrix argument that may open new possibilities in mathematical research for new researchers (see [1],[13],[16],[17])

## 2. Beta Function

The beta function is a special function introduced by Leonhard Euler and Adrien-Marie Legendre. It is denoted by $B(z_1, z_2)$ and defined as

$$B(z_1, z_2) = \int_0^1 t^{z_1-1}(1-t)^{z_2-1}\, dt, \tag{1}$$

Where $z_1$ and $z_2$ are any complex number inputs such that $\mathfrak{Re}(z_1) > 0, \mathfrak{Re}(z_2) > 0$.

The beta function is symmetric in nature, that is

$$B(z_1, z_2) = B(z_2, z_1)$$

for all inputs $z_1$ and $z_2$.

The beta and gamma functions are related through the following relationship:

$$B(z_1, z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1 + z_2)} \tag{2}$$

Where $\Gamma(z)$ denotes the gamma function. The beta function $B(z_1, z_2)$ has an important role in various mathematical fields. Its role extends to integral calculus, special functions, and probability theory. Generally, it is used to compute various integrals and probability distributions.

The beta function is also closely related to binomial coefficients. When $z_1$, (or $z_2$, by symmetry) is a positive integer, then from the definition of the gamma function:

$$B(z_1, z_2) = \frac{(z_1 - 1)!(z_2 - 1)!}{(z_1 + z_2 - 1)!} \tag{3}$$

Furthermore, let $\mathcal{C}$ and $\mathcal{D}$ be the two square matrices in $\mathbb{C}^{m\times m}$ satisfying (11). Then the beta function $(\mathcal{C},\mathcal{D})$, of matrix argument([13],[15]) is defined by

$$B(\mathcal{C},\mathcal{D}) = \int_0^1 t^{\mathcal{C}-I}(1-t)^{\mathcal{D}-I}\,dt \tag{4}$$

If $\mathcal{C}$ and $\mathcal{D}$ are diagonalizable matrices in $\mathbb{C}^{m\times m}$, and $\mathcal{C}\mathcal{D}=\mathcal{D}\mathcal{C}$ [19], then

$$B(\mathcal{C},\mathcal{D}) = \Gamma(\mathcal{C})\Gamma(\mathcal{D})\Gamma^{-1}(\mathcal{C}+\mathcal{D}) \tag{5}$$

## 3. Gamma Function

The gamma function [15] is a mathematical function that extends the factorial function to complex and real numbers. It is denoted by the Greek letter gamma ($\Gamma$) and is defined by convergent improper integral as

$$\Gamma(n) = \int_0^\infty e^{-t}t^{n-1}\,dt \quad \text{for} \quad \mathfrak{Re}(n) > 0 \tag{6}$$

where $n$ is the complex or real number for which the gamma function is being evaluated. Additionally, it holds that:

$$\Gamma(n) = (n-1)! \quad \text{for all positive integers } n.$$

**Some Properties of Beta and Gamma Function [8]**

(i) $B(z_1,z_2) = B(z_2,z_1)$

(ii) $B(z_1,z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}$

(iii) $B(z_1,z_2) = 2\int_0^{\frac{\pi}{2}} \sin^{2z_1-1}\theta\cos^{2z_2-1}\theta\,d\theta, \quad \mathfrak{Re}(z_1)>0, \quad \mathfrak{Re}(z_2)>0.$

(iv) $\Gamma(z+1) = z\Gamma(z)$

(v) $\Gamma(1-z_1)\Gamma(z_1) = \frac{\pi}{\sin(\pi z_1)}, \quad z_1 \notin \mathbb{Z}$. (Euler's reflection formula.)

(vi) $\Gamma(n)\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2n-1}}\Gamma(2n), \quad n>0$. (Legendre Duplication formula.) [23]

(vii) $\Gamma(1-z_1)\Gamma(z_1) = B(z_1,1-z_1) = \int_0^1 \left(\frac{t}{1-t}\right)^{z_1-1}\frac{dt}{1-t}$
where the integral converges for $0<\mathfrak{Re}(z)<1$.

(viii) $\Gamma_0(\mathcal{C}) = \Gamma(\mathcal{C}), \quad \Gamma_{k,0}(\mathcal{C}) = \Gamma_k(\mathcal{C}), \quad \Gamma_{1,b}(\mathcal{C}) = \Gamma_b(\mathcal{C}),$

(ix) $B_0(\mathcal{C},\mathcal{D}) = B(\mathcal{C},\mathcal{D}), \quad B_{k,0}(\mathcal{C},\mathcal{D}) = B_k(\mathcal{C},\mathcal{D}), \quad B_{1,r}(\mathcal{C},\mathcal{D}) = B_r(\mathcal{C},\mathcal{D}).$
where $\mathcal{C}$ and $\mathcal{D}$ are matrices in $\mathbb{C}^{m\times m}$ satisfying (11), and $r\in\mathbb{R}_0^+$, $k\in\mathbb{R}^+$.

## 4. The Laplace Transform:

The Laplace transform [10],[9] is a mathematical operation that converts a function from the time domain to the frequency (or complex) domain. It is used in solving differential equations and analyzing systems with varying inputs or initial conditions.

The Laplace transform $\mathcal{L}\{f(t)\}$ of a continuous-time function $f(t)$, defined on an interval $[0,\infty)$, is given by:

$$\mathcal{L}[f(t)] = \mathcal{F}(s) = \int_0^{\infty} e^{-st} f(t)\, dt \tag{7}$$

provided the integral exists for the complex parameter $s$, which can be real or complex, and $f(t) = \mathcal{L}^{-1}[\mathcal{F}(s)]$ is called the inverse Laplace transform of $\mathcal{F}(s)$.

Additionally, $s = \sigma + i\omega$ is a Laplace parameter or complex frequency domain parameter where $\sigma, \omega \in \mathbb{R}$. The inverse Laplace transform takes a function in the $s$-domain back into the time domain.

The Laplace transform possesses various properties that make it useful for the solution of linear ordinary and partial differential equations, such as superposition, differentiation, integration, and the convolution theorem. Additionally, the Laplace transform provides insights about system stability and transfer functions in control theory.

