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NULL CONTROLLABILITY OF FRACTIONAL DIFFERENTIAL SYSTEM WITH NONLOCAL INITIAL CONDITION

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Abstract. In this paper, we examine the conditions of exact null controllability of fractional dynamical system with nonlocal initial condition in infinite dimensional setting. The fractional derivatives used in the system are in Caputo sense and order of the derivatives are taken as $r \in (0,1)$. Schauder's fixed point theorem is used to prove null controllability with the help of the null controllability of the associated linear system.

1. Introduction

Fractional differential systems (FDS's) gain more and more importance in the recent decades because of their ability to model real world problems in a more efficient way in comparison to integer order systems. The controllability problem of FDS's are studied by many authors in numerous articles [5, 9, 15, 19, 17, 13, 12, 1, 8, 11]. There are different types of controllability, namely, exact controllability, approximate controllability, null controllability, trajectory controllability etc. The null controllability of a dynamical system means that the system can be steered to zero state from an arbitrary initial state by means of some control inputs. Dauer et al. [7] studied the null controllability of semilinear integer order systems in Hilbert space. They studied the following integro-differential system

$$\begin{aligned} x'(t) =& Ax(t) + Bu(t) + \int_0^t f(\tau, x_\tau) d\tau, \quad t \in [0,T], \\ x_0(\mu) =& \phi(\mu), \quad \mu \in [-k, 0]. \end{aligned}$$

Also in their paper [22] the authors derived conditions for null controllability of the following nonlocal system of integer order

$$y'(t) = A(t)y(t) + Bv(t) + f(t, y(g(t))), \quad t \in [0,T],$$

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$$y(0) + h(t) = y_0.$$

In case of fractional systems, Ahmed [2] studied null controllability of stochastic fractional system with Hilfer derivatives, Nirmala et al. [10] investigated null controllability of fractional dynamical systems with constrained control in finite dimensional case, while Sathiyaraj et al. [20, 21] studied Hilfer FDS with Brownian motion and delayed FDS in finite dimension. For more articles regarding null controllability of FDS in different settings we refer to [3, 4, 18, 23] and the references therein.

Motivated by the above mentioned studies, we consider the following nonlocal FDS for investigating null controllability

$$\begin{aligned} {}^{C}D^{r}_{0+}u(t) &= Au(t) + Bv(t) + g(t, u(t)), \quad t \in (0, a]; \\ u(0) + h(u) &= u_0. \end{aligned} \tag{1}$$

Let $\mathcal{J} = [0, a]$ and ${}^{C}D^{r}_{0+}$ denotes the regularized Caputo derivative of order r with $0 < r < 1$. We take two Hilbert spaces $\mathbb{U}$ and $\mathbb{V}$ such that the state variable $u(.) \in \mathbb{U}$ and the control function $v(.) \in \mathbb{V}$. The corresponding norms in these Hilbert spaces are taken to be the usual supremum norm. We take $v \in L^2(\mathcal{J}; \mathbb{V})$ which ia a Banach space of admissible controls endowed with the norm $\|v\| = \big(\int_0^a \|v(s)\|^2 ds\big)^{\frac{1}{2}}$. A generates a strongly continuous semigroup $\{\mathcal{Q}(t)\}_{0\le t\le a}$, B is a bounded linear operator from $\mathbb{V}$ to $\mathbb{U}$. $g : \mathcal{J} \times C(\mathcal{J}, \mathbb{U}) \to \mathbb{U}$ and $h : C(\mathcal{J}, \mathbb{U}) \to \mathbb{U}$ are two functions which will be specified later.

The main aim of our study is to extend the ideas presented in the articles [7] and [22] into fractional framework with somewhat different set of conditions. The fundamental difference of this work is that we consider fractional differential system instead of integer order systems as done in the aforementioned papers. Further nonlocal initial condition is taken into account in contrast with the local condition in [7]. The assumption that g is strongly measurable as mentioned in [22] is replaced by Lipschitz continuity in this work.

The rest of this paper is organized as follows: in Section 2 we list some important results and definitions of fractional calculus and semigroup theory, main theoretical results are discussed in the Section 3, an example is provided in Section 4 and conclusion can be found in Section 5.

2. Preliminaries

Here we state some definitions with few relevant results of fractional calculus.

