Journal of Fractional Calculus and Applications
Vol. 15(2) July 2024, No.15.
ISSN: 2090-5858.
ISSN 2090-584X (print)
http://jfca.journals.ekb.eg/

SOME NEW GENERAL SUMMATION FORMULAS CONTIGUOUS TO THE KUMMER'S FIRST SUMMATION THEOREM

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Abstract. Due to the great success of hypergeometric functions of one variable, a number of hypergeometric functions of two or more variables have been introduced and explored. The aim of this paper is to provide the extensions and generalizations of Kümmer's first summation theorem for the higher-order hypergeometric series, where numeratorial and denominatorial parameters differ by positive integers, in the form of

$$_{r+2}F_{r+1}[a,\ b,\ \{n_r+\zeta_r\};\ 1+a-b+m,\ \{\zeta_r\};\ -1],$$

with suitable convergence conditions. Where ζ_r is set of complex or real numbers, $\{n_r\}$ is set of positive integers and suitable restrictions on the value of m.

1. Introduction

The enormous popularity and broad usefulness of the hypergeometric function $_2F_1$ and the generalized hypergeometric functions $_pF_q$ $(p, q \in \mathbb{N}_0)$ of one variable have inspired and stimulated a large number of researchers to introduce and investigate hypergeometric functions of two or more variables (see, e.g., [3, 16, 7, 23]). A serious, significant, and systematic study of the hypergeometric functions of two variables was initiated by Appell [2], who offered the so-called Appell functions F_1, F_2, F_3, and F_4 which are generalizations of the Gauss hypergeometric function. The confluent forms of the Appell functions were studied by Humbert [9]. A complete list of these functions can be seen in the standard literature. Later, the four Appell functions and their confluent forms were further generalized by Kampé de Fériet, who introduced more general hypergeometric functions of two variables. The notation defined and introduced by Kampé de Fériet for his double-hypergeometric functions of superior order was subsequently abbreviated by Burchnall and Chaundy [4].

2020 *Mathematics Subject Classification.* $33C05, 33C20, 33D99, 33B20, 33B15.$

Key words and phrases. Gamma function, Pochhammer symbol, Generalized hypergeometric function, Kummer's theorem, Hypergeometric summation theorems.

Submitted Jan. 16, 2024. Revised May 8, 2024.

A natural generalization of the Gaussian hypergeometric series ${}_2F_1[\alpha,\beta;\gamma;z]$ is accomplished by introducing any arbitrary number of numerator and denominator parameters. Thus, the resulting series

$$ {}_pF_q\left[\begin{array}{ll}(\alpha_p); & \\ & z\\ (\beta_q); & \end{array}\right] = {}_pF_q\left[\begin{array}{ll}\alpha_1,\alpha_2,\ldots,\alpha_p; & \\ & z\\ \beta_1,\beta_2,\ldots,\beta_q; & \end{array}\right] = \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(\alpha_j)_n}{\prod_{j=1}^{q}(\beta_j)_n}\frac{z^n}{n!}, \tag{1}$$

where $(\alpha)_p$ is the Pochhammer symbol defined for $(\alpha, p\in\mathbb{C})$ (see, [24, p.2 and p.5])

$$\begin{aligned}(\alpha)_p &= \frac{\Gamma(\alpha+p)}{\Gamma(\alpha)}\\ &= \begin{cases}1 & \left(p=0\ ;\ \alpha\in\mathbb{C}\backslash\mathbb{Z}_0^-\right),\\ \alpha(\alpha+1)\cdots(\alpha+n-1) & (p=n\in\mathbb{N}\ ;\ \alpha\in\mathbb{C}),\end{cases}\end{aligned} \tag{2}$$

it being understood that $(0)_0=1$ (see, e.g., [21, 25]) and assumed tacitly that the Gamma quotient exists. Here an empty product is interpreted as 1, and it is assumed that the variable z, the numerator parameters $\alpha_1, \ldots, \alpha_p$, and the denominator parameters $\beta_1, \ldots, \beta_q$ take on complex values, provided that

$$(\beta_j\in\mathbb{C}\backslash\mathbb{Z}_0^-;\ j=1,\ldots,q). \tag{3}$$

For $p=q+1$, it has a branch cut discontinuity in the complex z-plane running from 1 to ∞. If $p\le q$ the series (1) converges for each $z\in\mathbb{C}$. For some recent results on this subject, especially on transformations, summations and some applications, see [14, 17].