**Properties of Laplace Transformation:**

i.If $s > 0$, then

$$\mathcal{L}[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$$

ii.

$$\mathcal{L}^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{\Gamma(n)}$$

**Convolution Theorem:** The Laplace convolution [10] of two functions $f(t)$ and $g(t)$ is defined on $[0,\infty)$ as follows:
If

$$\mathcal{L}[f(t)] = \mathcal{F}(s) \quad \text{and} \quad \mathcal{L}[g(t)] = \mathcal{G}(s).$$

$$\mathcal{L}^{-1}[\mathcal{F}(s)] = f(t) \quad \text{and} \quad \mathcal{L}^{-1}[\mathcal{G}(s)] = g(t).$$

Then,

$$\mathcal{L}[\mathcal{F}(s) * \mathcal{G}(s)] = \int_0^t f(x) g(t-x)\, dx = f(t) * g(t),$$

$$\int_0^t f(x) g(t-x)\, dx = \mathcal{L}^{-1}[\mathcal{L}\{\mathcal{F}(s)\} \cdot \mathcal{L}\{\mathcal{G}(s)\}]. \tag{8}$$

## 5. An Extension of the Beta and Gamma Functions of Matrix Arguments

In their research, Chaudhary and Zubir ([2],[6]) introduced the Gamma function as follows,

$$\Gamma_b(x) = \int_0^\infty t^{x-1} e^{-t-bt^{-1}}\, dt \quad \text{where} \quad (\mathfrak{Re}(x) > 0,\ \mathfrak{Re}(b) \geq 0) \tag{9}$$

Chaudhry et al. [5] have extended the beta function as follows,

$$B_b(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} \exp\left[-\frac{b}{t(1-t)}\right] dt \tag{10}$$

where $\mathfrak{Re}(x) > 0, \mathfrak{Re}(y) > 0.\mathfrak{Re}(b) \geq 0)$.

On the basis of the above definitions (9) and (10), we define the extension of beta function, the extended k-Beta function, the extension of gamma function, and the extension of k-Gamma functions.

Drawing inspiration from the definitions and concepts outlined by Ghazi and Praveen in their research ([11],[13],[16]) we consider the following:

For $n \in \mathbb{N}$, let $\mathbb{C}^{m\times m}$ represent the set of all square matrices of order $m$ with entries in complex numbers $\mathbb{C}$. Suppose $\rho(S)$ represents the set of all eigenvalues of $S \in \mathbb{C}^{n\times n}$.([11],[13]) For $S \in \mathbb{C}^{n\times n}$, let

$$\alpha(S) = \max\{\mathfrak{Re}(z) \mid z \in \rho(S)\} \quad \text{and} \quad \beta(S) = \min\{\mathfrak{Re}(z) \mid z \in \rho(S)\}.$$

$$\text{Let } S \text{ be the matrix in } \mathbb{C}^{m\times m} \text{ such that } \mathfrak{Re}(z) > 0 \text{ for } z \in \rho(S) \tag{11}$$

Here we are providing some basic concepts of beta, gamma, k-beta, and k-gamma functions and their extensions.

**The Extended Gamma function:** Let's assume $\mathcal{C}$ be a matrix such that $\mathcal{C} \in \mathbb{C}^{m\times m}$ satisfying (11), then the extension ([13],[16]) of the Gamma function of matrix argument can be defined as,

$$\Gamma_r(\mathcal{C}) = \int_0^\infty t^{\mathcal{C}-I} e^{(-t-rt^{-1})}\, dt \tag{12}$$

where $\mathfrak{Re}(\mathcal{C}) > 0$, $\mathfrak{Re}(r) \geq 0$.

**The Extended Beta function:** Let's assume $\mathcal{C}$ and $\mathcal{D}$ be the matrices such that $\mathcal{C}, \mathcal{D} \in \mathbb{C}^{m\times m}$ satisfying (11), then the extension [11] of the beta function of matrix argument can be defined as,

$$B_r(\mathcal{C},\mathcal{D}) = \int_0^1 t^{\mathcal{C}-I}(1-t)^{\mathcal{D}-I} \exp\left[-\frac{r}{t(1-t)}\right] dt \tag{13}$$

where $r \in \mathbb{R}_0^+$.

**The Extended $k$-Gamma function:** Let's assume $\mathcal{C}$ be a matrix such that $\mathcal{C} \in \mathbb{C}^{m\times m}$ satisfying (11), then we can define the extension [16] of the $k$-Gamma function of matrix argument as,

$$\Gamma_{k,r}(\mathcal{C}) = \int_0^\infty t^{\mathcal{C}-I} \exp\left(-\frac{t^k}{k} - \frac{r^k}{kt^k}\right) dt \tag{14}$$

where $k \in \mathbb{R}^+$, $r \in \mathbb{R}_0^+$.

**The Extended $k$-Beta function:** Let's assume $\mathcal{C}$ and $\mathcal{D}$ be the matrices such that $\mathcal{C}, \mathcal{D} \in \mathbb{C}^{m\times m}$ satisfying (11), then we can define the extension [13] of the beta function of matrix argument as,

$$B_{k,r}(\mathcal{C},\mathcal{D}) = \frac{1}{k}\int_0^1 t^{\mathcal{C}/k-I}(1-t)^{\mathcal{D}/k-I} \exp\left[-\frac{r^k}{kt(1-t)}\right] dt \tag{15}$$

where $r \in \mathbb{R}_0^+, k \in \mathbb{R}^+$.

## 6. Characteristic of Matrix Argument's Extended $k$-Gamma Function

Here we introduce some characteristics belonging to the matrix argument's extended $k$-Gamma function.

**Theorem 1:** Let $r \in \mathbb{R}_0^+$ and $\mathcal{C}$ and $\mathcal{D}$ are matrices belonging to $\mathbb{C}^{m\times m}$ satisfying the condition (11).([13],[16]) Then,

$$\Gamma_{k,r}(\mathcal{C}) \cdot \Gamma_{k,r}(\mathcal{D}) = \frac{1}{k}\int_0^\infty y^{\frac{\mathcal{C}+\mathcal{D}}{k}-I} e^{-\frac{y}{k}} B_{(k,\frac{r}{\sqrt[k]{y}})}(\mathcal{C},\mathcal{D})\, dy. \tag{16}$$

**Proof:** By the definition of the matrix argument's extended $k$-Gamma function, we have

$$\Gamma_{k,r}(\mathcal{C}) = \int_0^\infty t^{\mathcal{C}-I} \exp\left[-\frac{t^k}{k} - \frac{r^k}{kt^k}\right] dt, \quad k \in \mathbb{R}^+, \quad r \in \mathbb{R}_0^+.$$

Then,

$$\Gamma_{k,r}(\mathcal{C})\Gamma_{k,r}(\mathcal{D}) = \int_0^\infty\int_0^\infty u^{\mathcal{C}-I}v^{\mathcal{D}-I} \exp\left[-\frac{u^k}{k} - \frac{v^k}{k} - \frac{r^k}{ku^k} - \frac{r^k}{kv^k}\right] du\, dv.$$

Let, $x = \frac{u^k}{u^k+v^k}$ and $y = u^k + v^k$, the Jacobian is given by

$$\mathcal{J}(x,y) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

$$\mathcal{J}(x,y) = \frac{1}{k^2}.x^{\frac{1}{k}-1}(1-x)^{\frac{1}{k}-1}y^{\frac{2}{k}-1}.$$

Then,

$$\Gamma_{k,r}(\mathcal{C})\Gamma_{k,r}(\mathcal{D}) = \int_0^\infty\int_0^\infty u^{\mathcal{C}-I}v^{\mathcal{D}-I} \exp\left[-\frac{y}{k} - \frac{r^k}{kx}\right] \frac{1}{k}x^{\frac{1}{k}-1}(1-x)^{\frac{1}{k}-1}y^{\frac{2}{k}-1} dy\, dx.$$

