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# FEKETE-SZEGÖ INEQUALITIES FOR CERTAIN CLASS OF ANALYTIC FUNCTIONS OF COMPLEX ORDER DEFINED BY CONVOLUTION

R. M. EL-ASHWAH, M. K. AOUF AND F. M. ABDULKAREM

ABSTRACT. In this paper, we obtain Fekete-Szegö inequalities for a certain class of analytic functions
$f(z)$ for which $1+\frac{1}{b}\left[\left(\frac{(f*g)(z)}{z}\right)^{\alpha}\left(\frac{(f*h)(z)}{z}\right)^{\beta}-1\right]\prec\varphi(z)$ ($b\neq 0$, complex , $\alpha$ and $\beta$ are real numbers). Sharp bounds for the Fekete-Szegö functional $\left|a_3-\mu a_2^2\right|$ are obtained.

## 1. INTRODUCTION

Let $\mathcal{A}$ denoted the class of functions $f(z)$ which are analytic in the open unit disc $U=\{z:z\in\mathbb{C}$ and $|z|<1\}$ of the form:

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k\ (z\in U). \tag{1.1}$$

Let $\mathcal{S}$ be the family of functions $f(z)\in\mathcal{A}$, which are univalent. A function $f(z)\in\mathcal{S}$ is said to be starlike of order $\rho$, denoted by $\mathcal{S}^*(\rho)$, if and only if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\}>\rho\ (0\leq\rho<1;z\in U). \tag{1.2}$$

A function $f(z)\in\mathcal{S}$ is said to be convex of order $\rho$, denoted by $\mathcal{K}(\rho)$, if and only if

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\rho\ (0\leq\rho<1;z\in U). \tag{1.3}$$

The classes $\mathcal{S}^*(\rho)$ and $\mathcal{K}(\rho)$ were defined by Robertson [29]. From (1.2) and (1.3) it follows that

$$f(z)\in\mathcal{K}(\rho)\Leftrightarrow zf'(z)\in\mathcal{S}^*(\rho). \tag{1.4}$$

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A function $f(z) \in \mathcal{S}$ is said to be close-to-convex of order $\rho$, denoted by $\mathcal{C}(\rho)$ if and only if

$$\Re\left\{\frac{f'(z)}{g'(z)}\right\} > \rho \ (0 \leq \rho < 1; g \in \mathcal{K}; z \in U), \tag{1.5}$$

where $\mathcal{C}(0) = \mathcal{C}$ (see Kaplan [11]).
We note that:

$$\mathcal{S}^*(0) = \mathcal{S}^* \text{ and } \mathcal{K}(0) = \mathcal{K}$$

and

$$\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{C} \subset \mathcal{S}.$$

In Bieberbach [6] proved that $f(z) \in \mathcal{S}$, then $\left|a_3 - a_2^2\right| \leq 1$.
A classical theorem of Fekete-Szegö [9] states that, for $f(z) \in \mathcal{S}$ given by (1.1),

$$\left|a_3 - \mu a_2^2\right| \leq 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right) \qquad \text{if } 0 \leq \mu \leq 1, \tag{1.6}$$

holds for any normalized univalent function of the form (1.1) in the open unit disc $U$ and for $0 \leq \mu \leq 1$. This inequality is sharp for each $\mu$(see [9]) . The coefficient functional

$$\phi_\mu(f) = a_3 - \mu a_2^2 = \frac{1}{6}\left(f'''(0) - \frac{3\mu}{2}[f''(0)]\right), \tag{1.7}$$

on normalized analytic functions in $U$ represents various geometric quantities for example, when $\mu = 1$, $\phi_1(f) = a_3 - a_2^2$, becomes $\mathcal{S}_f(0)/6$, where $\mathcal{S}_f$ denoted the Schwarzian derivative $(f''/f')' - (f''/f')^2/2$ of locally univalent functions in $U$. In literature, there exists a large number of results about inequalities for $\phi_\mu(f)$ corresponding to various subclass of $\mathcal{S}$. The problem of maximising the absolute value of the functional is called the Fekete-Szegö problem (see [9] and [13]) , Koepf [14], solved the Fekete-Szegö problem for close-to-convex functions and the largest real number $\mu$ for which $\phi_\mu(f)$ is maximised by the Koebe function $z/(1-z)^2$ is $\mu = 1/3$ (see [14] and [19]), this result was generalized for functions that are close-to-convex.

Given two functions $f$ and $g$, which are analytic in $U$, we say that the function $f(z)$ is subordinate to $g(z)$ in $U$ and write $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$, analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ $(z \in U)$. Indeed it is known that $f(z) \prec g(z) \Rightarrow f(0) = g(0)$ and $f(U) \subset g(U)$.

In particular, if $g(z)$ is univalent function in $\mathbb{U}$, then

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U) \ \ \text{(see [21])}.$$

Let $\varphi(z)$ be an analytic function with positive real part on $U$ satisfies $\varphi(0) = 1$ and $\varphi'(0) > 0$ which maps $U$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. Let $\mathcal{S}^*(\varphi)$ be the class of functions $f(z) \in \mathcal{S}$ for which

$$\frac{zf'(z)}{f(z)} \prec \varphi(z) \ (z \in U), \tag{1.8}$$

and $\mathcal{K}(\varphi)$ be the class of functions $f(z) \in \mathcal{S}$ for which

$$1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \ (z \in U). \tag{1.9}$$

The classes of $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ were introduced and studied by Ma and Minda [20]. The class $\mathcal{S}^*(\rho)$ of starlike functions of order $\rho$ and the class $\mathcal{K}(\rho)$ of convex functions of order $\rho$ $(0 \leq \rho < 1)$ are the special cases of $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$, respectively, when $\varphi(z) = \dfrac{1+(1-2\rho)z}{1-z}$ $(0 \leq \rho < 1)$.