Definition 2.1. [14] *For the function $\psi : [0, \infty) \to \mathbb{R}$, the fractional Riemann-Liouville (R-L) integral of order $r > 0$, with lower limit 0 is defined as*

$$I^{r}_{0+}\psi(\zeta) = \frac{1}{\Gamma(r)} \int_0^{\zeta} (\zeta - \vartheta)^{r-1} \psi(\vartheta) d\vartheta,$$

assuming the right hand side of the equation is defined on the interval $[0, \infty)$ point wise.

Definition 2.2. [14] *The fractional R-L derivative of the function $\psi : [0, \infty) \to \mathbb{R}$ of order $r > 0$, with lower limit 0 is defined as*

$${}^{RL}D^{r}_{0+}\psi(\zeta) = \frac{1}{\Gamma(k - r)} \big(\frac{d}{d\zeta}\big)^{k} \int_0^{\zeta} (\zeta - \vartheta)^{k-r-1} \psi(\vartheta) d\vartheta,$$

where $k-1 < r \leq k$, *with* $k \in \mathbb{N}$.

Definition 2.3. [14] *The fractional Caputo derivative of the function* $\psi : [0,\infty) \to \mathbb{R}$ *of order* $r > 0$ *is defined by*

$$^{C}D^{r}_{0+}\psi(\zeta) = \frac{1}{\Gamma(k-r)} \int_0^{\zeta} (\zeta-\vartheta)^{k-r-1}\psi^{(k)}(\vartheta)d\vartheta,$$

where $k-1 < r \leq k$, $k \in \mathbb{N}$.

Let $\{\mathcal{Q}\}_{t\geq 0}$ be the semigroup associated with the operator A. Then the following operators are defined to present the mild solution of (1)

$$S_r(t)u = \int_0^t \phi_r(\mu)\mathcal{Q}(t^r\mu)ud\mu,$$

$$P_r(t)u = r\int_0^t \mu\phi_r(\mu)\mathcal{Q}(t^r\mu)ud\mu.$$

Here $\phi_r(\mu) = \frac{1}{r}\mu^{-1-\frac{1}{r}}\psi_r(\mu^{-\frac{1}{r}})$ is called probability density function which satisfies $\phi_r(\mu) \geq 0$ and $\int_0^\infty \phi_r(\mu)d\mu = 1$. Also the function ψ is defined as

$$\psi_r(\mu) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\mu^{-nr-1}\frac{\Gamma(nr+1)}{n!}\sin(n\pi r), \quad \mu \in (0,\infty).$$

Let $\|\mathcal{Q}(t)\| \leq M$ for all $t \geq 0$. Then we have the following Lemma

Lemma 2.1. [16, 25]

(i) *For fixed* $t \geq 0$, $S_r(t)$ *and* $P_r(t)$ *are linear bounded operators on* $\mathbb{U}$ *and* $\|S_r(t)\| \leq M$, $\|P_r(t)\| \leq \frac{M}{\Gamma(r)}$.
(ii) *If* $\mathcal{Q}(t)$ *is compact in* $\mathbb{U}$, *then* $S_r(t)$ *and* $P_r(t)$ *are also compact for* $t > 0$.
(iii) $S_r(t)$ *and* $P_r(t)$ *are continuous for* $t > 0$.

On the basis of Lemma 4.21 of [24] we define the mild solution of (1) as

Definition 2.4. *A function* $u \in C(\mathcal{J},\mathbb{U})$ *is said to be a mild solution of the problem (1) if it satisfies the following integral equation:*

$$u(t) = S_r(t)\big[u_0 - h(u)\big] + \int_0^t (t-\tau)^{r-1}P_r(t-\tau)\Big[Bv(\tau)+g(\tau,u(\tau))\Big]d\tau, \quad t \in \mathcal{J}. \quad (2)$$

Definition 2.5 (Null controllability)**.** *The system (1) is said to be null controllable if there exists a control* $v \in L^2(\mathcal{J},\mathbb{V})$ *such that with this control we have* $u(a) = 0$.

Lemma 2.2 (Schauder Fixed Point Theorem)**.** *Let* W *be a closed bounded and convex subset of a Banach space* $\mathbb{U}$ *and* $\mathcal{Y} : W \to W$ *be completely continuous, then* $\mathcal{Y}$ *has at least one fixed point in* W.