After integration, we get the desired result as,

$$\Gamma_{k,r}(\mathcal{C})\Gamma_{k,r}(\mathcal{D}) = \frac{1}{k}\int_0^\infty y^{\frac{\mathcal{C}+\mathcal{D}}{k}-I} e^{-\frac{y}{k}} B_{(k,\frac{r}{\sqrt[k]{y}})}(\mathcal{C},\mathcal{D})\, dy.$$

**Corollary 1.1:** If we put $r = 0$ in the above, we get,

$$\Gamma_k(\mathcal{C})\Gamma_k(\mathcal{D}) = \Gamma_k(\mathcal{C}+\mathcal{D})B_k(\mathcal{C},\mathcal{D}) \tag{17}$$

**Corollary 1.2:** If we take $\mathcal{C} = \mathcal{D}$, then the above theorem reduces to,

$$\Gamma_{k,r}(\mathcal{C})\Gamma_{k,r}(\mathcal{C}) = \frac{1}{k}\int_0^\infty y^{\frac{\mathcal{C}+\mathcal{C}}{k}-I}e^{-\frac{y}{k}}B_{(k,\frac{r}{\sqrt[k]{y}})}(\mathcal{C},\mathcal{D})dy.$$

$$[\Gamma_{k,r}(\mathcal{C})]^2 = \frac{1}{k}\int_0^\infty y^{\frac{2\mathcal{C}}{k}-I}e^{-\frac{y}{k}}B_{(k,\frac{r}{\sqrt[k]{y}})}(\mathcal{C},\mathcal{D})\,dy.$$

Also, by applying the Legendre-Duplication formula [23], we have

$$B_{(k,\frac{r}{\sqrt[k]{y}})}(\mathcal{C},\mathcal{C}) = \frac{\sqrt{n}\Gamma_k(\mathcal{C})}{2^{(2\mathcal{C}-1)}\Gamma_k\left(\mathcal{C}+\frac{1}{2}\right)}, \quad n > 0.$$

Thus,

$$[\Gamma_{k,b}(\mathcal{C})]^2 = \frac{1}{k}\frac{\sqrt{n}\Gamma_k(\mathcal{C})}{2^{(2\mathcal{C}-1)}\Gamma_k\left(\mathcal{C}+\frac{1}{2}\right)}\int_0^\infty y^{\frac{2\mathcal{C}}{k}-I}e^{-\frac{y}{k}}dy. \tag{18}$$

**Corollary 1.3:** If we take $r = 0$ and $\mathcal{C} = \mathcal{D}$ in Corollary 1.2, we get,

$$\Gamma_k(\mathcal{C})\Gamma_k(\mathcal{C}) = \Gamma_k(\mathcal{C}+\mathcal{C})B_k(\mathcal{C},\mathcal{C}).$$

$$\Gamma_k(\mathcal{C})]^2 = \Gamma_k(2\mathcal{C})B_k(\mathcal{C},\mathcal{C}).$$

Also,

$$B_k(\mathcal{C},\mathcal{C}) = \frac{\sqrt{n}\Gamma_k(\mathcal{C})}{2^{(2\mathcal{C}-1)}\Gamma_k\left(\mathcal{C}+\frac{1}{2}\right)}, \quad n > 0.$$

Then,

$$[\Gamma_k(\mathcal{C})]^2 = \Gamma_k(2\mathcal{C})\cdot\frac{\sqrt{n}\Gamma_k(\mathcal{C})}{2^{(2\mathcal{C}-1)}\Gamma_k\left(\mathcal{C}+\frac{1}{2}\right)}$$

Consequently,

$$\Gamma_k(\mathcal{C})\Gamma_k\left(\mathcal{C}+\frac{1}{2}\right) = \frac{\sqrt{n}\Gamma_k(2\mathcal{C})}{2^{(2\mathcal{C}-1)}} \tag{19}$$

**Theorem 2:** Let's assume $r \in \mathbb{R}_0^+$, and also let $\mathcal{C}$ and $\mathcal{D}$ be the matrices in $\mathbb{C}^{m\times m}$ satisfying the condition (11). Then,

$$\Gamma_{k,r}(\mathcal{C}+\mathcal{D})\cdot\Gamma_{k,r}(\mathcal{C}-\mathcal{D}) = \frac{2}{k}\int_0^\infty p^{\frac{4\mathcal{C}}{k}-I}e^{-\frac{p^2}{k}}B_{(k,\frac{r}{\sqrt[k]{p^2}})}\left(\mathcal{C}+\mathcal{D},\mathcal{C}-\mathcal{D}\right)dp. \tag{20}$$

**Proof:** The matrix argument's $k$-gamma function [13] has been defined as,

$$\Gamma_{k,r}(\mathcal{C}) = \int_0^\infty t^{\mathcal{C}-I}\exp\left[-\frac{t^k}{k}-\frac{r^k}{kt^k}\right]dt, \quad k \in \mathbb{R}^+, \quad r \in \mathbb{R}_0^+.$$

Then,

$$\Gamma_{k,r}(\mathcal{C}+\mathcal{D}) = \int_0^\infty t^{(\mathcal{C}+\mathcal{D})-I}\exp\left[-\frac{t^k}{k}-\frac{r^k}{kt^k}\right]dt$$

Put $t = u^2$, in the above equation we have $dt = 2u\,du$, and the limits $t = 0$ and $t = \infty$ change to $u = 0$ and $u = \infty$ respectively.

Then, we write:

$$\Gamma_{k,r}(\mathcal{C}+\mathcal{D})\cdot\Gamma_{k,r}(\mathcal{C}-\mathcal{D}) = 4\int_0^\infty \int_0^\infty u^{2(\mathcal{C}+\mathcal{D})-2I} v^{2(\mathcal{C}-\mathcal{D})-2I} \exp\left[-\frac{u^{2k}}{k} - \frac{v^{2k}}{k} - \frac{r^k}{ku^{2k}} - \frac{r^k}{kv^{2k}}\right] u\,v\,du\,dv.$$

$$\Gamma_{k,r}(\mathcal{C}+\mathcal{D})\cdot\Gamma_{k,r}(\mathcal{C}-\mathcal{D}) = 4\int_0^\infty \int_0^\infty u^{2(\mathcal{C}+\mathcal{D})-I} v^{2(\mathcal{C}-\mathcal{D})-I} \exp\left[-\frac{u^{2k}}{k} - \frac{v^{2k}}{k} - \frac{r^k}{ku^{2k}} - \frac{r^k}{kv^{2k}}\right] du\,dv.$$

Let $u = (p\cdot\cos\theta)^{\frac{1}{k}}$ and $v = (p\cdot\sin\theta)^{\frac{1}{k}}$, then the Jacobian $\mathcal{J}$ is given by,

$$\mathcal{J} = \frac{p^{\frac{2}{k}-1}(\sin\theta\cos\theta)^{\frac{1}{k}-1}}{k^2} \tag{21}$$