In [1] Ali et al. introduced the class $\mathcal{M}(\gamma,\varphi)$ of $\gamma-$convex function with respect to $\varphi$ consisting of functions $f(z) \in \mathcal{A}$ satisfying

$$\left[(1-\gamma)\frac{zf'(z)}{f(z)} + \gamma\left(1+\frac{zf''(z)}{f'(z)}\right)\right] \prec \varphi(z)(\gamma \geq 0).$$

For related results, refer ([1], [2], [3], [26], [27], [32]) and the references cited therein. Obradović [25] introduced the class of functions $f(z) \in \mathcal{A}$ satisfying

$$\Re\left\{f'(z)\left(\frac{z}{f(z)}\right)^{\lambda+1}\right\} > 0 \quad (0 < \lambda < 1).$$

Tuneksi and Darus [31] obtained Fekete-Szegö inequality for the class of functions $f(z) \in \mathcal{A}$ satisfying

$$\Re\left\{f'(z)\left(\frac{z}{f(z)}\right)^{\lambda+1}\right\} > \rho \quad (0 \leq \rho < 1; 0 < \lambda < 1).$$

The Hadamard product (or convolution) of $f(z)$, given by (1.1) and

$$g(z) = z + \sum_{k=2}^{\infty} g_k z^k, \quad h(z) = z + \sum_{k=2}^{\infty} h_k z^k, \tag{1.10}$$

are defined by

$$(f*g)(z) = z + \sum_{k=2}^{\infty} a_k g_k z^k = (g*f)(z)$$

and

$$(f*h)(z) = z + \sum_{k=2}^{\infty} a_k h_k z^k = (h*f)(z).$$

**Definition 1.** *Let $\alpha$ and $\beta$ are real numbers and $b \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$. The class $\mathcal{M}^b_{\alpha,\beta}(g,h,\varphi)$ consists of all analytic functions $f(z) \in \mathcal{A}$ satisfyies*

$$1 + \frac{1}{b}\left[\left(\tfrac{(f*g)(z)}{z}\right)^{\alpha}\left(\tfrac{(f*h)(z)}{z}\right)^{\beta} - 1\right] \prec \varphi(z), \; g_k > 0, \; h_k > 0, \; \alpha g_k + \beta h_k > 0. \tag{1.11}$$

We note that for suitable choices of $g$, $h$, $\alpha$, $\beta$, $b$ and $\varphi(z)$, we obtain the following subclasses:

**Remark 1.** *(i) $\mathcal{M}^1_{\alpha,\beta}(g,h,\varphi) = \mathcal{M}^{\alpha,\beta}_{g,h}(\varphi)$ (see Kumar and Kumar [15]);*
*(ii) $\mathcal{M}^1_{1,-1}(g,h,\varphi) = \mathcal{M}_{g,h}(\varphi)(g_k - h_k > 0)$ (see Murugusundaramoorthy et al. [22]);*
*(iii) $\mathcal{M}^1_{1,-1}(\frac{z}{(1-z)^2}, \frac{z}{(1-z)}, \varphi) = \mathcal{S}^*(\varphi)$ and $\mathcal{M}^1_{1,-1}(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}, \varphi) = \mathcal{K}(\varphi)$ (see Ma and Minda [20]);*