Here and elsewhere, $\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of complex numbers, real numbers, natural numbers, integers, positive and negative real numbers, respectively.

For more details of ${}_pF_q$ including its convergence, its various special and limiting cases, and its further diverse generalizations, one may be referred, for example see [3].

Kummer's First summation theorem [7, p. 852, Eq.(1.3)], see also [12, p. 134]:

$$ {}_2F_1\left[\begin{array}{ll}a,\ b; & \\ & -1\\ 1+a-b; & \end{array}\right] = \frac{\Gamma(1+a-b)\Gamma\left(1+(\frac{a}{2})\right)}{\Gamma\left(1+(\frac{a}{2})-b\right)\Gamma(1+a)}, \tag{4}$$

where $1+a-b\in\mathbb{C}\backslash\mathbb{Z}_0^-$, and $\Re(b)<1$.

Contiguous Kummer's summation theorems[16]

$$ {}_2F_1\left[\begin{array}{ll}a,\ b; & \\ & -1\\ a-b; & \end{array}\right] = \frac{\Gamma(\frac12)\Gamma(a-b)}{2^a}\left[\frac{1}{\Gamma(\frac{a}{2})\Gamma(\frac{a}{2}-b+\frac12)}+\frac{1}{\Gamma(\frac{a}{2}+\frac12)\Gamma(\frac{a}{2}-b)}\right], \tag{5}$$

$$\begin{aligned} {}_2F_1\left[\begin{array}{ll}a,\ b; & \\ & -1\\ a-b-1; & \end{array}\right] &= \frac{\Gamma(\frac12)\Gamma(a-b-1)}{2^a}\\ &\times\left[\frac{(a-b-1)}{\Gamma(\frac{a}{2}-b)\Gamma(\frac{a}{2}+\frac12)}+\frac{2}{\Gamma(\frac{a}{2}-b-\frac12)\Gamma(\frac{a}{2})}\right].\end{aligned} \tag{6}$$

Extension of Gauss' summation theorem[11, 16]

$$ {}_3F_2\left[\begin{array}{ll}a,\ b,\ d+1; & \\ & 1\\ c+1,\ d; & \end{array}\right] = \frac{\Gamma(c+1)\Gamma(c-a-b)}{\Gamma(c-a+1)\Gamma(c-b+1)}\left[(c-a-b)+\frac{ab}{d}\right], \tag{7}$$

provided $\Re(c-a-b)>0$ and $d\neq 0,-1,-2,...$

Extension of Kummer's summation theorem[11]

$$ {}_3F_2\left[\begin{array}{ll}a,\ b,\ d+1; & \\ & -1\\ 2+a-b,\ d; & \end{array}\right] = \frac{\Gamma(\frac12)\Gamma(2+a-b)}{2^a(1-b)}\left[\frac{(\frac{1+a-b}{d}-1)}{\Gamma(\frac{a}{2})\Gamma(\frac{a}{2}-b+\frac32)}+\frac{(1-\frac{a}{d})}{\Gamma(\frac{a}{2}+\frac12)\Gamma(\frac{a}{2}-b+1)}\right], \tag{8}$$

Summation formula given by Choi-Rathie-Malani[6, p. 1523, Eq.(2.2)], see also [22, p. 828, Th.(3)]

$$ {}_2F_1\left[\begin{array}{c} \alpha,\ \beta; \\ \\ 1+\alpha-\beta+j; \end{array} -1\right] = \frac{\Gamma(1+\alpha-\beta+j)}{2\Gamma(\alpha)(1-\beta)_j}\sum_{r=0}^{j}\left\{\binom{j}{r}\frac{(-1)^r\Gamma\left(\frac{r+\alpha}{2}\right)}{\Gamma\left(\frac{r+\alpha}{2}+1-\beta\right)}\right\} \quad (9) $$

where $\Re(\beta) < \left(\frac{2+j}{2}\right) \alpha, 1-\beta, 1+\alpha-\beta+j \in \mathbb{C}\backslash\mathbb{Z}_0^-, j \in \mathbb{N}_0$.