Here we have

$$\Gamma_{k,r}(\mathcal{C}+\mathcal{D})\cdot\Gamma_{k,r}(\mathcal{C}-\mathcal{D}) = \frac{4}{k^2}\int_0^\infty p^{\frac{2(\mathcal{C}+\mathcal{D})-I}{k}}\cdot p^{\frac{2(\mathcal{C}-\mathcal{D})-I}{k}}\cdot p^{\frac{2}{k}-I} e^{-\frac{p^2}{k}}\,dp$$

$$\cdot\int_0^{\frac{\pi}{2}} (\cos\theta)^{\frac{2(\mathcal{C}+\mathcal{D})-I}{k}} (\sin\theta)^{\frac{2(\mathcal{C}-\mathcal{D})-I}{k}} e^{\left(\frac{-r^k}{kp^2(\sin\theta\cos\theta)^2}\right)}\cdot(\sin\theta\cos\theta)^{\frac{1}{k}-1}\,d\theta.$$

.

$$= \frac{4}{k^2}\int_0^\infty p^{\frac{(2(\mathcal{C}+\mathcal{D})-I)+2(\mathcal{C}-\mathcal{D})-I+2I-k}{k}} e^{-\frac{p^2}{k}}\,dp$$

$$\cdot\int_0^{\frac{\pi}{2}} (\cos\theta)^{\frac{2(\mathcal{C}+\mathcal{D})-I}{k}} (\sin\theta)^{\frac{2(\mathcal{C}-\mathcal{D})-I}{k}} e^{\left(\frac{-r^k}{kp^2(\sin\theta\cos\theta)^2}\right)}\cdot(\sin\theta\cos\theta)^{\frac{1}{k}-1}\,d\theta$$

$$= 2\cdot\frac{1}{k}\int_0^\infty p^{\frac{4\mathcal{C}-2I+2I}{k}-I} e^{-\frac{p^2}{k}}\,dp\cdot$$

$$2\cdot\frac{1}{k}\int_0^{\frac{\pi}{2}} (\cos\theta)^{\frac{2(\mathcal{C}+\mathcal{D})}{k}-I} (\sin\theta)^{\frac{2(\mathcal{C}-\mathcal{D})}{k}-I} e^{\left(\frac{-r^k}{kp^2(\sin\theta\cos\theta)^2}\right)}\,d\theta$$

$$= 2\cdot\frac{1}{k}\int_0^\infty p^{\frac{4\mathcal{C}}{k}-I} e^{-\frac{p^2}{k}}\,dp\cdot$$

$$2\cdot\frac{1}{k}\int_0^{\frac{\pi}{2}} (\cos\theta)^{\frac{2(\mathcal{C}+\mathcal{D})}{k}-I} (\sin\theta)^{\frac{2(\mathcal{C}-\mathcal{D})}{k}-I} e^{\left(\frac{-r^k}{kp^2(\sin\theta\cos\theta)^2}\right)}\,d\theta$$

$$= 2\cdot\frac{1}{k}\int_0^\infty p^{\frac{4\mathcal{C}}{k}-I} e^{-\frac{p^2}{k}}\,dp\cdot$$

$$2\cdot\frac{1}{k}\int_0^{\frac{\pi}{2}} (\cos\theta)^{\frac{2(\mathcal{C}+\mathcal{D})}{k}-I} (\sin\theta)^{\frac{2(\mathcal{C}-\mathcal{D})}{k}-I} e^{\left(\frac{-a^k}{k(\sin\theta\cos\theta)^2}\right)}\,d\theta$$

Where,

$$a^k = \frac{r^k}{\left(p^{2/k}\right)^k}$$

or

$$a = \frac{r}{k\sqrt{p^2}}$$

$$\Gamma_{k,r}(\mathcal{C}+\mathcal{D})\Gamma_{k,r}(\mathcal{C}-\mathcal{D}) = \frac{2}{k}\int_0^\infty p^{\frac{4C}{k}-I} e^{-\frac{p^2}{k}} B_{(k,\frac{r}{\sqrt[k]{p^2}})}(\mathcal{C}+\mathcal{D},\mathcal{C}-\mathcal{D})\,dp.$$

Proved.

## 7. Characteristic of Matrix Argument's Extended k-Beta Function:

Here we introduce some characteristics of the matrix argument's beta function, extended beta, and extended k-beta function.

**Theorem 3:** Let $k \in \mathbb{R}^+$, $r \in \mathbb{R}_0^+$, and let $\mathcal{C}$ and $\mathcal{D}$ be matrices in $\mathbb{C}^{m\times m}$ satisfying the condition (11). Then, the matrix argument's beta function can be expressed as:

$$B_k(\mathcal{C},\mathcal{D}) = \frac{1}{k}\int_0^t t^{\frac{\mathcal{C}}{k}-I}(1-t)^{\frac{\mathcal{D}}{k}-I}dt. \quad (k \in \mathbb{R}^+)$$

Then prove that

$$B_k(\mathcal{C},\mathcal{D}) = \frac{1}{k}B\left(\frac{\mathcal{C}}{k},\frac{\mathcal{D}}{k}\right) \tag{22}$$

Proof: The matrix argument's k-beta function has been expended as, [13]

$$B_k(\mathcal{C},\mathcal{D}) = \frac{1}{k}\int_0^t t^{\frac{\mathcal{C}}{k}-I}(1-t)^{\frac{\mathcal{D}}{k}-I}dt. \quad (k \in \mathbb{R}^+)$$

Let $t = \frac{y}{n}$, then $dt = \frac{dy}{n}$. When $t = 0$, $y = 0$ and when $t = 1$, $y = n$.

Thus,

$$B_k(\mathcal{C},\mathcal{D}) = \frac{1}{k}\int_0^n \left(\frac{y}{n}\right)^{\frac{\mathcal{C}}{k}-I}(1-\frac{y}{n})^{\frac{\mathcal{D}}{k}-I}\frac{dy}{n}$$

$$B_k(\mathcal{C},\mathcal{D}) = \frac{1}{k}\frac{1}{n^{\frac{\mathcal{C}+\mathcal{D}}{k}-2I+I}}\int_0^n (y)^{\frac{\mathcal{C}}{k}-I}(n-y)^{\frac{\mathcal{D}}{k}-I}dy$$