*(iv)* $\mathcal{M}_{1,-1}^{1}(\frac{z}{(1-z)^2},\frac{z}{(1-z)},\frac{1+z}{1-z})=\mathcal{S}^*$ *and* $\mathcal{M}_{1,-1}^{1}(\frac{z+z^2}{(1-z)^3},\frac{z}{(1-z)^2},\frac{1+z}{1-z})=\mathcal{K}$ *(see Robertson* [29]*);*
*(v)* $\mathcal{M}_{1,-1}^{1-\rho}(\frac{z}{(1-z)^2},\frac{z}{(1-z)},\frac{1+z}{1-z})=\mathcal{S}^*(\rho)(0\le\rho<1)$ *and*
$\mathcal{M}_{1,-1}^{1-\rho}(\frac{z+z^2}{(1-z)^3},\frac{z}{(1-z)^2},\frac{1+z}{1-z})=\mathcal{K}(\rho)(0\le\rho<1)$ *(see Robertson* [29]*);*
*(vi)* $\mathcal{M}_{1,-1}^{b}(\frac{z}{(1-z)^2},\frac{z}{(1-z)},\varphi)=\mathcal{S}_b^*(\varphi)$ *and* $\mathcal{M}_{1,-1}^{b}(\frac{z+z^2}{(1-z)^3},\frac{z}{(1-z)^2},\varphi)=\mathcal{C}_b(\varphi)$ *(see Ravichandran et al.* [28]*);*
*(vii)* $\mathcal{M}_{1,-(\lambda+1)}^{1}(\frac{z}{(1-z)^2},\frac{z}{(1-z)},\frac{1+z}{1-z})=\mathcal{M}(\lambda)(0<\lambda<1)$ *(see Obradović* [25]*);*
*(iix)* $\mathcal{M}_{1,-1}^{b}(\frac{z}{(1-z)^2},\frac{z}{(1-z)},\frac{1+z}{1-z})=\mathcal{S}(b)(b\in\mathbb{C}^*)$ *(see Nasr and Aouf* [24]*, see also Aouf et al.* [4]*);*
*(ix)* $\mathcal{M}_{1,-1}^{b}(\frac{z+z^2}{(1-z)^3},\frac{z}{(1-z)^2},\frac{1+z}{1-z})=\mathcal{C}(b)(b\in\mathbb{C}^*)$ *(see Waitrowski* [32]*, see also Nasr and Aouf* [23]*):*
*(x)* $\mathcal{M}_{1,-1}^{\cos\eta e^{-i\eta}}(\frac{z}{(1-z)^2},\frac{z}{(1-z)},\frac{1+z}{1-z})=\mathcal{S}^\eta(|\eta|<\frac{\pi}{2})$ *(see Spacek* [30]*);*
*(xi)* $\mathcal{M}_{1,-1}^{(1-\rho)\cos\eta e^{-i\eta}}(\frac{z}{(1-z)^2},\frac{z}{(1-z)},\frac{1+z}{1-z})=\mathcal{S}^\eta(\rho)(0\le\rho<1,\ |\eta|<\frac{\pi}{2})$ *(see Keogh and Merkes* [12]*, see also Libera* [16], [17]*);*
*(xii)* $\mathcal{M}_{1,-1}^{(1-\rho)\cos\eta e^{-i\eta}}(\frac{z+z^2}{(1-z)^3},\frac{z}{(1-z)^2},\frac{1+z}{1-z})=\mathcal{C}^\eta(\rho)(0\le\rho<1,\ |\eta|<\frac{\pi}{2})$ *(see Chichra* [7]*);*
*(xiii)* $\mathcal{M}_{1,-1}^{b}\left(z+\sum_{k=2}^{\infty}k^{n+1}z^k,z+\sum_{k=2}^{\infty}k^nz^k,\varphi\right)=\mathcal{H}_{n,b}(\varphi)(n\in\mathbb{N}_0=\mathbb{N}\cup\{0\},\mathbb{N}=\{1,2,...\},\ b\in\mathbb{C}^*)$ *(see Aouf and Silverman* [5]*);*
*(xiv)*$\mathcal{M}_{1,-1}^{b}\left(z+\sum_{k=2}^{\infty}k[(\lambda-\rho)(\gamma-\delta)(k-1)+1]^nz^k,z+\sum_{k=2}^{\infty}[(\lambda-\rho)(\gamma-\delta)(k-1)+1]^nz^k,\varphi\right)$
$=\mathcal{G}_{\gamma,\delta,\lambda,\rho}^{n,b}(\varphi)(\gamma,\delta,\lambda,\rho>0,\ \gamma>\delta,\ \lambda>\rho,\ n\in\mathbb{N}_0,\ b\in\mathbb{C}^*)$ *(see Aouf et al.* [3]*);*
*(xv)*$\mathcal{M}_{1,-1}^{1}\left(z+\sum_{k=n+1}^{\infty}k\left[\frac{1+(\lambda_1+\lambda_2)(k-1)}{1+\lambda_2(k-1)}\right]^mC(n,k)z^k,z+\sum_{k=n+1}^{\infty}\left[\frac{1+(\lambda_1+\lambda_2)(k-1)}{1+\lambda_2(k-1)}\right]^mC(n,k)z^k,\varphi\right)$
$=\mathcal{M}_{\lambda_1,\lambda_2}^{n,m}(\varphi)(m,n\in\mathbb{N}_0,\ \lambda_2\ge\lambda_1\ge0)$ *(see Eljamal and Darus* [8]*);*
*(xvi)*$\mathcal{M}_{1,-1}^{1}\left(z+\sum_{k=2}^{\infty}k[(\lambda-\rho)(\gamma-\delta)(k-1)+1]^nz^k,z+\sum_{k=2}^{\infty}[(\lambda-\rho)(\gamma-\delta)(k-1)+1]^nz^k,\varphi\right)$
$=\mathcal{M}_{\gamma,\delta,\lambda,\rho}^{n}(\varphi)(\gamma,\delta,\lambda,\rho>0,\gamma>\delta,\ \lambda>\rho,n\in\mathbb{N}_0)$ *(see Ramadan and Darus*[26]*).*

Also, we note that:

**Remark 2.** *(i) Putting* $\alpha=\beta=1$ *and* $b=\cos\eta e^{-i\eta}(|\eta|<\frac{\pi}{2})$. *Then, we have*

$$\begin{aligned}\mathcal{M}_{1,1}^{\cos\eta e^{-i\eta}}(g,h,\varphi)&=\mathcal{M}^\eta(g,h,\varphi)\\&=\left\{f\in\mathcal{A}:\frac{e^{i\eta}\left(\frac{(f*g)(z)}{z}\right)\left(\frac{(f*h)(z)}{z}\right)-i\sin\eta}{\cos\eta}\prec\varphi(z)\right.\\&\qquad\left.(|\eta|<\frac{\pi}{2};z\in U)\right\};\end{aligned}$$

*(ii) Putting* $\alpha=\beta=1$ *and* $b=(1-\rho)\cos\eta e^{-i\eta}(0\le\rho<1,\ |\eta|<\frac{\pi}{2})$. *Then, we have*

$$\begin{aligned}\mathcal{M}_{1,1}^{(1-\rho)\cos\eta e^{-i\eta}}(g,h,\varphi) &= \mathcal{M}_{\rho}^{\eta}(g,h,\varphi)\\ &= \left\{ f\in\mathcal{A}: \frac{e^{i\eta}\left(\frac{(f*g)(z)}{z}\right)\left(\frac{(f*h)(z)}{z}\right)-\rho\cos\eta-i\sin\eta}{(1-\rho)\cos\eta}\prec\varphi(z)\right.\\ &\qquad \left.(0\le\rho<1;|\eta|<\frac{\pi}{2};z\in U)\right\};\end{aligned}$$