Motivated by the work of Andrews [1], Bailey[3], Carlson[5], Choi[6], Erdélyi et al. [8], Kim et al.[10], Miller et al. [13], Minton [15], Prudnikov et al.[16], Qureshi[18, 19, 20], Rakha-Rathie [22], Slatter [23], Srivastava[24, 25] and Vidunas [27], we mention some summation theorems for ${}_6F_5[-1]$ and ${}_5F_4[-1]$ in Section 2. In Section 3, we have given the summation theorems for ${}_4F_3[-1]$. In Section 4, we have given the summation theorems for ${}_3F_2[-1]$. The proof of summation theorems can be derived by using the formula given by Choi-Rathie-Malani (9), series rearrangement technique and Pochhammer symbol identities.

Any values of numerator and denominator parameters in sections 2,3 and 4, leading to the results which do not make sense are tacitly excluded.

2. Summation Theorems for ${}_6F_5[-1]$ and ${}_5F_4[-1]$

Theorem 2.1. *The following theorem holds true*

$$ {}_6F_5\left[\begin{array}{c} a,\ b,\ c+1,\ d+1,\ g+1,\ h+1; \\ \\ 1+a-b+m,\ c,\ d,\ g,\ h; \end{array} -1\right] = \frac{\Gamma(1+a-b+m)}{2\Gamma(a)(1-b)_m}\left\{\sum_{r=0}^{m}\binom{m}{r}\frac{(-1)^r\Gamma\left(\frac{r+a}{2}\right)}{\Gamma\left(\frac{r+a+2-2b}{2}\right)}+\right. $$

$$ +\frac{(1+cdg+cdh+cgh+dgh+cd+cg+ch+dg+dh+gh+c+g+h+d)}{cdgh}\sum_{r=0}^{m+1}\binom{m+1}{r}\times $$

$$ \times\frac{(-1)^r\Gamma\left(\frac{r+a+1}{2}\right)}{\Gamma\left(\frac{r+a+1-2b}{2}\right)}+\frac{(7+cd+cg+ch+dg+dh+gh+3c+3d+3g+3h)}{cdgh}\sum_{r=0}^{m+2}\binom{m+2}{r}\times $$

$$ \times\frac{(-1)^r\Gamma\left(\frac{r+a+2}{2}\right)}{\Gamma\left(\frac{r+a-2b}{2}\right)}+\frac{(6+c+d+g+h)}{cdgh}\sum_{r=0}^{m+3}\binom{m+3}{r}\frac{(-1)^r\Gamma\left(\frac{r+a+3}{2}\right)}{\Gamma\left(\frac{r+a-1-2b}{2}\right)}+ $$

$$ \left.+\frac{1}{cdgh}\sum_{r=0}^{m+4}\binom{m+4}{r}\frac{(-1)^r\Gamma\left(\frac{r+a+4}{2}\right)}{\Gamma\left(\frac{r+a-2-2b}{2}\right)}\right\}, \quad (10) $$

where $\Re(b) < \left(\frac{-2+m}{2}\right)$; $a, b, c, d, g, h, 1+a-b+m \in \mathbb{C}\backslash\mathbb{Z}_0^-$, and $m \in \mathbb{N}_0$.
The proof of the Theorem 2.1 can be obtained by using formula (9) with the aid of following Pochhammer symbol identity. The involved details are omitted.

$$ \frac{(c+1)_r(d+1)_r(g+1)_r(h+1)_r}{(c)_r(d)_r(g)_r(h)_r} $$

$$ = \left[1+\frac{(1+cdg+cdh+cgh+dgh+cd+cg+ch+dg+dh+gh+c+d+g+h)r}{cdgh}+\right. $$

$$ +\frac{(7+cd+cg+ch+dg+dh+gh+3c+3d+3g+3h)r(r-1)}{cdgh}+ $$

$$ \left.+\frac{(6+c+d+g+h)r(r-1)(r-2)}{cdgh}+\frac{r(r-1)(r-2)(r-3)}{cdgh}\right] \quad (11) $$