By the Laplace convolution theorem [9],

$$B_k(\mathcal{C},\mathcal{D}) = \frac{1}{k}\frac{1}{n^{\frac{\mathcal{C}+\mathcal{D}}{k}-I}}L^{-1}\left[L\left\{n^{(\frac{\mathcal{C}}{k}-I)}\right\} * L\left\{n^{(\frac{\mathcal{D}}{k}-I)}\right]\right.$$

$$B_k(\mathcal{C},\mathcal{D}) = \frac{1}{k}\frac{1}{n^{\frac{\mathcal{C}+\mathcal{D}}{k}-I}}L^{-1}\left[\frac{\Gamma(\frac{C}{k}-I+I)\Gamma(\frac{D}{k}-I+I)}{s^{\frac{\mathcal{C}}{k}-I+I}s^{\frac{D}{k}-I+I}}\right]$$

$$B_k(\mathcal{C},\mathcal{D}) = \frac{1}{k}\frac{1}{n^{\frac{\mathcal{C}+\mathcal{D}}{k}-I}}\cdot\Gamma(\frac{C}{k})\cdot\Gamma(\frac{D}{k})L^{-1}\left[\frac{1}{s^{\frac{\mathcal{C}+\mathcal{D}}{k}}}\right]$$

$$B_k(\mathcal{C},\mathcal{D}) = \frac{1}{k}\frac{1}{n^{\frac{\mathcal{C}+\mathcal{D}}{k}-I}} \cdot \Gamma(\frac{C}{k}) \cdot \Gamma(\frac{D}{k}) \cdot \frac{n^{\frac{\mathcal{C}+\mathcal{D}}{k}-I}}{\Gamma\frac{(\mathcal{C}+\mathcal{D})}{k}}$$

$$B_k(\mathcal{C},\mathcal{D}) = \frac{1}{k}\Gamma(\frac{C}{k}) \cdot \Gamma(\frac{D}{k}) \cdot \frac{1}{\Gamma\frac{(\mathcal{C}+\mathcal{D})}{k}}$$

$$B_k(\mathcal{C},\mathcal{D}) = \frac{1}{k}B\left(\frac{\mathcal{C}}{k},\frac{\mathcal{D}}{k}\right)$$

Proved.

**Theorem 4:** Let $r \in \mathbb{R}_0^+$ and let $\mathcal{C}$ and $\mathcal{D}$ be the matrices in $\mathbb{C}^{m\times m}$ satisfying the condition (11), then

$$B_r(\mathcal{C},\mathcal{D}) = 2\int_0^{\frac{\pi}{2}} \sin\theta^{2C-I}\cos\theta^{2D-I}\exp\left[-\frac{r}{(\sin\theta\cos\theta)^2}\right]d\theta. \tag{23}$$

And

$$\Gamma_r(\mathcal{C})\Gamma_r(\mathcal{D}) = 2\int_0^{\infty} x^{2(\mathcal{C}+\mathcal{D})-I}e^{-x^2}B_{r/x}(\mathcal{C},\mathcal{D})\,dx. \tag{24}$$

**Proof:** The matrix argument's beta function has been expended as [19],

$B_r(\mathcal{C},\mathcal{D}) = \int_0^1 t^{C-I}\ (1-t)^{D-I}exp\left[-\frac{r}{t(1-t)}\right]dt$, where $r \in \mathbb{R}_0^+$.

Replacing $t = \sin^2\theta$, then $dt = 2\sin\theta\cos\theta\,d\theta$. Also, when $t = 0, \theta = 0$ and when $t = 1, \theta = \frac{\pi}{2}$, we have:

$$B_r(\mathcal{C},\mathcal{D}) = \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{(C-I)}(1-\sin^2\theta)^{(D-I)}exp\left[-\frac{r}{\sin^2\theta(1-\sin^2\theta)}\right]2\sin\theta\cos\theta\,d\theta.$$

$$B_r(\mathcal{C},\mathcal{D}) = 2\int_0^{\frac{\pi}{2}} (\sin^2\theta)^{(C-I)}(1-\sin^2\theta)^{(D-I)}\sin\theta\cos\theta\exp\left[-\frac{r}{\sin^2\theta(1-\sin^2\theta)}\right]d\theta.$$

$$B_r(\mathcal{C},\mathcal{D}) = 2\int_0^{\frac{\pi}{2}} \sin\theta^{2C-I}\cos\theta^{2D-I}\exp\left[-\frac{r}{(\sin\theta\cos\theta)^2}\right]d\theta.$$

Now, we know that the extended gamma function [2], $\Gamma_r(\mathcal{C})$ is given by

$$\Gamma_r(\mathcal{C}) = \int_0^{\infty} t^{C-I}\,e^{-t-bt^{-1}}\,dt$$

where $r \in \mathbb{R}_0^+$.

Putting $t = u^2$ in the above equation, we have $dt = 2u\,du$. When $t = 0$, $u = 0$ and when $t = \infty$, $u = \infty$. Thus,

$$\Gamma_r(\mathcal{C}) = \int_0^{\infty} u^{2C-2I}e^{-u^2-\frac{r}{u^2}}2u\,du$$

$$\Gamma_r(\mathcal{C}) = 2\int_0^{\infty} u^{2C-I}e^{-u^2-\frac{r}{u^2}}\,du$$

Similarly, for $\Gamma_r(\mathcal{D})$:

$$\Gamma_r(\mathcal{D}) = 2\int_0^\infty v^{2D-I} e^{-v^2-\frac{r}{v^2}}\, dv.$$

Then, we have

$$\Gamma_r(\mathcal{C})\Gamma_r(\mathcal{D}) = 4\int_0^\infty \int_0^\infty u^{2\mathcal{C}-I} v^{2\mathcal{D}-I}\, e^{-u^2-\frac{r}{u^2}} e^{-v^2-\frac{r}{v^2}}\, du\, dv.$$

$$\Gamma_r(\mathcal{C})\Gamma_r(\mathcal{D}) = 4\int_0^\infty \int_0^\infty u^{2\mathcal{C}-I} v^{2\mathcal{D}-I}\, e^{-u^2-v^2-\frac{r}{u^2}-\frac{r}{v^2}}\, du\, dv. \tag{25}$$

Let $u = x \cdot \cos\theta$ and $v = x \cdot \sin\theta$. Then, we have:

$$\frac{\partial u}{\partial x} = \cos\theta \quad \text{and} \quad \frac{\partial v}{\partial x} = \sin\theta.$$

$$\frac{\partial u}{\partial \theta} = -x \cdot \sin\theta \quad \text{and} \quad \frac{\partial v}{\partial \theta} = x \cdot \cos\theta.$$

The value of the Jacobian $\mathcal{J}$ is given by:

$$\mathcal{J} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial \theta} & \frac{\partial v}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -x\cdot\sin\theta & x\cdot\cos\theta \end{vmatrix} = x. \tag{26}$$

Now, we have:

$$\Gamma_r(\mathcal{C})\Gamma_r(\mathcal{D}) = 4\int_0^\infty \int_0^{\frac{\pi}{2}} x^{2\mathcal{C}-I} \cos\theta^{2\mathcal{C}-I}\, x^{2\mathcal{D}-I} \sin\theta^{2\mathcal{D}-I}\, exp\left[-x^2 - \frac{r}{x^2\cos^2\theta} - \frac{r}{x^2\sin^2\theta}\right] x\, dx\, d\theta.$$

$$\Gamma_r(\mathcal{C})\Gamma_r(\mathcal{D}) = 2\int_0^\infty x^{2(\mathcal{C}+\mathcal{D})-I} e^{-x^2}\, dx \cdot 2\int_0^{\frac{\pi}{2}} \cos\theta^{2\mathcal{C}-I}\, \sin\theta^{2\mathcal{D}-I}\, exp\left[-\frac{r}{x^2(\cos\theta\sin\theta)^2}\right] d\theta.$$

Therefore, we have:

$$\Gamma_r(\mathcal{C})\Gamma_r(\mathcal{D}) = 2\int_0^\infty x^{2(\mathcal{C}+\mathcal{D})-I} e^{-x^2} B_{r/x}(\mathcal{D},\mathcal{C})\, dx.$$

and since $\mathcal{B}(m,n) = \mathcal{B}(n,m)$,

$$\Gamma_r(\mathcal{C})\Gamma_r(\mathcal{D}) = 2\int_0^\infty x^{2(\mathcal{C}+\mathcal{D})-I} e^{-x^2} B_{r/x}(\mathcal{C},\mathcal{D})\, dx.$$

proved.