*(iii) Putting* $b=(1-\rho)\cos\eta e^{-i\eta}(0\le\rho<1,\ |\eta|<\frac{\pi}{2})$. *Then, we have*

$$\begin{aligned}\mathcal{M}_{\alpha,\beta}^{(1-\rho)\cos\eta e^{-i\eta}}(g,h,\varphi) &= \mathcal{M}_{\alpha,\beta,\rho}^{\eta}(g,h,\varphi)\\ &= \left\{ f\in\mathcal{A}: \frac{e^{i\eta}\left(\frac{(f*g)(z)}{z}\right)^{\alpha}\left(\frac{(f*h)(z)}{z}\right)^{\beta}-\rho\cos\eta-i\sin\eta}{(1-\rho)\cos\eta}\prec\varphi(z)\right.\\ &\qquad \left.(\alpha,\beta\in\mathbb{R};0\le\rho<1;|\eta|<\frac{\pi}{2};z\in U)\right\};\end{aligned}$$

*(iv) Putting* $\alpha=1,\ \beta=-1$ *and* $b=(1-\rho)\cos\eta e^{-i\eta}(0\le\rho<1,\ |\eta|<\frac{\pi}{2})$. *Then, we have*

$$\begin{aligned}\mathcal{M}_{1,-1}^{(1-\rho)\cos\eta e^{-i\eta}}(g,h,\varphi) &= \mathcal{N}_{\rho}^{\eta}(g,h,\varphi)\\ &= \left\{ f\in\mathcal{A}: \frac{e^{i\eta}\left(\frac{(f*g)(z)}{(f*h)(z)}\right)-\rho\cos\eta-i\sin\eta}{(1-\rho)\cos\eta}\prec\varphi(z)\right.\\ &\qquad \left.(|\eta|<\frac{\pi}{2};z\in U)\right\};\end{aligned}$$

*(v) Putting* $\beta=0$ *and* $b=(1-\rho)\cos\eta e^{-i\eta}(0\le\rho<1,\ |\eta|<\frac{\pi}{2})$. *Then, we have*

$$\begin{aligned}\mathcal{M}_{\alpha,0}^{(1-\rho)\cos\eta e^{-i\eta}}(g,h,\varphi) &= \mathcal{M}_{\alpha,\rho}^{\eta}(g,h,\varphi)\\ &= \left\{ f\in\mathcal{A}: \frac{e^{i\eta}\left(\frac{(f*g)(z)}{z}\right)^{\alpha}-\rho\cos\eta-i\sin\eta}{(1-\rho)\cos\eta}\prec\varphi(z)\right.\end{aligned}$$

$$\left.(\alpha \in \mathbb{R}; 0 \leq \rho < 1; |\eta| < \tfrac{\pi}{2}; z \in U)\right\};$$

*(vi) Putting* $\alpha = 0$ *and* $b = (1-\rho)\cos\eta e^{-i\eta}(0 \leq \rho < 1,\ |\eta| < \frac{\pi}{2})$*. Then, we have*

$$\begin{aligned}\mathcal{M}_{0,\beta}^{(1-\rho)\cos\eta e^{-i\eta}}(g,h,\varphi) &= \mathcal{M}_{\beta,\rho}^{\eta}(g,h,\varphi) \\ &= \left\{ f \in \mathcal{A} : \frac{e^{i\eta}\left(\frac{(f*h)(z)}{z}\right)^{\beta} - \rho\cos\eta - i\sin\eta}{(1-\rho)\cos\eta} \prec \varphi(z) \right. \\ &\qquad \left. (\beta \in \mathbb{R}; 0 \leq \rho < 1; |\eta| < \frac{\pi}{2}; z \in U) \right\}.\end{aligned}$$

In this paper, we obtain the Fekete-Szegö inequalities for functions in the class $\mathcal{M}_{\alpha,\beta}^{b}(g,h,\varphi)$.

## 2. Fekete-Szegö problem

To prove our results, we need the following lemmas.

**Lemma 1.** *[*[12], [18]*]. If* $p(z) = 1 + c_1 z + c_2 z^2 + ...$*, is a function with positive real part in* $U$ *and* $\mu$ *is a complex number, then*

$$\left|c_2 - \mu c_1^2\right| \leq 2\max\{1; |2\mu - 1|\}.$$

*The result is sharp for the functions given by*

$$p(z) = \frac{1+z^2}{1-z^2} \quad \text{and} \quad p(z) = \frac{1+z}{1-z}.$$

**Lemma 2.** [20]*. If* $p(z) = 1 + c_1 z + c_2 z^2 + ...$*, is a function with positive real part in* $U$*, then*

$$\left|c_2 - \nu c_1^2\right| \leq \begin{cases} -4\nu + 2 & if\ \nu \leq 0, \\ 2 & if\ 0 \leq \nu \leq 1, \\ 4\nu - 2 & if\ \nu \geq 1. \end{cases}$$

*When* $\nu < 0$ *or* $\nu > 1$*, the equality holds if and only if* $p_1(z) = \dfrac{1+z}{1-z}$ *or one of its rotations. If* $0 < \nu < 1$*, then the equality holds if and only if* $p_2(z) = \dfrac{1+z^2}{1-z^2}$ *or one of its rotations. If* $\nu = 0$*, the equality holds if and only if*

$$p_3(z) = \left(\frac{1}{2} + \frac{1}{2}\gamma\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\gamma\right)\frac{1-z}{1+z}\ (0 \leq \gamma \leq 1),$$

*or one of its rotations. If $\nu = 1$, the equality holds if and only if*

$$\frac{1}{p_4(z)} = \left(\frac{1}{2} + \frac{1}{2}\gamma\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\gamma\right)\frac{1-z}{1+z} \ (0 \le \gamma \le 1).$$

*or one of its rotations. Also the above upper bound is sharp and it can be improved as follows when $0 < \nu < 1$ :*

$$\left|c_2 - \nu c_1^2\right| + \nu \left|c_1\right|^2 \le 2 \ (0 < \nu \le \frac{1}{2}),$$

*and*

$$\left|c_2 - \nu c_1^2\right| + (1-\nu)\left|c_1\right|^2 \le 2 \ (\frac{1}{2} < \nu < 1).$$

Unless otherwise mentioned, we assume throughout this paper that $b \in \mathbb{C}^*$and $\alpha$, $\beta \in \mathbb{R}$.