3. Main results

We take the following assumptions

A1: The semigroup $\{\mathcal{Q}(t)\}$ is compact.
A2: $g : \mathcal{J} \times C(\mathcal{J},\mathbb{U}) \to \mathbb{U}$ is continuous and there exists functions $\alpha(.) \in L^1(\mathcal{J},\mathbb{R}^+)$ and $\beta(.) \in L^1(C(\mathcal{J},\mathbb{U}),\mathbb{R}^+)$ such that

$$\|g(t,u(t))\| \leq \alpha(t)\beta(u), \quad \forall (t,u) \in \mathcal{J}\times C(\mathcal{J},\mathbb{U}).$$

A3: The function $h : C(\mathcal{J}, \mathbb{U}) \to \mathbb{U}$ is continuous and there exists a constant $L_h > 0$ such that $\|h(u)\| \leq L_h \|u\|$.

A4: The associated linear system

$$\begin{aligned} {}^C D^r_{0+} u(t) &= Au(t) + Bv(t) + g(t), \quad t \in (0, a]; \\ u(0) &= u_0, \end{aligned} \tag{3}$$

where $g \in L^2(\mathcal{J}, \mathbb{U})$ is exactly null controllable on $\mathcal{J}$ in $\mathbb{U}$.

Now define the following operators

(i) $\mathcal{L}^a_0 : L^2(\mathcal{J}, \mathbb{V}) \to \mathbb{U}$ such that

$$\mathcal{L}^a_0(v) = \int_0^a (a-\tau)^{r-1} P_r(a-\tau) Bv(\tau) d\tau \ \text{ and,}$$

$\mathcal{L}_0$ is the restriction of $\mathcal{L}^a_0$ to $[\ker \mathcal{L}^a_0]^\perp$.

(ii) $\mathcal{N}^a_0 : \mathbb{U} \times L^2(\mathcal{J}, \mathbb{V}) \to \mathbb{U}$ such that

$$\mathcal{N}^a_0(x, f) = S_r(a)x + \int_0^a (a-\tau)^{r-1} P_r(a-\tau) f(\tau) d\tau.$$

The following Lemmas are useful for our main results.

Lemma 3.3. [6, 22] *The linear system (3) is exactly null controllable on $\mathcal{J}$ if*

$$Im\, \mathcal{L}_0 \supset Im\, \mathcal{N}^a_0.$$

Lemma 3.4. [6, 22] *The linear system (3) is exactly null controllable in $\mathcal{J}$ if and only if there exists a positive integer γ such that*

$$\|(\mathcal{L}^a_0)^* u\| \geq \gamma \|(\mathcal{N}^a_0)^* u\|, \quad \forall u \in \mathbb{U},$$

where $*$ denotes the transpose.

Lemma 3.5. *Let the system (3) be exactly null controllable in $\mathcal{J}$, then the linear operator $(\mathcal{L}_0)^{-1}(\mathcal{N}^a_0) : \mathbb{U} \times L^2(\mathcal{J}, \mathbb{U}) \to L^2(\mathcal{J}, \mathbb{V})$ is bounded. Further the control*

$$\begin{aligned} v(t) &= -(\mathcal{L}_0)^{-1}(\mathcal{N}^a_0(u_0, g))(t) \\ &= -(\mathcal{L}_0)^{-1}\Big[S_r(a)u_0 + \int_0^a (a-\tau)^{r-1} P_r(a-\tau) g(\tau) d\tau\Big] \end{aligned}$$

transfer the system from u_0 to 0.

Proof. Let us first symbolize $\mathcal{H} : \mathbb{U} \times L^2(\mathcal{J}, \mathbb{U}) \to L^2(\mathcal{J}, \mathbb{V})$ by

$$\mathcal{H}(x, f) = (\mathcal{L}_0)^{-1} \mathcal{N}^a_0(x, f).$$

From the definition, we see that $\mathcal{L}^a_0$ is a bounded linear operator. The null space of $\mathcal{L}^a_0$ is defined by $\ker \mathcal{L}^a_0 = \{u \in L^2(\mathcal{J}, \mathbb{U}) : \mathcal{L}^a_0 = 0\}$ and and its orthogonal compliment by $[\ker \mathcal{L}^a_0]^\perp$.

Observe that the operator $\mathcal{N}^a_0(x, f)$ is bounded by virtue of the boundedness of $S_r(t)$, $P_r(t)$ and f in finite time. Since $\mathcal{L}_0 : [\ker \mathcal{L}^a_0]^\perp \to \text{Im}\, \mathcal{L}^a_0$, so $\mathcal{L}_0^{-1}$ is bijective and by inverse mapping theorem it is bounded if both $[\ker \mathcal{L}^a_0]^\perp$ and Im $\mathcal{L}^a_0$ are Banach spaces.