**Corollary 4.1:** If $\mathcal{C} = \mathcal{D}$ then,

$$\Gamma_r(\mathcal{C})\Gamma_r(\mathcal{C}) = 2\int_0^\infty x^{2(\mathcal{C}+\mathcal{C})-I} e^{-x^2} \cdot B_{r/x}(\mathcal{C},\mathcal{C})\, dx,$$

$$[\Gamma_r(\mathcal{C})]^2 = 2\int_0^\infty x^{4\mathcal{C}-I} e^{-x^2} \cdot B_{r/x}(\mathcal{C},\mathcal{C})\, dx. \tag{27}$$

By the Legendre-Duplication formula [23] we have:

$$B_{r/x}(\mathcal{C},\mathcal{C}) = \frac{\sqrt{n}\,\Gamma(\mathcal{C})}{2^{2\mathcal{C}-1}\,\Gamma\left(\mathcal{C}+\frac{1}{2}\right)}, \quad n>0. \tag{28}$$

Thus,

$$[\Gamma_r(\mathcal{C})]^2 = \frac{\sqrt{n}\,\Gamma(\mathcal{C})}{2^{2\mathcal{C}-2}\,\Gamma\left(\mathcal{C}+\frac{1}{2}\right)} \int_0^\infty x^{4\mathcal{C}-I} e^{-x^2}\,dx. \tag{29}$$

**Corollary 4.2:** Also, we have the extension of the k-beta function as [16]
If $k \in \mathbb{R}^+$, $r \in \mathbb{R}_0^+$, Let $\mathcal{C}$ and $\mathcal{D}$ are matrices in $\mathbb{C}^{m\times m}$ satisfying the condition (11), then

$$B_{k,r}(\mathcal{C},\mathcal{D}) = \frac{2}{k}\int_0^{\frac{\pi}{2}} \sin\theta^{\frac{2C}{k}-I}\cos\theta^{\frac{2D}{k}-I}\, exp\left[-\frac{r^k}{k(\sin\theta\cos\theta)^2}\right] d\theta.$$

Putting $r=0$ in the above equation, we get:

$$B_{k,0}(\mathcal{C},\mathcal{D}) = \frac{2}{k}\int_0^{\frac{\pi}{2}} \sin\theta^{\frac{2C}{k}-I}\cos\theta^{\frac{2D}{k}-I}\, d\theta.$$

$$B_{k,0}(\mathcal{C},\mathcal{D}) = \frac{2}{k}\cdot\frac{\Gamma\left(\frac{2\mathcal{C}}{2k}\right)\Gamma\left(\frac{2\mathcal{D}}{2k}\right)}{2\Gamma\left(\frac{2\mathcal{C}+2\mathcal{D}}{2k}\right)}.$$

Therefore,

$$B_k\mathcal{C},\mathcal{D}) = \frac{1}{k}B\left(\frac{\mathcal{C}}{k},\frac{\mathcal{D}}{k}\right). \tag{30}$$

Since

$$B_{k,0}(\mathcal{C},\mathcal{D}) = B_k(\mathcal{C},\mathcal{D})$$

This (30) is the desired relation between the beta and extended beta functions.

**Theorem 5:** Let $r \in \mathbb{R}_0^+$. Also, let $\mathcal{C}$ and $\mathcal{D}$ be the matrices in $\mathbb{C}^{m\times m}$ such that $\mathcal{C},\mathcal{D},\mathcal{C}+\mathcal{D}$, and $\mathcal{C}-\mathcal{D}$ satisfy the condition (11) then,

$$B_r(\mathcal{C},\mathcal{D}+I) + B_r(\mathcal{C}+I,\mathcal{D}) = B_r(\mathcal{C},\mathcal{D}). \tag{31}$$

**Proof:** The matrix argument's beta function has been expanded as follows:

$$B_r(\mathcal{C},\mathcal{D}) = \int_0^1 t^{\mathcal{C}-I}(1-t)^{\mathcal{D}-I}\exp\left[-\frac{r}{t(1-t)}\right]dt.$$

where $r \in \mathbb{R}_0^+$.

Then,

$$B_r(\mathcal{C},\mathcal{D}+I) = \int_0^1 t^{\mathcal{C}-I}(1-t)^{\mathcal{D}+I-I}\exp\left[-\frac{r}{t(1-t)}\right]dt,$$

and

$$B_r(\mathcal{C}+I,\mathcal{D}) = \int_0^1 t^{\mathcal{C}+I-I}(1-t)^{\mathcal{D}-I} \exp\left[-\frac{r}{t(1-t)}\right] dt.$$

Then,

$$B_r(\mathcal{C},\mathcal{D}+I)+B_r(\mathcal{C}+I,\mathcal{D}) = \int_0^1 \left\{t^{\mathcal{C}-I}(1-t)^{\mathcal{D}} + t^{\mathcal{C}}(1-t)^{\mathcal{D}-I}\right\} \exp\left[-\frac{r}{t(1-t)}\right] dt.$$

$$B_r(\mathcal{C},\mathcal{D}+I)+B_r(\mathcal{C}+I,\mathcal{D}) = \int_0^1 \left\{t^{\mathcal{C}-I}(1-t)^{\mathcal{D}-I}(1-t) + t^{\mathcal{C}-I}(1-t)^{\mathcal{D}-I}t\right\} \exp\left[-\frac{r}{t(1-t)}\right] dt.$$

$$B_r(\mathcal{C},\mathcal{D}+I)+B_r(\mathcal{C}+I,\mathcal{D}) = \int_0^1 t^{\mathcal{C}-I}(1-t)^{\mathcal{D}-I}\{1-t+t\} \exp\left[-\frac{r}{t(1-t)}\right] dt.$$

Thus, we have:

$$B_r(\mathcal{C},\mathcal{D}+I)+B_r(\mathcal{C}+I,\mathcal{D}) = \int_0^1 t^{\mathcal{C}-I}(1-t)^{\mathcal{D}-I} \exp\left[-\frac{r}{t(1-t)}\right] dt,$$

$$B_r(\mathcal{C},\mathcal{D}+I)+B_r(\mathcal{C}+I,\mathcal{D}) = B_r(\mathcal{C},\mathcal{D}).$$

Proved.