**Theorem 1.** *Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + ..., B_1 > 0$. If $f(z)$ given by (1.1) belongs to the class $\mathcal{M}^b_{\alpha,\beta}(g, h, \varphi)$ and $\mu$ is a complex number, then*

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| \le \ & \frac{B_1 |b|}{2(\alpha g_3 + \beta h_3)} \max\left\{1, \left| -\frac{B_2}{B_1} \right.\right. \\ & \left.\left. + \frac{[(\alpha(\alpha-1)g_2^2 + \beta(\beta-1)h_2^2 + 2\alpha\beta g_2 h_2 + 2\mu(\alpha g_3 + \beta h_3)]b}{2(\alpha g_2 + \beta h_2)^2} B_1 \right|\right\}.\end{aligned} \tag{2.1}$$

*The result is sharp.*

*Proof.* If $f(z) \in \mathcal{M}^b_{\alpha,\beta}(g, h, \varphi)$, then there is a Schwarz function $w(z)$ in $U$ with $w(0) = 0$ and $|w(z)| < 1$ in $U$ and such that

$$1 + \frac{1}{b}\left[\left(\frac{(f*g)(z)}{z}\right)^{\alpha}\left(\frac{(f*h)(z)}{z}\right)^{\beta} - 1\right] = \varphi(w(z)). \tag{2.2}$$

Define the function $p_1(z)$ by

$$p_1(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + .... \tag{2.3}$$

Since $w(z)$ is a Schwarz function, we see that $\Re\{p_1(z)\} > 0$ and $p_1(0) = 1$. Define

$$p(z) = 1 + \frac{1}{b}\left[\left(\frac{(f*g)(z)}{z}\right)^{\alpha}\left(\frac{(f*h)(z)}{z}\right)^{\beta} - 1\right] = 1 + b_1 z + b_2 z^2 + .... \tag{2.4}$$

In view of (2.2), (2.3) and (2.4), we have

$$p(z) = \varphi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right). \tag{2.5}$$

Since

$$\frac{p_1(z) - 1}{p_1(z) + 1} = \frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right) z^2 + \left(c_3 + \frac{c_1^3}{4} - c_1 c_2\right) z^3 + ...\right].$$

Therefore, we have

$$\varphi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right) = 1 + \frac{1}{2} B_1 c_1 z + \left[\frac{1}{2} B_1 \left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4} B_2 c_1^2\right] z^2 + ..., \tag{2.6}$$

and from this equation and (2.4), we obtain

$$b_1 = \frac{1}{2}B_1 c_1,$$

and

$$b_2 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2.$$

A computation shows that

$$\left(\frac{(f*g)(z)}{z}\right)^{\alpha} = 1 + \alpha a_2 g_2 z + \left(\alpha a_3 g_3 + \frac{\alpha(\alpha-1)}{2}a_2^2 g_2^2\right)z^2 + ...,$$

and

$$\left(\frac{(f*h)(z)}{z}\right)^{\beta} = 1 + \beta a_2 h_2 z + \left(\alpha a_3 h_3 + \frac{\beta(\beta-1)}{2}a_2^2 h_2^2\right)z^2 + ... \ .$$

Substituting these in (2.4) and comaring coefficients, we have

$$bb_1 = (\alpha g_2 + \beta h_2)a_2, \tag{2.7}$$

and

$$bb_2 = (\alpha g_3 + \beta h_3)a_3 + [(\alpha(\alpha-1)g_2^2 + \beta(\beta-1)h_2^2 + 2\alpha\beta g_2 h_2]\frac{a_2^2}{2} \tag{2.8}$$

or, iquivalently, we have

$$a_2 = \frac{B_1 c_1 b}{2(\alpha g_2 + \beta h_2)}, \tag{2.9}$$

and

$$a_3 = \frac{2(\alpha g_2+\beta h_2)^2[2(c_2 - \frac{1}{2}c_1^2)B_1 + B_2 c_1^2]b - [(\alpha(\alpha-1)g_2^2 + \beta(\beta-1)h_2^2 + 2\alpha\beta g_2 h_2]B_2 c_1^2 b^2}{8(\alpha g_3 + \beta h_3)(\alpha g_2 + \beta h_2)^2}. \tag{2.10}$$

From (2.9) and (2.10), we have

$$a_3 - \mu a_2^2 = \frac{B_1 b}{2(\alpha g_3 + \beta h_3)}[c_2 - \nu c_1^2], \tag{2.11}$$

where

$$\nu = \frac{1}{2}\left[1 - \frac{B_2}{B_1} + \frac{[(\alpha(\alpha-1)g_2^2 + \beta(\beta-1)h_2^2 + 2\alpha\beta g_2 h_2 + 2\mu(\alpha g_3 + \beta h_3)]b}{2(\alpha g_2 + \beta h_2)^2}B_1\right]. \tag{2.12}$$

Our result now follows by an application of Lemma 1. The result is sharp for the functions

$$1 + \frac{1}{b}\left[\left(\frac{(f*g)(z)}{z}\right)^{\alpha}\left(\frac{(f*h)(z)}{z}\right)^{\beta} - 1\right] = \varphi(z^2), \tag{2.13}$$

and

$$1 + \frac{1}{b}\left[\left(\frac{(f*g)(z)}{z}\right)^{\alpha}\left(\frac{(f*h)(z)}{z}\right)^{\beta} - 1\right] = \varphi(z). \tag{2.14}$$