Obviously $[\ker \mathcal{L}^a_0]^\perp$ is closed and hence a Banach space but the same can't be said about Im $\mathcal{L}^a_0$. Consider the sequence $< x_n, f_n >$ in $\mathbb{U} \times L^2(\mathcal{J}, \mathbb{U})$ such that

$\lim_{n\to\infty} < x_n, f_n > \to < x, f >$. Also let $\mathcal{H}(x_n, f_n)$ converges in $\mathbb{V}$ and $-v = \lim_{n\to\infty} \mathcal{H}(x_n, f_n)$. The closeness of $[\ker \mathcal{L}_0^a]^\perp$ implies that $v \in [\ker \mathcal{L}_0^a]^\perp$. Now

$$\mathcal{L}_0^a(v) + \mathcal{N}_0^a(x, f) = \lim_{n\to\infty} \Big[-\mathcal{L}_0^a \mathcal{H}(x_n, f_n) + \mathcal{N}_0^a(x_n, f_n)\Big] = 0,$$

by the continuity of $\mathcal{L}_0^a$ and $\mathcal{N}_0^a$. So $-v = -\mathcal{L}_0^{-1}\mathcal{N}_0^a(x, f) = -\mathcal{H}(x, f)$ and so $\mathcal{H}$ is closed. By closed graph theorem, we see that $\mathcal{H}$ is bounded.

For the other part, we can directly compute $v(t)$ into the mild solution of the linear system (3) to get $u(a) = 0$. □

Theorem 3.1. *Assume that the conditions $A1-A4$ are satisfied, then the nonlocal system (1) is exactly null controllable in $\mathcal{J}$ provided*

$$ML_h + \frac{M}{\Gamma(r)}\|B\|\Big(\frac{a^r}{r}\Big)^{\frac{1}{2}}\|\mathcal{H}\|\|u_0\|L_h < 1.$$

Proof. For any $u \in \mathbb{U}$, we choose the control as

$$\begin{aligned} v(t) &= -(\mathcal{L}_0)^{-1}\Big[\mathcal{N}_0^a(u_0 - h(u), g)\Big](t) \\ &= -(\mathcal{L}_0)^{-1}\Big[S_r(a)(u_0 - h(u)) + \int_0^a (a-\tau)^{r-1}P_r(a-\tau)g(\tau, u(\tau))d\tau\Big](t) \\ &= -\mathcal{H}(u_0 - h(u), g)(t). \end{aligned} \tag{4}$$

Obviously $v(t)$ is well defined as $u_0 - h(u) \in \mathbb{U}$ and $g(t, u(t)) \in L^2(\mathcal{J}, \mathbb{U})$. To show that this control steers the system (1) from $u_0 - h(u)$ to 0 at time $t = a$ we compute directly the value of $v(t)$ into the mild solution as given in (2).

$$\begin{aligned} u(a) =& S_r(a)[u_0 - h(u)] - \int_0^a (t-\tau)^{r-1}P_r(t-\tau)B\mathcal{H}(u_0 - h(u), g)(\tau)d\tau \\ &+ \int_0^a (t-\tau)^{r-1}P_r(t-\tau)g(\tau, u(\tau))d\tau \\ =& S_r(a)[u_0 - h(u)] - \int_0^a (t-\tau)^{r-1}P_r(t-\tau)B(\mathcal{L}_0)^{-1}\Big[S_r(a)[u_0 - h(u)] \\ &+ \int_0^a (t-\tau)^{r-1}P_r(t-\tau)g(\tau, u(\tau))d\tau\Big]d\tau \\ &+ \int_0^a (t-\tau)^{r-1}P_r(t-\tau)g(\tau, u(\tau))d\tau \\ =& 0. \end{aligned}$$

Consider the set

$$\mathcal{W}_k = \{u \in \mathbb{U} : u(0) = u_0 - h(u), \|u\| \le k\}.$$

Obviously $\mathcal{W}_k$ is convex, closed and bounded.

