Journal of Fractional Calculus and Applications
Vol. 16(1) Jan 2025, No. 1
ISSN: 2090-5858
ISSN: 2090-584X(print)
https://jfca.journals.ekb.eg/

SUBORDINATION FACTOR SEQUENCE RESULTS FOR STARLIKE AND CONVEX CLASSES DEFINED BY A GENERALIZED OPERATOR

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Abstract. In this investigations, we generalize the multiplier operator analytic and univalent functions in the form $f(z)=z+\sum_{k=2}^{\infty}a_kz^k$ defined in the open unit disc $U=\{z:z\in\mathbb{C}$ and $|z|<1\}$. This new operator contains many other operators which were defined by many authors such as Cho and kim [8], Cho and Srivastava [9], Cătaş et al. [7], Uralegaddi and Samanatha [13], Aouf et al. ([4], with $w=0$) and others for different values of its parameters. Using the principle of subordination and this new operator, we define two subclasses of starlike and convex functions $S_n^*(\lambda,s,A,B,\alpha)$ and $C_n^*(\lambda,s,A,B,\alpha)$ respectively, which in turn generalize many other classes for the special values of the parameters. Using the definition and the lemma of Wilf [14], we obtain many results of subordinating factor sequence for these classes which lead to obtaining that also for the special subclasses by using the technique of Attiya [5], Frasin [10] and recently by Aouf and Mostafa [2, 3].

1 Introduction

Denote by $\hat{A}$ the class of analytic functions of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k \qquad (z\in U=\{z:z\in\mathbb{C} \text{ and } |z|<1\})\,. \tag{1}$$

For two functions $f,g\in\hat{A}$, $f(z)$ is subordinate to $g(z)$ $(f(z)\prec g(z))$, if there exists a function $\omega(z)$, analytic in U with $\omega(0)=0$ and $|\omega(z)|<1$, $f(z)=g(\omega(z))$ and if $g(z)$ is univalent in U, then (see [6, 11])

$$f(z)\prec g(z)\Longleftrightarrow f(0)=g(0)\quad f(U)\subset g(U). \tag{2}$$

For $\wp$, $\lambda>0$, $\Upsilon,s\geq 0$ and $n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $\mathbb{N}=\{1,2,3,...\}$, we define the operator $I_{\wp,\Upsilon}^n(s,\lambda):\hat{A}\longrightarrow\hat{A}$ by

2020 *Mathematics Subject Classification.* : 30C45,30C50, 30C55
Key words and phrases. Subordination, factor sequence, regular function, convex and starlike functions.

Submitted March 4, 2024. Revised July 5, 2024.

$$I^0_{\wp,\Upsilon}(s,\lambda)f(z)=f(z),$$
$$I^1_{\wp,\Upsilon}(s,\lambda)f(z)=\left(1-\frac{\wp+\Upsilon}{s+\lambda}\right)f(z)+\frac{\wp+\Upsilon}{s+\lambda}zf'(z)$$
$$=z+\sum_{k=2}^{\infty}\left[\frac{s+\lambda+(\wp+\Upsilon)(k-1)}{s+\lambda}\right]a_kz^k,$$
$$I^2_{\wp,\Upsilon}(s,\lambda)f(z)=\left(1-\frac{\wp+\Upsilon}{s+\lambda}\right)I^1_{\wp,\Upsilon}(s,\lambda)f(z)+\frac{\wp+\Upsilon}{s+\lambda}z\left(I^1_{\wp,\Upsilon}(s,\lambda)f(z)\right)'$$
$$=z+\sum_{k=2}^{\infty}\left[\frac{s+\lambda+(\wp+\Upsilon)(k-1)}{s+\lambda}\right]^2a_kz^k$$

and (in general) for $n\in\mathbb{N}$,

$$I^n_{\wp,\Upsilon}(s,\lambda)f(z)=\left(1-\frac{\wp+\Upsilon}{s+\lambda}\right)I^{n-1}_{\wp,\Upsilon}(s,\lambda)f(z)+\frac{\wp+\Upsilon}{s+\lambda}z\left(I^{n-1}_{\wp,\Upsilon}(s,\lambda)f(z)\right)'$$
$$=z+\sum_{k=2}^{\infty}\Psi^n(k,\lambda,s)a_kz^k, \tag{3}$$

where

$$\Psi^n(k,\lambda,s)=\left[\frac{s+\lambda+(\wp+\Upsilon)(k-1)}{s+\lambda}\right]^n,\ n\in\mathbb{N}_0. \tag{4}$$

We note that
(i) $I^n_{\wp,0}(s,1)f(z)=D^n_{s,\wp}f(z)$ (Cătaş et al. [7]);
(ii) $I^n_{\wp,0}(0,1)f(z)=D^n_{\wp}f(z)$ (Al-Oboudi [1]);
(iii) $I^n_{1,0}(0,1)f(z)=D^nf(z)$ (Sălăgean [12]).