This completes the proof of Theorem 1. □

**Remark 3.** *(i) Taking $b = 1$ in Theorem 1, we obtain the result obtaind by Kumer and Kumer [[15], Theorm 2.4];*
*(ii) Taking $\alpha = 1$, $\beta = -1$, $g = \dfrac{z}{(1-z)^2}$ and $h = \dfrac{z}{(1-z)}$ in Theorem 1, we improve the result obtaind by Ravichandran et al. [[24], Theorm 4.1];*
*(iii) Taking $\alpha = 1$, $\beta = -1$, $g = z + \sum_{k=2}^{\infty} k^{n+1} z^k$ and $h = z + \sum_{k=2}^{\infty} k^n z^k (n \in \mathbb{N}_0)$ in Theorem 1, we obtain the result obtaind by Aouf and Silverman [[5], Theorm 1];*
*(iv) Taking $\alpha = 1$, $\beta = -1$, $g = z + \sum_{k=2}^{\infty} k[(\lambda-\rho)(\gamma-\delta)(k-1)+1]^n z^k$ and $h = z + \sum_{k=2}^{\infty}[(\lambda-\rho)(\gamma-\delta)(k-1)+1]^n z^k (\gamma, \delta, \lambda, \rho > 0,\ \gamma > \delta,\ \lambda > \rho,\ n \in \mathbb{N}_0,$ $b \in \mathbb{C}^*)$ in Theorem 1, we obtain the result obtaind by Aouf et al. [[3], Theorm 1];*
*(v) Taking $\alpha = 1$, $\beta = -1, b = (1-\rho)\cos\eta e^{-i\eta}(0 \le \rho < 1, |\eta| < \frac{\pi}{2})$, $g = \frac{z}{(1-z)^2}$ and $h = \frac{z}{(1-z)}$ in Theorem 1, we the result obtaind by Keogh and Merkes [[14], Theorm 1];*
*(vi) Taking $\alpha = 1$, $\beta = -1$, $g = z + \sum_{k=2}^{\infty} k^{n+1} z^k$ and $h = z + \sum_{k=2}^{\infty} k^n z^k (n \in \mathbb{N}_0)$ in Theorem 1, we obtain the result obtaind by Goyal and Kumar [[10], Corollary 2.10].*

Also by specializing the parameters in Theorem 1, we obtain the following new sharp results.

Putting $\varphi(z) = \dfrac{1+Az}{1+Bz}(-1 \le B < A \le 1)$, $B_1 = A - B$ and $B_2 = -B(A-B)$ in Theorem 1, we obtain the following corollary:

**Corollary 1.** *If $f(z)$ given by (1.1) belongs to the class $\mathcal{M}_{\alpha,\beta}^{b}(g,h,\varphi)$ and $\mu$ is a complex number, then*

$$\begin{aligned} \left|a_3 - \mu a_2^2\right| \quad &\le \quad \frac{(A-B)\,|b|}{2(\alpha g_3 + \beta h_3)} \max\left\{1, \left|B\right.\right. \\ &+ \left.\left.\frac{[(\alpha(\alpha-1)g_2^2 + \beta(\beta-1)h_2^2 + 2\alpha\beta g_2 h_2 + 2\mu(\alpha g_3 + \beta h_3)]}{2(\alpha g_2 + \beta h_2)^2}(A-B)b\right|\right\}. \end{aligned}$$

*The result is sharp.*

Putting $b = \cos\eta e^{-i\eta}(|\eta| < \frac{\pi}{2})$ in Theorem 1, we obtain the following corollary:

**Corollary 2.** *If $f(z)$ given by (1.1) belongs to the class $\mathcal{M}_{\alpha,\beta}^{\eta}(g,h,\varphi)$ and $\mu$ is a complex number, then*

$$\begin{aligned} \left|a_3 - \mu a_2^2\right| \quad &\le \quad \frac{B_1 \cos\eta}{2(\alpha g_3 + \beta h_3)} \max\left\{1, \left|-\frac{B_2}{B_1}e^{i\eta}\right.\right. \\ &+ \left.\left.\frac{[(\alpha(\alpha-1)g_2^2 + \beta(\beta-1)h_2^2 + 2\alpha\beta g_2 h_2 + 2\mu(\alpha g_3 + \beta h_3)]}{2(\alpha g_2 + \beta h_2)^2} B_1 \cos\eta\right|\right\}. \end{aligned}$$

*The result is sharp.*

Putting $b = (1-\rho)\cos\eta e^{-i\eta}(0 \le \rho < 1,\ |\eta| < \frac{\pi}{2})$ in Theorem 1, we obtain the following corollary:

**Corollary 3.** *If $f(z)$ given by (1.1) belongs to the class $\mathcal{M}^{\eta}_{\alpha,\beta,\rho}(g,h,\varphi)$ and $\mu$ is a complex number, then*

$$\begin{aligned}\left|a_3-\mu a_2^2\right| &\leq \frac{B_1(1-\rho)\cos\eta}{2(\alpha g_3+\beta h_3)}\max\left\{1,\left|-\frac{B_2}{B_1}e^{i\eta}\right.\right.\\ &\left.\left.+\frac{[(\alpha(\alpha-1)g_2^2+\beta(\beta-1)h_2^2+2\alpha\beta g_2h_2+2\mu(\alpha g_3+\beta h_3)]}{2(\alpha g_2+\beta h_2)^2}B_1(1-\rho)\cos\eta\right|\right\}.\end{aligned}$$

*The result is sharp.*

Putting $\alpha=b=1$ and $\beta=-1$ in Theorem 1, we obtain the following corollary:

**Corollary 4.** *If $f(z)$ given by (1.1) belongs to the class $\mathcal{M}_{g,h}(\varphi)$ and $\mu$ is a complex number, then*

$$\begin{aligned}\left|a_3-\mu a_2^2\right| &\leq \frac{B_1}{2(g_3-h_3)}\max\left\{1,\left|-\frac{B_2}{B_1}\right.\right.\\ &\left.\left.+\frac{[-h_2^2-g_2h_2+\mu(g_3-\beta h_3)]}{(g_2-h_2)^2}B_1\right|\right\}.\end{aligned}$$

*The result is sharp.*

Using Lemma 2, we have the following theorem.