The control defined by (4) is bounded, as we see for $u \in \mathcal{W}_k$,

$$\begin{aligned} \|v\| =& \Big(\int_0^a \|\mathcal{H}(u_0 - h(u), g)(s)\|^2\, ds\Big)^{\frac{1}{2}} \\ &\le \|\mathcal{H}\| \left[\|u_0\| + \|h(u)\| + \left(\int_0^a \Big(\int_0^s \|g(\tau, u(\tau))\|\, d\tau\Big)^2 ds\right)^{\frac{1}{2}} \right] \end{aligned}$$

$$
\begin{aligned}
&\leq \|\mathcal{H}\| \left[\|u_0\| + L_h k + \frac{M}{\Gamma(r)} \left(\int_0^a \Big(\int_0^s \alpha(\tau)\beta(u) \Big)^2 ds \right)^{\frac{1}{2}} \right] \\
&\leq \|\mathcal{H}\| \left[\|u_0\| + L_h k + \frac{M}{\Gamma(r)} \Big(\frac{a^r}{r}\Big)^{\frac{1}{2}} \|\alpha\| \, \psi(k) \right] \\
&= M_v \ \ \text{(say)}.
\end{aligned}
$$

Now our job is to prove that the solution of (1) with respect to the control given by (4) exists in $\mathcal{J}$. For any arbitrary $u(.)$ and $t \in \mathcal{J}$ define the operator $\mathcal{Y}$ on $C(\mathcal{J}, \mathbb{U})$ by

$$
(\mathcal{Y}u)(t) = S_r(t)[u_0 - h(u)] + \int_0^t (t-\tau)^{r-1} P_r(t-\tau) \Big[B\mathcal{H}(u_0 - h(u), g)(\tau) + g(\tau, u(\tau)) \Big] d\tau.
$$

We will show that $\mathcal{Y}$ has a fixed point in $\mathcal{J}$ by using Schauder fixed point theorem, which implies the existence of the mild solution of (1) with the control defined by (4). We split the proof into several steps.

Step 1: We claim that there exists $k \in \mathbb{R}^+$ such that $\mathcal{Y}(\mathcal{W}_k) \subset \mathcal{W}_k$. Let this be not true, then for each $k \in \mathbb{R}^+, \exists\, u_k(.) \in \mathcal{W}_k$ with the condition that $\mathcal{Y}(u_k) \notin \mathcal{W}_k$, which implies that $\|\mathcal{Y}u_k(t)\| > k$ for some $t \in \mathcal{J}$. Here t is dependent on k.

Now

$$
\begin{aligned}
k &< \|\mathcal{Y}(u_k)(t)\| \\
&\leq \|S_r(t)[u_0 - h(u)]\| + \int_0^t \Big\| (t-\tau)^{r-1} P_r(t-\tau) \Big\| \\
&\quad \times \Big[\Big\| B\mathcal{H}\big(u_0 - h(u), g\big)(\tau) + g(\tau, u(\tau)) \Big] d\tau \Big\| \\
&\leq M \|u_0 - h(u)\| + \frac{M}{\Gamma(r)} \int_0^a (t-\tau)^{r-1} \Big[\|B\| \, \|\mathcal{H}(u_0 - h(u), g)(\tau)\| \\
&\quad + \|g(\tau, u(\tau))\| \Big] d\tau \\
&\leq M\|u_0\| + M L_h \|u\| + \frac{M}{\Gamma(r)} \|B\| \left(\int_0^a \Big((t-\tau)^{r-1} \|\mathcal{H}(u_0 - h(u), g)(\tau)\| \Big)^2 d\tau \right)^{\frac{1}{2}} \\
&\quad + \frac{M}{\Gamma(r)} \int_0^t \int_0^\tau (t-\tau)^{r-1} \|g(s, u(s))\| \, ds d\tau \\
&\leq M\|u_0\| + M L_h k + \frac{M}{\Gamma(r)} \|B\| \left[\Big(\frac{a^r}{r}\Big)^{\frac{1}{2}} \Big(\|\mathcal{H}\| \|u_0\| L_h k + \Big(\frac{a^r}{r}\Big)^{\frac{1}{2}} \Big) \right. \\
&\quad \left. + \frac{M}{\Gamma(r)} \|\alpha\| \psi(k) \right] + \frac{M}{\Gamma(r)} \Big(\frac{a^r}{r}\Big)^{\frac{1}{2}} \|\alpha\| \psi(k).
\end{aligned}
$$

Dividing both sides by k and letting $k \to \infty$ we have

$$
1 \leq M L_h + \frac{M}{\Gamma(r)} \|B\| \Big(\frac{a^r}{r}\Big)^{\frac{1}{2}} \|\mathcal{H}\| \|u_0\| L_h,
$$

which contradicts the assumption of the theorem. So $\mathcal{Y}$ maps $\mathcal{W}_k$ into itself.