**Corollary 5.1:** If $\mathcal{C}=\mathcal{D}$, then the above theorem reduces to

$$B_r(\mathcal{C},\mathcal{C}+I)+B_r(\mathcal{C}+I,\mathcal{C}) = B_r(\mathcal{C},\mathcal{C}).$$

Also, by the property of the beta function [12], $B(m,n)=B(n,m)$, we have

$$2B_r(\mathcal{C},\mathcal{C}+I) = B_r(\mathcal{C},\mathcal{C}) \tag{32}$$

By the Legendre duplication formula [23],

$$B_r(\mathcal{C},\mathcal{C}) = \frac{\sqrt{n}\Gamma_r(\mathcal{C})}{2^{2C-I}\Gamma_r\left(\mathcal{C}+\frac{1}{2}\right)}, \quad n>0,$$

Then,

$$2B_r(\mathcal{C}+I,\mathcal{C}) = \frac{\sqrt{n}\Gamma_r(\mathcal{C})}{2^{2C-I}\Gamma_r\left(\mathcal{C}+\frac{1}{2}\right)}, \quad n>0, \tag{33}$$

Also, by the property of the beta function, $B(m,n)=B(n,m)$, we have

$$B_r(\mathcal{C}+I,\mathcal{C}) = \frac{1}{2}\cdot\frac{\sqrt{n}\Gamma_r(\mathcal{C})}{2^{2C-I}\Gamma_r\left(\mathcal{C}+\frac{1}{2}\right)}, \quad n>0,$$

$$B_r(\mathcal{C}+I,\mathcal{C}) = \frac{\sqrt{n}\Gamma_r(\mathcal{C})}{2^{2C}\Gamma_r\left(\mathcal{C}+\frac{1}{2}\right)}, \quad n>0, \tag{34}$$

**Theorem 6:** Let $r\in\mathbb{R}_0^+$, and let $\mathcal{C}$ and $\mathcal{D}$ be matrices in $\mathbb{C}^{m\times m}$ such that $\mathcal{C},\mathcal{D},\mathcal{C}+\mathcal{D}$, and $\mathcal{C}-\mathcal{D}$ satisfy the condition (11), and $I$ is the identity matrix in $\mathbb{C}^{m\times m}$. Then the extended beta function of matrix arguments is given by

$$B_r(\mathcal{C},\mathcal{D}) = \sum_{n=0}^{\infty} B_r(\mathcal{C}+nI,\mathcal{D}+I). \tag{35}$$

**Proof:** From the definition of the extended beta function, we have

$$B_r(\mathcal{C},\mathcal{D})=\int_0^1 t^{\mathcal{C}-I}(1-t)^{\mathcal{D}-I}\exp\left[-\frac{r}{t(1-t)}\right]dt,$$

where $r\in\mathbb{R}_0^+$.

$$B_r(\mathcal{C},\mathcal{D})=\int_0^1 t^{\mathcal{C}-I}(1-t)^{\mathcal{D}}(1-t)^{-1}\exp\left[-\frac{r}{t(1-t)}\right]dt,$$

Using the matrix identity $(1-t)^{-I}=\sum_{n=0}^{\infty}t^{nI}$ for $|t|<1$, then

$$B_r(\mathcal{C},\mathcal{D})=\int_0^1 t^{\mathcal{C}-I}(1-t)^{\mathcal{D}}\sum_{n=0}^{\infty}t^{nI}\exp\left[-\frac{r}{t(1-t)}\right]dt.$$

By interchanging the order of integration and summation, we obtain:

$$B_r(\mathcal{C},\mathcal{D})=\sum_{n=0}^{\infty}\int_0^1 t^{\mathcal{C}+nI-I}(1-t)^{\mathcal{D}+I-I}\exp\left[-\frac{r}{t(1-t)}\right]dt.$$

Then

$$B_r(\mathcal{C},\mathcal{D})=\sum_{n=0}^{\infty}B_r(\mathcal{C}+nI,\mathcal{D}+I).$$

Proved.

**Theorem 7:** Let $k\in\mathbb{R}^+$ and $r\in\mathbb{R}_0^+$. Also, let $\mathcal{C}$ and $\mathcal{D}$ be matrices in $\mathbb{C}^{m\times m}$ such that $\mathcal{C},\mathcal{D},\mathcal{C}+I,\mathcal{D}+I$ satisfy the condition (11), and let $I$ be the identity matrix in $\mathbb{C}^{m\times m}$. Then the extended $k$-beta function is given by

$$B_{k,r}(\mathcal{C},\mathcal{D})=\sum_{n=0}^{\infty}B_{k,r}\left(\mathcal{C}+nkI,\mathcal{D}+kI\right). \tag{36}$$

**Proof:** According to the definition of the extended k-beta function, we have

$$B_{k,r}(\mathcal{C},\mathcal{D})=\frac{1}{k}\int_0^1 t^{\frac{\mathcal{C}}{k}-I}(1-t)^{\frac{\mathcal{D}}{k}-I}\exp\left[-\frac{r^k}{kt(1-t)}\right]dt,$$

where $r\in\mathbb{R}_0^+$ and $k\in\mathbb{R}^+$.

$$B_{k,r}(\mathcal{C},\mathcal{D})=\frac{1}{k}\int_0^1 t^{\frac{\mathcal{C}}{k}-I}(1-t)^{\frac{\mathcal{D}}{k}}(1-t)^{-I}\exp\left[-\frac{r^k}{kt(1-t)}\right]dt.$$

Using the identity $(1-t)^{-I}=\sum_{n=0}^{\infty}t^nI$ (valid for $|t|<1$), we have:

$$B_{k,r}(\mathcal{C},\mathcal{D})=\frac{1}{k}\int_0^1 t^{\frac{\mathcal{C}}{k}-I}(1-t)^{\frac{\mathcal{D}}{k}}\sum_{n=0}^{\infty}t^{nI}\exp\left[-\frac{r^k}{kt(1-t)}\right]dt.$$

Through the manipulation of the sequences of integration, summation, and term rearrangement, we get:

$$B_{k,r}(\mathcal{C},\mathcal{D}) = \sum_{n=0}^{\infty} \frac{1}{k} \int_0^1 t^{\frac{\mathcal{C}}{k}+nI-I} (1-t)^{\frac{\mathcal{D}}{k}+I-I} \exp\left[-\frac{r^k}{kt(1-t)}\right] dt.$$

Thus, we conclude:

$$B_{k,r}(\mathcal{C},\mathcal{D}) = \sum_{n=0}^{\infty} B_{k,r}(\mathcal{C}+nkI, \mathcal{D}+kI).$$

Proved.