**Theorem 2.** *Let $\varphi(z)=1+B_1z+B_2z^2+...,(B_i>0,i\in\mathbb{N},\ b>0)$. If $f(z)$ given by (1.1) belongs to the class $\mathcal{M}^{b}_{\alpha,\beta}(g,h,\varphi)$ and $\mu$ is a real number, then*

$$\left|a_3-\mu a_2^2\right|\leq\begin{cases}\dfrac{B_1\zeta b}{(\alpha g_3+\beta h_3)} & if\ \ \mu\leq\sigma_1,\\ \dfrac{B_1 b}{(\alpha g_3+\beta h_3)} & if\ \ \sigma_1\leq\mu\leq\sigma_2,\\ \dfrac{-B_1\zeta b}{(\alpha g_3+\beta h_3)} & if\ \ \mu\geq\sigma_2,\end{cases}\tag{2.15}$$

*where*

$$\zeta=\frac{B_2}{B_1}-\frac{[(\alpha(\alpha-1)g_2^2+\beta(\beta-1)h_2^2+2\alpha\beta g_2h_{23})+2\mu(\alpha g_3+\beta h_3)]}{2(\alpha g_2+\beta h_2)^2}B_1b,$$

$$\sigma_1=\frac{2(B_2-B_1)(\alpha g_2+\beta h_2)^2-[(\alpha(\alpha-1)g_2^2+\beta(\beta-1)h_2^2+2\alpha\beta g_2h_2)]B_1^2b}{2(\alpha g_3+\beta h_3)B_1^2b},\tag{2.16}$$

*and*

$$\sigma_2=\frac{2(B_2+B_1)(\alpha g_2+\beta h_2)^2-[(\alpha(\alpha-1)g_2^2+\beta(\beta-1)h_2^2+2\alpha\beta g_2h_2]B_1^2b}{2(\alpha g_3+\beta h_3)B_1^2b}.\tag{2.17}$$

*The result is sharp.*

*Proof.* To show that the bounds are sharp, we define the functions $K_{\varphi n}(z)$ $(n\geq 2)$ by

$$1+\frac{1}{b}\left[\left(\frac{(K_{\varphi n}*g)(z)}{z}\right)^{\alpha}\left(\frac{(K_{\varphi n}*h)(z)}{z}\right)^{\beta}-1\right]=\varphi(z^{n-1}),\ K_{\varphi n}(0)=0=K'_{\varphi n}(0)-1,$$

and the functions $F_\eta(z)$ and $G_\eta(z)$ $(0 \leq \eta \leq 1)$ by

$$1+\frac{1}{b}\left[\left(\frac{(F_\eta * g)(z)}{z}\right)^{\alpha}\left(\frac{(F_\eta * h)(z)}{z}\right)^{\beta}-1\right]=\varphi\left(\frac{z(z+\eta)}{1+\eta z}\right),\ F_\eta(0)=0=F_\eta'(0)-1,$$

and

$$1+\frac{1}{b}\left[\left(\frac{(G_\eta * g)(z)}{z}\right)^{\alpha}\left(\frac{(G_\eta * h)(z)}{z}\right)^{\beta}-1\right]=\varphi\left(-\frac{z(z+\eta)}{1+\eta z}\right),\ G_\eta(0)=0=G_\eta'(0)-1.$$

Cleary the functions $K_{\varphi n}(z), F_\eta(z)$ and $G_\eta(z) \in \mathcal{M}^b_{\alpha,\beta}(g,h,\varphi)$. Also we write $K_\varphi = K_{\varphi 2}$.
If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if $f$ is $K_\varphi$ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then the equality holds if $f$ is $K_{\varphi 3}$ or one of its rotations. If $\mu = \sigma_1$, then the equality holds if and only if $f$ is $F_\eta$ or one of its rotations. If $\mu = \sigma_2$, then the equality holds if and only if $f$ is $G_\eta$ or one of its rotations. This completes the proof of Theorem 2. □

**Remark 4.** *(i) Taking $b = 1$ in Theorem 2, we obtain the result obtaind by Kumer and Kumer ([15], Theorm 2.1]);*
*(ii) Taking $\alpha = 1$, $\beta = -1$, $g = z + \sum_{k=2}^{\infty} k^{n+1} z^k$ and $h = z + \sum_{k=2}^{\infty} k^n z^k (n \in \mathbb{N}_0)$ in Theorem 2, we obtain the result obtaind by Aouf and Silverman ([5], Theorm 2];*
*(iii) Taking $\alpha = 1$, $\beta = -1$, $g = z + \sum_{k=2}^{\infty} k[(\lambda-\rho)(\gamma-\delta)(k-1)+1]^n z^k$ and $h = z + \sum_{k=2}^{\infty} [(\lambda-\rho)(\gamma-\delta)(k-1)+1]^n z^k (\gamma,\delta,\lambda,\rho > 0,\ \gamma > \delta,\ \lambda > \rho,\ n \in \mathbb{N}_0,$ $b \in \mathbb{C}^*)$ in Theorem 2, we obtain the result obtaind by Aouf et al. ([3], Theorm 2];*
*(iv) Taking $\alpha = 1$, $\beta = -1$, $b = 1$, $g = z + \sum_{k=n+1}^{\infty} k\left[\frac{1+(\lambda_1+\lambda_2)(k-1)}{1+\lambda_2(k-1)}\right]^m C(n,k) z^k$ and $h = z + \sum_{k=n+1}^{\infty}\left[\frac{1+(\lambda_1+\lambda_2)(k-1)}{1+\lambda_2(k-1)}\right]^m C(n,k) z^k (m,n \in \mathbb{N}_0,\ \lambda_2 \geq \lambda_1 \geq 0)$ in Theorem 2, we obtain the result obtaind by Eljamal and Darus ([8],Theorm 2.1]);*
*(v) Taking $\alpha = 1$, $\beta = -1$ and $b = 1$ in Theorem 2, we the result obtaind by Murugusundaramoorthy et al. ([22],Theorm 2.1]).*