Step 2: $\mathcal{Y}$ maps $\mathcal{W}_k$ into equicontinuous sets of $C(\mathcal{J}, \mathbb{U})$.

Let $0 < t_1 < t_2 \leq a$ and $u \in \mathcal{W}_k$ be any arbitrary element. Then

$$\begin{aligned}\mathcal{Y}(u)(t_1) - \mathcal{Y}(u)(t_2) =& \Big(S_r(t_1) - S_r(t_2)\Big)[u_0 - h(u)] \\ & - \int_{t_1}^{t_2} (t_2 - \tau)^{r-1} P_r(t_2 - \tau) \times B\mathcal{H}(u_0 - h(u), g)(\tau) d\tau \\ & + \int_0^{t_1} \Big[(t_1 - \tau)^{r-1} P_r(t_1 - \tau) - (t_2 - \tau)^{r-1} P_r(t_2 - \tau)\Big] \\ & \times B\mathcal{H}(u_0 - h(u), g)(\tau) d\tau \\ & - \int_{t_1}^{t_2} (t_2 - \tau)^{r-1} P_r(t_2 - \tau) \int_0^{\tau} g(s, u(s)) ds d\tau \\ & + \int_0^{t_1} \Big[(t_1 - \tau)^{r-1} P_r(t_1 - \tau) - (t_2 - \tau)^{r-1} P_r(t_2 - \tau)\Big] \\ & \times \int_0^{\tau} g(s, u(s)) ds d\tau\end{aligned}$$

Taking norm on both sides

$$\begin{aligned}\|\mathcal{Y}(u)(t_1) - \mathcal{Y}(u)(t_2)\| \leq & \|S_r(t_1) - S_r(t_2)\| \big(\|u_0\| + \|h(u)\|\big) \\ & + \|B\| \frac{M}{\Gamma(r)} \frac{a^r}{r} \int_{t_1} t_2 \|\mathcal{H}(u_0 - h(u), g)(\tau)\| d\tau \\ & + \|B\| \int_0^{t_1} \big\|(t_1 - \tau)^{r-1} P_r(t_1 - \tau) - (t_2 - \tau)^{r-1} P_r(t_2 - \tau)\big\| \\ & \times \|\mathcal{H}(u_0 - h(u), g)(\tau)\| d\tau + \frac{M}{\Gamma(r)} \frac{a^r}{r} \int_{t_1+\epsilon}^{t_2+\epsilon} \int_0^{\tau} \alpha(s)\beta(u) ds d\tau \\ & + \int_0^{t_1} \big\|(t_1 - \tau)^{r-1} P_r(t_1 - \tau) - (t_2 - \tau)^{r-1} P_r(t_2 - \tau)\big\| \\ & \times \int_0^{\tau} \alpha(\tau)\beta(u) ds d\tau \\ \leq & \|S_r(t_1) - S_r(t_2)\| \big(\|u_0\| + \|h(u)\|\big) \\ & + \|B\| \frac{M}{\Gamma(r)} \frac{a^r}{r} \int_{t_1}^{t_2} \|\mathcal{H}(u_0 - h(u), g)(\tau)\| d\tau \\ & + \|B\| \int_0^{t_1} \big\|(t_1 - \tau)^{r-1} P_r(t_1 - \tau) - (t_2 - \tau)^{r-1} P_r(t_2 - \tau)\big\| \\ & \times \|\mathcal{H}(u_0 - h(u), g)(\tau)\| d\tau + \frac{M}{\Gamma(r)} \frac{a^r}{r} \int_{t_1}^{t_2} \int_0^{\tau} \alpha(s) ds d\tau \psi(k) \\ & + \int_0^{t_1} \big\|(t_1 - \tau)^{r-1} P_r(t_1 - \tau) - (t_2 - \tau)^{r-1} P_r(t_2 - \tau)\big\| \\ & \times \int_0^{\tau} \alpha(s) ds d\tau \psi(k) \\ = & E_1 + E_2 + E_3 + E_4 + E_5.\end{aligned}$$

Clearly E_1 and $E_3 \to 0$ as $t_1 \to t_2$, by the property of compactness of S_r and P_r and by Lebasgues dominated convergence theorem. $E_2 \to 0$ and $E_4 \to 0$ as $t_1 \to t_2$ is obvious. By compactness of P_r we see that $\|P_r(t_1 - \tau) - P_r(t_2 - \tau)\| \to$

0, so by Lebesgue's dominated convergence theorem $E_5 \to 0$. This means that $\|\mathcal{Y}(u)(t_1) - \mathcal{Y}(u)(t_2)\| \to 0$ so $\mathcal{Y}$ is equicontinuous.