Now by using the new operator and subordination definition, we define the following classes $S^*_n(\lambda,s,A,B,\alpha)$ and $C^*_n(\lambda,s,A,B,\alpha)$ as follows:

Definition 1. If $f\in\hat{A}$, $\wp$, $\lambda>0$, $\Upsilon,s\geq 0$, $-1\leq B<A\leq 1$, $0\leq\alpha<1$ and $n\in\mathbb{N}_0$, then $f\in S^*_n(\lambda,s,A,B,\alpha)$ if it satisfies that

$$\frac{1}{1-\alpha}\left(\frac{z\left(I^n_{\wp,\Upsilon}(s,\lambda)f(z)\right)'}{I^n_{\wp,\Upsilon}(s,\lambda)f(z)}-\alpha\right)\prec\frac{1+Az}{1+Bz},\ z\in U, \tag{5}$$

or, equivalently,

$$\left|\frac{\frac{z\left(I^n_{\wp,\Upsilon}(s,\lambda)f(z)\right)'}{I^n_{\wp,\Upsilon}(s,\lambda)f(z)}-1}{B\frac{z\left(I^n_{\wp,\Upsilon}(s,\lambda)f(z)\right)'}{I^n_{\wp,\Upsilon}(s,\lambda)f(z)}-[B+(1-\alpha)(A-B)]}\right|<1,\ z\in U.$$

We note also that:
(i) $S^*_n(1,s,A,B,\alpha)=S^*_n(s,A,B,\alpha)$;
(ii) $S^*_n(1,0,A,B,\alpha)=S^*_n(A,B,\alpha)$;
(iii) $S^*_n(\lambda,s,1,-1,\alpha)=S^*_n(\lambda,s,\alpha)$.

Definition 2. If $f \in \hat{A}$, $\wp, \lambda > 0$, $\Upsilon, s \geq 0$, $-1 \leq B < A \leq 1$, $0 \leq \alpha < 1$ and $n \in \mathbb{N}_0$, then $f \in C_n^*(\lambda, s, A, B, \alpha)$ if it satisfies the following:

$$\frac{1}{1-\alpha}\left(\frac{\left(z\left(I_{\wp,\Upsilon}^n(s,\lambda)f(z)\right)'\right)'}{\left(I_{\wp,\Upsilon}^n(s,\lambda)f(z)\right)'}-\alpha\right) \prec \frac{1+Az}{1+Bz},\ z \in U, \tag{6}$$

or, equivalently,

$$\left|\frac{\frac{\left(z\left(I_{\wp,\Upsilon}^n(s,\lambda)f(z)\right)'\right)'}{\left(I_{\wp,\Upsilon}^n(s,\lambda)f(z)\right)'}}{B\frac{\left(z\left(I_{\wp,\Upsilon}^n(s,\lambda)f(z)\right)'\right)'}{\left(I_{\wp,\Upsilon}^n(s,\lambda)f(z)\right)'}-(1-\alpha)(A-B)}\right| < 1,\ z \in U.$$

We note that:
(i) $C_n^*(1, s, A, B, \alpha) = C_n^*(s, A, B, \alpha)$;
(ii) $C_n^*(1, 0, A, B, \alpha) = C_n^*(A, B, \alpha)$;
(iii) $C_n^*(\lambda, s, 1, -1, \alpha) = C_n^*(\lambda, s, \alpha)$.

From (5) and (6), we have

$$f \in C_n^*(\lambda, s, A, B, \alpha) \Leftrightarrow zf'(z) \in S_n^*(\lambda, s, A, B, \alpha). \tag{7}$$

To obtain our main result we need the following definition and lemma:

Definition 3. [14] A sequence $\{c_k\}_{k=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if, whenever of the form (1) is regular, univalent and convex in U, we have the subordination given by

$$\sum_{k=1}^{\infty} c_k a_k z^k \prec f(z) \quad (z \in U;\ a_1 = 1).$$

Lemma 1. [14] The sequence $\{c_k\}_{k=1}^{\infty}$ is a subordinating factor sequence if and only if

$$\operatorname{Re}\left\{1 + 2\sum_{k=1}^{\infty} c_k z^k\right\} > 0, \quad (z \in U).$$

2 Main results

Theorem 1. If $f \in \hat{A}$, satisfies the following condition:

$$\sum_{k=2}^{\infty}\left[(k-1)(1-B)+(1-\alpha)(A-B)\right]\Psi^n(k,\lambda,s)\,a_k \leq (A-B)(1-\alpha). \tag{8}$$

then $f \in S_n^*(\lambda, s, A, B, \alpha)$.