Also, using Lemma 2, we have the following theorem.

**Theorem 3.** *Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + ..., (B_i > 0, i \in \mathbb{N},\ b > 0)$ and*

$$\sigma_3 = \frac{2B_2(\alpha g_2 + \beta h_2)^2 - [(\alpha(\alpha-1)g_2^2 + \beta(\beta-1)h_2^2 + 2\alpha\beta g_2 h_2)]B_1^2 b}{2(\alpha g_3 + \beta h_3)B_1^2 b}.$$

*If $f(z)$ given by (1.1) belongs to the class $\mathcal{M}^b_{\alpha,\beta}(g,h,\varphi)$ and $\mu$ is a real number, then we have*
*(i) If $\sigma_1 \leq \mu \leq \sigma_3$, then*

$$\left|a_3 - \mu a_2^2\right| + R_1 \leq \frac{bB_1}{(\alpha g_3 + \beta h_3)},$$

*where*

$$R_1 = \frac{2(B_1 - B_2)(\alpha g_2 + \beta h_2)^2 + [(\alpha(\alpha-1)g_2^2 + \beta(\beta-1)h_2^2 + 2\alpha\beta g_2 h_2 + 2\mu(\alpha g_3 + \beta h_3)]B_1^2 b}{2(\alpha g_3 + \beta h_3) b B_1^2}\left|a_2\right|^2.$$

*(ii) If $\sigma_3 \leq \mu \leq \sigma_2$, then*

$$\left|a_3 - \mu a_2^2\right| + R_2 \leq \frac{bB_1}{(\alpha g_3 + \beta h_3)},$$

*where*

$$R_2 = \frac{2(B_1+B_2)(\alpha g_2+\beta h_2)^2+[(\alpha(\alpha-1)g_2^2+\beta(\beta-1)h_2^2+2\alpha\beta g_2h_2+2\mu(\alpha g_3+\beta h_3)]B_1^2b}{2(\alpha g_3+\beta h_3)bB_1^2}\left|a_2\right|^2,$$

*where* $\sigma_1$ *and* $\sigma_2$ *are given by (2.16) and (2.17).*

**Remark 5.** *(i) Taking* $b=1$ *in Theorem 3, we obtain the result obtaind by Kumer and Kumer [[15], Remark 2.2];*
*(ii) Taking* $\alpha=1$, $\beta=-1$, $g=z+\sum_{k=2}^{\infty}k^{n+1}z^k$ *and* $h=z+\sum_{k=2}^{\infty}k^nz^k(n\in\mathbb{N}_0)$ *in Theorem 3, we obtain the result obtaind by Aouf and Silverman [[5], Theorm 2];*
*(iii) Taking* $\alpha=1$, $\beta=-1$, $g=z+\sum_{k=2}^{\infty}k[(\lambda-\rho)(\gamma-\delta)(k-1)+1]^nz^k$ *and* $h=z+\sum_{k=2}^{\infty}[(\lambda-\rho)(\gamma-\delta)(k-1)+1]^nz^k(\gamma,\delta,\lambda,\rho>0,\ \gamma>\delta,\ \lambda>\rho,\ n\in\mathbb{N}_0,$ $b\in\mathbb{C}^*)$ *in Theorem 3, we obtain the result obtaind by Aouf et al. [[3], Theorm 3];*
*(iv) Taking* $\alpha=1$, $\beta=-1$ *and* $b=1$ *in Theorem 3, we obtain the result obtaind by Murugusundaramoorthy et al. [[22], Theorm 2.1].*

**Remark 6.** *Specializing the parameters* $\alpha$, $\beta$ *and* $b$, *we obtain results corresponding to the classes* $\mathcal{M}^\eta(g,h,\varphi)$, $\mathcal{M}_\rho^\eta(g,h,\varphi)$, $\mathcal{M}_{\alpha,\beta,\rho}^\eta(g,h,\varphi)$, $\mathcal{N}_\rho^\eta(g,h,\varphi)$, $\mathcal{M}_{\alpha,\rho}^\eta(g,h,\varphi)$ *and* $\mathcal{M}_{\beta,\rho}^\eta(g,h,\varphi)$, *mentioned in the introdution.*

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R. M. El-Ashwah, Department of Mathematics, Faculty of Science, University of Damietta, New Damietta 34517, Egypt

*E-mail address*: `r_elashwah@yhoo.com`

M. K. Aouf, Department of Mathematics, Faculty of Science, University of Mansoura, Mansoura 35516, Egypt

*E-mail address*: `mkaouf127@yahoo.com`

F. M. Abdulkarem, Department of Mathematics, Faculty of Science, University of Damietta, New Damietta 34517, Egypt

*E-mail address*: `fammari76@gmail.com`