Step 3: For any $t \in \mathcal{J}$ construct the set

$$\mathcal{E}(t) = \{(\mathcal{Y}(u))(t) : u(.) \in \mathcal{W}_k\}.$$

We show that $\mathcal{E}$ is relatively compact.

For $t = 0$, $\mathcal{E} = \{u_0 - h(u)\}$ and as $h(u)$ is bounded in $\mathbb{U}$, so it is true for $t = 0$.

Define for $0 < \epsilon < t$

$$\mathcal{E}_\epsilon(t) = \{u_\epsilon(t) : u(.) \in \mathcal{W}_k\}$$

such that

$$u_\epsilon(t) = S_r(t)[u_0 - h(u)] + \int_0^{t-\epsilon} (t-\tau)^{r-1} P_r(t-\tau) \Big[B\mathcal{H}(u_0 - h(u), g)(\tau) + g(\tau, u(\tau))\Big] d\tau.$$

As we know $S_r(t)$ and $P_r(t)$ are compact operators, so the set $\mathcal{E}_\epsilon(t)$ is relatively compact in $\mathbb{U}$ for any ϵ with $0 < \epsilon < t$. Now for any $u(.) \in \mathcal{W}_k$

$$\begin{aligned}
\|\mathcal{Y}(u)(t) - u_\epsilon(t)\| &\leq \left\| \int_0^{\epsilon} (t-\tau)^{r-1} P_r(t-\tau) \Big[B\mathcal{H}(u_0 - h(u), g)(\tau) + g(\tau, u(\tau))\Big] d\tau \right\| \\
&\leq \frac{M}{\Gamma(r)} \Big(\frac{\epsilon^r}{r}\Big)^{\frac{1}{2}} \Big[M_v + \Big(\frac{\epsilon^r}{r}\Big)^{\frac{1}{2}} \|\alpha\| \psi(k)\Big] \\
&\to 0 \quad \text{as} \quad \epsilon \to 0^+.
\end{aligned}$$

So the set $\mathcal{E}_\epsilon(t)$ is arbitrarily close to $\mathcal{E}(t)$. Hence for each $t \in \mathcal{J}$, $\mathcal{E}(t)$ is relatively compact in $\mathbb{U}$.

Step 4: $\mathcal{Y}$ is continuous.

Consider the sequence $\{u_n\}$ with $u_n \in \mathbb{U}$ be such that $u_n \to \bar{u}$ as $n \to \infty$. Now

$$\begin{aligned}
\|\mathcal{Y}(u_n)(t) - \mathcal{Y}(\bar{u})(t)\| \leq{}& \big\|S_r(t)\big[h(u_n) - h(\bar{u})\big]\big\| + \int_0^t (t-\tau)^{r-1} P_r(t-\tau) \\
&\times \|B\mathcal{H}(u_0 - h(u_n), g(u_n))(\tau) - B\mathcal{H}(u_0 - h(\bar{u}), g(\bar{u}))(\tau)\| \, d\tau \\
&+ \int_0^t (t-\tau)^{r-1} P_r(t-\tau) \, \|g(\tau, u_n(\tau)) - g(\tau, \bar{u}(\tau))\| \, d\tau
\end{aligned}$$

From the property that g and h are continuous, $g(u_n) \to g(\bar{u})$ and $h(u_n) \to h(\bar{u})$ as $n \to \infty$. So the right hand side of the above expression tends to 0 as $n \to \infty$, implying that $\mathcal{Y}$ is continuous. So by Ascoli-Arzela theorem of infinite dimensional version, $\mathcal{Y}$ is a completely continuous operator on $C(\mathcal{J}, \mathbb{U})$.

Thus, all the requisites of Schauder fixed point theorem are satisfied, hence $\mathcal{Y}$ has at least one fixed point which is a mild solution of (1).