Proof. If (8) holds, then we have

$$\left|z\left(I^n_{\wp,\Upsilon}(s,\lambda)f(z)\right)' - I^n_{\wp,\Upsilon}(s,\lambda)f(z)\right| - \left|Bz\left(I^n_{\wp,\Upsilon}(s,\lambda)f(z)\right)'\right.$$

$$\left. - \left[B+(1-\alpha)(A-B)\right]I^n_{\wp,\Upsilon}(s,\lambda)f(z)\right| = \left|\sum_{k=2}^{\infty}(k-1)\Psi^n(k,\lambda,s)a_kz^k\right|$$

$$-\left|(A-B)(1-\alpha)z+\sum_{k=2}^{\infty}\left[(A-B)(1-\alpha)-B(k-1)\right]\Psi^n(k,\lambda,s)a_kz^k\right|$$

$$\le \sum_{k=2}^{\infty}(k-1)\Psi^n(k,\lambda,s)|a_k| - (A-B)(1-\alpha)$$

$$+\sum_{k=2}^{\infty}\left[(A-B)(1-\alpha)-B(k-1)\right]\Psi^n(k,\lambda,s)|a_k|$$

$$=\sum_{k=2}^{\infty}\left[(k-1)(1-B)+(1-\alpha)(A-B)\right]\Psi^n(k,\lambda,s)|a_k| - (A-B)(1-\alpha) \le 0.$$

Corollary 1. If $f \in S^*_n(\lambda,s,A,B,\alpha)$, then

$$|a_k| \le \frac{(A-B)(1-\alpha)}{\left[(k-1)(1-B)+(1-\alpha)(A-B)\right]\Psi^n(k,\lambda,s)} \quad (k \ge 2). \tag{9}$$

The result is sharp for

$$f(z) = z + \frac{(A-B)(1-\alpha)}{\left[(k-1)(1-B)+(1-\alpha)(A-B)\right]\Psi^n(k,\lambda,s)}z^k \quad (k \ge 2). \tag{10}$$

Similarly, we can prove the following theorem for the class $C^*_n(\lambda,s,A,B,\alpha)$ by using (7).

Theorem 2. If $f \in \hat{A}$, satisfies the following condition:

$$\sum_{k=2}^{\infty}k\left[(k-1)(1-B)+(1-\alpha)(A-B)\right]\Psi^n(k,\lambda,s)a_k \le (A-B)(1-\alpha). \tag{11}$$

Then $f \in C^*_n(\lambda,s,A,B,\alpha)$.

Corollary 2. If $f \in C^*_n(\lambda,s,A,B,\alpha)$, then

$$|a_k| \le \frac{(A-B)(1-\alpha)}{k\left[(k-1)(1-B)+(1-\alpha)(A-B)\right]\Psi^n(k,\lambda,s)} \quad (k \ge 2). \tag{12}$$

The result is sharp for

$$f(z) = z + \frac{(A-B)(1-\alpha)}{k\left[(k-1)(1-B)+(1-\alpha)(A-B)\right]\Psi^n(k,\lambda,s)}z^k \quad (k \ge 2). \tag{13}$$

Let $\bar{S}^*_n(\lambda,s,A,B,\alpha)$ denote the class of functions $f \in \hat{A}$ whose coefficients satisfy the condition (8). We note that:

$$\bar{S}_n^*(\lambda, s, A, B, \alpha) \subseteq S_n^*(\lambda, s, A, B, \alpha),$$

and $\bar{S}_n^*(s, A, B, \alpha)$, $\bar{S}_n^*(A, B, \alpha)$ and $\bar{S}_n^*(\lambda, s, \alpha)$ denote the classes of functions $f \in \hat{A}$ whose coefficients satisfy the condition (8), with the corresponding values of the parameters.

Using the technique of Aouf and Mostsfa [2, 3], we prove the following results.