**Theorem 8:** Let $r \in \mathbb{R}_0^+$. Also, let $\mathcal{C}$ and $\mathcal{D}$ be the matrices in $\mathbb{C}^{m\times m}$ such that $\mathcal{C}, \mathcal{D}$, and $I-\mathcal{D}$ satisfy the condition (11), and $I$ is the identity matrix in $\mathbb{C}^{m\times m}$. Then,

$$B_r(\mathcal{C}, I-\mathcal{D}) = \sum_{n=0}^{\infty} \frac{(\mathcal{D})_n}{n!} B_r(\mathcal{C}, I) \tag{37}$$

**Proof:** By the definition of the extended beta function by Ghazi et al. [13], the k-beta function of matrix argument can be defined as follows:

$$B_r(\mathcal{C},\mathcal{D}) = \sum_{n=0}^{\infty} \int_0^1 t^{\mathcal{C}-I}(1-t)^{\mathcal{D}-I} \exp\left[-\frac{r}{t(1-t)}\right] dt. \quad (r \in \mathbb{R}_0^+).$$

Then,

$$B_r(\mathcal{C}, I-\mathcal{D}) = \sum_{n=0}^{\infty} \int_0^1 t^{\mathcal{C}-I}(1-t)^{I-D-I} \exp\left[-\frac{r}{t(1-t)}\right] dt.$$

$$B_r(\mathcal{C}, I-\mathcal{D}) = \int_0^1 t^{\mathcal{C}-I}(1-t)^{-\mathcal{D}} \exp\left[-\frac{r}{t(1-t)}\right] dt.$$

Now, by using the matrix identity,

$$(1-t)^{-\mathcal{D}} = \sum_{n=0}^{\infty} \frac{(\mathcal{D})_n}{n!},$$

we have,

$$B_r(\mathcal{C}, I-\mathcal{D}) = \int_0^1 t^{\mathcal{C}-I} \sum_{n=0}^{\infty} \frac{(\mathcal{D})_n}{n!} \exp\left[-\frac{r}{t(1-t)}\right] dt.$$

Through the manipulation of the sequences of integration, summation, and term rearrangement, we obtain

$$B_r(\mathcal{C}, I-\mathcal{D}) = \sum_{n=0}^{\infty} \frac{(\mathcal{D})_n}{n!} \int_0^1 t^{\mathcal{C}-I}(1-t)^{I-I} \exp\left[-\frac{r}{t(1-t)}\right] dt.$$

Thus,

$$B_r(\mathcal{C}, I-\mathcal{D}) = \sum_{n=0}^{\infty} \frac{(\mathcal{D})^n}{n!} B_r(\mathcal{C}, I)$$

Proved.

## 8. Conclusion

: In conclusion, this paper extensively explores the various expressions of beta and gamma functions of matrix arguments. By applying multiple properties of these functions, we have discovered and proven numerous results that have considerable value in addressing novel problems and paving the way for future researchers. Our investigation also encompasses important properties of these extended functions, including integral relationships, transforms, and representations. We have successfully established a fresh connection between the beta and gamma functions by leveraging the Laplace transformation and convolution formula. We hope these findings will find practical utility across diverse domains such as statistics, number theory, and more. Through these efforts, we offer valuable insights and contribute to further expanding the field of study in this area.

## 9. Suggestion

We believe that these new results can now be used as a way of proving various useful results in a more general context.

## References

[1] Abdalla, M., Boulaaras, S., Akel, M., Idris, S. A., and Jain, S. (2021). Certain fractional formulas of the extended k-hypergeometric functions. Advances in Difference Equations, 2021, 1-10.

[2] Altın, A., Özergin, E., and Özarslan, M. A. (2011). Extension of gamma, beta, and hypergeometric functions. Journal of Computational and Applied Mathematics, 235(16), 4601-4610.

[3] Aomoto, K., Kita, M., Kohno, T., and Iohara, K. (2011). Theory of hypergeometric functions. Tokyo: Springer. Springer Monographs in Mathematics.

[4] Butler, R. W., and Wood, A. T. (2002). Laplace approximations for hypergeometric functions with matrix argument. The Annals of Statistics, 30(4), 1155-1177.

[5] Chaudhry, M. A., Qadir, A., Rafique, M., and Zubair, S. M. (1997). Extension of Euler's beta function. Journal of computational and applied mathematics, 78(1), 19-32.

[6] Chaudhry, M. A., and Zubair, S. M. (1994). Generalized incomplete gamma functions with applications. Journal of Computational and Applied Mathematics, 55(1), 99-124.

[7] Choi, J., and Srivastava, H. M. (2009). Integral representations for the Gamma function, the Beta function, and the Double Gamma function. Integral Transforms and Special Functions, 20(11), 859-869.

[8] Constantine, A. G., and Muirhead, R. J. (1972). Partial differential equations for hypergeometric functions of two argument matrices. Journal of Multivariate Analysis, 2(3), 332-338.

[9] Davies, B., and Martin, B. (1979). Numerical inversion of the Laplace transform: a survey and comparison of methods. Journal of computational physics, 33(1), 1-32.

[10] Feng, Q., and Yuan, S. (2023). The explicit solutions for a class of fractional Fourier-Laplace convolution

[11] Gross, K. I., and Richards, D. S. P. (1987). Special functions of matrix arguments. I. Algebraic induction, zonal polynomials, and hypergeometric functions. Transactions of the American Mathematical Society, 301(2), 781-811.

[12] Dufresne, D. (1998). Algebraic properties of beta and gamma distributions and applications. Advances in Applied Mathematics, 20(3), 285-299.

[13] Gazi S Khammash, Praveen Agarwal, and June Sang Choi. Extended K-Gamma and K-Beta Functions of Matrix Arguments Mathematics 2020, 8, 1715: DOI: 10.3390/math8101715.

[14] James, A.T. (1975). Special functions of matrix and single argument in statistics. In Theory and Applications of Special Functions; Askey, R.A., Ed.; Academic Press: New York, NY, USA, 1975; pp. 497–520.

[15] Jodar, L., and Cortés, J. C. (1998). Some properties of Gamma and Beta matrix functions. Applied Mathematics Letters, 11(1), 89-93.

[16] Khammash, G. S., Agarwal, P., and Choi, J. (2020). Extended k-gamma and k-beta functions of matrix arguments. Mathematics, 8(10), 1715.

[17] Korkmaz-Duzgun, D., and Erkus-Duman, E. (2019). Generating functions for k-hypergeometric functions. Int. J. Appl. Phys. Math, 9, 119-126.

[18] Koev, P., Edelman, A. (2006). The efficient evaluation of the hypergeometric function of a matrix argument. Mathematics of Computation, 75(254), 833-846.

[19] Mubeen, S., Rahman, G., and Arshad, M. (2015). K-gamma k-beta matrix functions and their properties. J. Math. Comput. Sci., 5(5), 647-657.

[20] Muirhead, R. J. (1970). Systems of partial differential equations for hypergeometric functions of matrix argument. The Annals of Mathematical Statistics, 991-1001.

[21] Terras, A. (1985). Special functions for the symmetric space of positive matrices. SIAM journal on mathematical analysis, 16(3), 620-640.

[22] Vilenkin, N. J., Klimyk, A. U., Vilenkin, N. J., Klimyk, A. U. (1992). Group Representations and Special Functions of a Matrix Argument. Representation of Lie Groups and Special Functions: Volume 3: Classical and Quantum Groups and Special Functions, 251-360.

[23] Park, I. H., and Seo, T. Y. (1998). A proof of the Legendre duplication formula for the gamma function. East Asian mathematical journal, 14(2), 321-327.

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