□

4. Application

Consider the following fractional differential equation

$$\begin{aligned}
{}^{C}D^r_{0+} y(t,z) &= y_{zz}(t,z) + by(\frac{t}{4}, z) + Bv(t,z), \quad t \in \mathcal{J} = [0,1]; \\
y(t,0) &= y(t,\pi) = 0, \quad t \in \mathcal{J}, \\
y(0,z) &+ \sum_{j=1}^{p} c_j y(t_j, z) = y_0,
\end{aligned} \tag{5}$$

where $t_j \in (0,1), j = 1, 2, \ldots, p$.

We take the both spaces $\mathbb{U} = \mathbb{V} = L^2[0,\pi]$, and define $u(t) = y(t,z)$. Now denote the operator A as $Au = u''$ with its domain is defined as

$$D(A) = \{u \in \mathbb{U} : u, u' \text{ are absolutely continuous and } u'' \in \mathbb{U}, u(0) = u(\pi) = 0\}.$$

Then A has eigenvalues $-n^2$, $\forall n \in \mathbb{N}$ and has a discrete spectrum. If we let $\mu_n = n^2\pi^2$ and $\varphi_n(z) = \sqrt{\frac{2}{\pi}}\sin(n\pi z)$ for each $n \in \mathbb{N}$, then $\{-\mu_n; \varphi_n\}_{n=1}^{\infty}$ is the eigensystem of A and the set $\{\varphi_n\}_{n=0}^{\infty}$ forms an orthogonal basis of $\mathbb{U}$.

Now

$$Az = \sum_{n=1}^{\infty} n^2 < z, \varphi_n > \varphi_n, \text{ and}$$

$$\mathcal{Q}(t)z = \sum_{n=0}^{\infty} e^{-n^2 t} < z, \varphi_n > \varphi_n.$$

Which shows that A generates a strongly continuous semigroup $\{\mathcal{Q}\}_{t\geq 0}$ which can be easily verified to be compact, self-adjoint and analytic. So the assumption $A1$ is satisfied.

Moreover, $\|\mathcal{Q}(t)\| \leq e^{-t} \leq 1 = M$. Let us take $g(t, u(t)) = by(t/4, z)$ and the set $\mathcal{W}_k = \{u \in \mathbb{U} : \|u\| \leq k\}$, then for $t \in \mathcal{J}, u \in \mathcal{W}_k$, we have

$$\|g(t, u(t))\| = \|by(t/4, z)\| \leq b \int_0^1 \|u(\tau)\|^2 d\tau = \alpha(t)\beta(u),$$

for $t \in \mathcal{J}$ and $u \in \mathcal{W}_k \in \mathbb{U}$ satisfying assumption $A2$.

Taking $h(u) = \sum_{j=1}^{p} c_j y(t,z)$, $u_0 = y_0$, we see that $\|h(u)\| \leq L_h\|u\|$ where $L_h = \max_{1\leq j\leq p} c_j$, thereby satisfying assumption $A3$.

To see that the assumption $A4$ is satisfied, we have to show that the associated linear system of (5) is exactly null controllable. By virtue of the Lemma 3, we must find some $\gamma > 0$ such that

$$\|(\mathcal{L}_0^a)^* u\| \geq \gamma \|(\mathcal{N}_0^a)^* u\|, \quad \forall u \in \mathbb{U}.$$

Or equivalently

$$\int_0^1 \left\|(1-\tau)^{r-1} B^* P_r^*(1-\tau)\right\|^2 d\tau \geq \gamma\Big[\|S_r^*(1)\|^2 + \int_0^1 \left\|(1-\tau)^{r-1} P_r^*(1-\tau)\right\|^2 d\tau\Big].$$

Following the method applied in [6], we can find that $\gamma = \frac{1}{2}$ and hence $A4$ is satisfied. For the inequality mentioned in the Theorem 3.1, it can be achieved by suitable choice of c_j. Hence all the requisites of Theorem 3.1 are satisfied and hence the system (5) is null controllable.

5. Conclusion

In this article we dealt with a nonlocal fractional dynamical system with Caputo derivative of order $0 < r < 1$. The conditions of null controllability of this system are established using the conditions of null controllability of the corresponding linear system. By use of fractional calculus, semigroup theory and fixed point theorem we achieved the results. Finally with the help of an example the theoretical results are illustrated.

Our future work will include investigation of exact null controllability of finite and infinite dimensional fractional system with multiple delays in control and state variable.

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