Theorem 3. If $f \in \bar{S}_n^*(\lambda, s, A, B, \alpha)$, then for each convex and regular function $\varphi(z)$, we have

$$\frac{[(1-B)+(1-\alpha)(A-B)]\Psi^n(2,\lambda,s)}{2\{[(1-B)+(1-\alpha)(A-B)]\Psi^n(2,\lambda,s)+(A-B)(1-\alpha)\}}(f*\varphi)(z) \prec \varphi(z) \tag{14}$$

and

$$\operatorname{Re}\{f(z)\} > -\frac{[(1-B)+(1-\alpha)(A-B)]\Psi^n(2,\lambda,s)+(A-B)(1-\alpha)}{[(1-B)+(1-\alpha)(A-B)]\Psi^n(2,\lambda,s)}. \tag{15}$$

The constant factor $\frac{[(1-B)+(1-\alpha)(A-B)]\Psi^n(2,\lambda,s)}{2\{[(1-B)+(1-\alpha)(A-B)]\Psi^n(2,\lambda,s)+(A-B)(1-\alpha)\}}$ is the best estimate.

Proof. Let $f \in \bar{S}_n^*(\lambda, s, A, B, \alpha)$ and $\varphi(z) = z + \sum_{k=2}^{\infty} c_k z^k$ is convex and regular function. Then we have

$$\frac{[(1-B)+(1-\alpha)(A-B)]\Psi^n(2,\lambda,s)}{2\{[(1-B)+(1-\alpha)(A-B)]\Psi^n(2,\lambda,s)+(A-B)(1-\alpha)\}}(f*\varphi)(z)$$

$$= \frac{[(1-B)+(1-\alpha)(A-B)]\Psi^n(2,\lambda,s)}{2\{[(1-B)+(1-\alpha)(A-B)]\Psi^n(2,\lambda,s)+(A-B)(1-\alpha)\}}\left(z + \sum_{k=2}^{\infty} c_k a_k z^k\right). \tag{16}$$

By applying Definition 3, the subordination (14) will hold if the sequence

$$\left\{\frac{[(1-B)+(1-\alpha)(A-B)]\Psi^n(2,\lambda,s)}{2\{[(1-B)+(1-\alpha)(A-B)]\Psi^n(2,\lambda,s)+(A-B)(1-\alpha)\}}a_k\right\}_{k=1}^{\infty}, \tag{17}$$

is a subordinating factor sequence, with $a_1 = 1$. From Lemma 1, we obtain

$$\operatorname{Re}\left\{1 + \sum_{k=1}^{\infty} \frac{[(1-B)+(1-\alpha)(A-B)]\Psi^n(2,\lambda,s)}{\{[(1-B)+(1-\alpha)(A-B)]\Psi^n(2,\lambda,s)+(A-B)(1-\alpha)\}}a_k z^k\right\} > 0. \tag{18}$$

Since

$$\mho(k) = [(k-1)(1-B)+(1-\alpha)(A-B)]\Psi^n(k,\lambda,s),$$

is an increasing function of k $(k \geq 2)$, we obtain

$$\operatorname{Re}\left\{1+\sum_{k=1}^{\infty}\frac{[(1-B)+(1-\alpha)(A-B)]\Psi^{n}(2,\lambda,s)}{\{[(1-B)+(1-\alpha)(A-B)]\Psi^{n}(2,\lambda,s)+(A-B)(1-\alpha)\}}a_k z^k\right\}$$

$$=\operatorname{Re}\left\{1+\frac{[(1-B)+(1-\alpha)(A-B)]\Psi^{n}(2,\lambda,s)}{\{[(1-B)+(1-\alpha)(A-B)]\Psi^{n}(2,\lambda,s)+(A-B)(1-\alpha)\}}z\right.$$

$$\left.+\frac{\sum\limits_{k=2}^{\infty}[(1-B)+(1-\alpha)(A-B)]\Psi^{n}(2,\lambda,s)\,a_k z^k}{[(1-B)+(1-\alpha)(A-B)]\Psi^{n}(2,\lambda,s)+(A-B)(1-\alpha)}\right\}$$

$$\geq 1-\frac{[(1-B)+(1-\alpha)(A-B)]\Psi^{n}(2,\lambda,s)}{[(1-B)+(1-\alpha)(A-B)]\Psi^{n}(2,\lambda,s)+(A-B)(1-\alpha)}r$$

$$-\frac{\sum\limits_{k=2}^{\infty}[(k-1)(1-B)+(1-\alpha)(A-B)]\Psi^{n}(k,\lambda,s)\,|a_k|\,r^k}{[(1-B)+(1-\alpha)(A-B)]\Psi^{n}(2,\lambda,s)+(A-B)(1-\alpha)}$$

$$>1-\frac{[(1-B)+(1-\alpha)(A-B)]\Psi^{n}(2,\lambda,s)}{[(1-B)+(1-\alpha)(A-B)]\Psi^{n}(2,\lambda,s)+(A-B)(1-\alpha)}r$$

$$-\frac{(A-B)(1-\alpha)}{[(1-B)+(1-\alpha)(A-B)]\Psi^{n}(2,\lambda,s)+(A-B)(1-\alpha)}r>0 \qquad (|z|=r<1),$$

where we used the result (8) of Theorem 1.

To prove inequality (15) taking the convex and regular function $\varphi(z)=\frac{z}{1-z}=z+\sum\limits_{k=2}^{\infty}z^k$. To prove the sharpness of the constant factor $\frac{[(1-B)+(1-\alpha)(A-B)]\Psi^{n}(2,\lambda,s)}{2\{[(1-B)+(1-\alpha)(A-B)]\Psi^{n}(2,\lambda,s)+(A-B)(1-\alpha)\}}$, we assume the function $f_0(z)\in S_n^*(\lambda,s,A,B,\alpha)$ given by

$$f_0(z)=z-\frac{(A-B)(1-\alpha)}{[(1-B)+(1-\alpha)(A-B)]\Psi^{n}(2,\lambda,s)}z^2. \tag{19}$$

From (14), we have

$$\frac{[(1-B)+(1-\alpha)(A-B)]\Psi^{n}(2,\lambda,s)}{2\{[(1-B)+(1-\alpha)(A-B)]\Psi^{n}(2,\lambda,s)+(A-B)(1-\alpha)\}}f_0(z)\prec\frac{z}{1-z}. \tag{20}$$

Furthermore, it can easily be validate for $f_0(z)$ given by (19) that

$$\min_{|z|\leq r}\left\{\operatorname{Re}\left(\frac{[(1-B)+(1-\alpha)(A-B)]\Psi^{n}(2,\lambda,s)}{2\{[(1-B)+(1-\alpha)(A-B)]\Psi^{n}(2,\lambda,s)+(A-B)(1-\alpha)\}}\right)f_0(z)\right\}=-\frac{1}{2}, \tag{21}$$

which shows that the factor $\frac{[(1-B)+(1-\alpha)(A-B)]\Psi^{n}(2,\lambda,s)}{2\{[(1-B)+(1-\alpha)(A-B)]\Psi^{n}(2,\lambda,s)+(A-B)(1-\alpha)\}}$ is the best possible.

Let $\bar{C}_n^*(\lambda,s,A,B,\alpha)$ denote the class of functions $f\in\hat{A}$ whose coefficients satisfy the condition (8). We note that:

$$\bar{C}_n^*(\lambda, s, A, B, \alpha) \subseteq C_n^*(\lambda, s, A, B, \alpha),$$

and $\bar{C}_n^*(s, A, B, \alpha)$, $\bar{C}_n^*(A, B, \alpha)$ and $\bar{C}_n^*(\lambda, s, \alpha)$ denote the classes of functions $f \in \hat{A}$ whose coefficients satisfy the condition (11), with the corresponding values of the parameters.

Similarly, we can prove the following theorem for the class $\bar{C}_n^*(\lambda, s, A, B, \alpha)$.

Theorem 4. If $f \in \bar{C}_n^*(\lambda, s, A, B, \alpha)$, then for each convex and regular function $\varphi(z)$, we have

$$\frac{[(1-B)+(1-\alpha)(A-B)]\Psi^n(2,\lambda,s)}{\{2[(1-B)+(1-\alpha)(A-B)]\Psi^n(2,\lambda,s)+(A-B)(1-\alpha)\}}(f*\varphi)(z) \prec \varphi(z) \tag{22}$$

and

$$\operatorname{Re}\{f(z)\} > -\frac{2[(1-B)+(1-\alpha)(A-B)]\Psi^n(2,\lambda,s)+(A-B)(1-\alpha)}{2[(1-B)+(1-\alpha)(A-B)]\Psi^n(2,\lambda,s)}. \tag{23}$$

The constant factor $\frac{[(1-B)+(1-\alpha)(A-B)]\Psi^n(2,\lambda,s)}{\{2[(1-B)+(1-\alpha)(A-B)]\Psi^n(2,\lambda,s)+(A-B)(1-\alpha)\}}$ is the best estimate.

Remark. Taking different values of the parameters λ, s, A, B, α in Theorems 1,2,3,4 we have also results for the corresponding subclasses.

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