Theorem 2.2. *The following theorem holds true*

$$ {}_5F_4\left[\begin{array}{c} a,\ b,\ c+1,\ d+1,\ g+1; \\ \\ 1+a-b+m,\ c,\ d,\ g; \end{array} -1\right] = \frac{\Gamma(1+a-b+m)}{2\Gamma(a)(1-b)_m}\left\{\sum_{r=0}^{m}\binom{m}{r}\frac{(-1)^r\Gamma\left(\frac{r+a}{2}\right)}{\Gamma\left(\frac{r+a+2-2b}{2}\right)}+\right. $$

$$ +\frac{(1+c+g+d+cd+cg+dg)}{cdg}\sum_{r=0}^{m+1}\binom{m+1}{r}\frac{(-1)^r\Gamma\left(\frac{r+a+1}{2}\right)}{\Gamma\left(\frac{r+a+1-2b}{2}\right)}+\frac{(c+d+g+3)}{cdg}\times $$

$$ \left.\times\sum_{r=0}^{m+2}\binom{m+2}{r}\frac{(-1)^r\Gamma\left(\frac{r+a+2}{2}\right)}{\Gamma\left(\frac{r+a-2b}{2}\right)}+\frac{1}{cdg}\sum_{r=0}^{m+3}\binom{m+3}{r}\frac{(-1)^r\Gamma\left(\frac{r+a+3}{2}\right)}{\Gamma\left(\frac{r+a-1-2b}{2}\right)}\right\}, \quad (12) $$

where $\Re(b) < \left(\frac{-1+m}{2}\right)$; $a, b, c, d, g, 1+a-b+m \in \mathbb{C}\setminus\mathbb{Z}_0^-$, and $m \in \mathbb{N}_0$.

Proof of the Theorem 2.2:
Using the definition of Pochhammer symbol

$$ (\alpha)_p = \frac{\Gamma(\alpha+p)}{\Gamma(\alpha)}, $$

we can see that

$$ \frac{(c+1)_r(d+1)_r(g+1)_r}{(c)_r(d)_r(g)_r} $$

$$ = \left[1+\frac{(1+c+d+g+cd+cg+dg)r}{cdg}+\frac{(c+d+g+3)r(r-1)}{cdg}+\frac{r(r-1)(r-2)}{cdg}\right], \quad (13) $$

Using the equation (13) in the Theorem 2.2, after some simplification, we come to

$$ {}_5F_4\left[\begin{array}{c} a,\ b,\ c+1,\ d+1,\ g+1; \\ \\ 1+a-b+m,\ c,\ d,\ g; \end{array} -1\right] $$

$$ = {}_2F_1\left[\begin{array}{c} a,\ b\ ; \\ \\ 1+a-b+m\ ; \end{array} -1\right]+\frac{(1+c+d+g+cd+cg+dg)}{cdg}\sum_{r=1}^{\infty}\frac{(a)_r(b)_r(-1)^r}{(1+a-b+m)_r(r-1)!}+ $$

$$ +\frac{(c+d+g+3)}{cdg}\sum_{r=2}^{\infty}\frac{(a)_r(b)_r(-1)^r}{(1+a-b+m)_r(r-2)!}+\frac{1}{cdg}\sum_{r=3}^{\infty}\frac{(a)_r(b)_r(-1)^r}{(1+a-b+m)_r(r-3)!}. \quad (14) $$

Replacing r by $r+1$ in the second term, r by $r+2$ in the third term and r by $r+3$ in the fourth term on the right hand side of the equation (14), we get

$$ {}_5F_4\left[\begin{array}{c} a,\ b,\ c+1,\ d+1,\ g+1; \\ \\ 1+a-b+m,\ c,\ d,\ g; \end{array} -1\right] $$

$$ = {}_2F_1\left[\begin{array}{c} a,\ b\ ; \\ \\ 1+a-b+m\ ; \end{array} -1\right]-\frac{ab(1+c+d+g+cd+cg+dg)}{cdg(1+a-b+m)}\,{}_2F_1\left[\begin{array}{c} a+1,\ b+1\ ; \\ \\ 2+a-b+m\ ; \end{array} -1\right] $$

$$ +\frac{(c+d+g+3)a(a+1)b(b+1)}{cdg(1+a-b+m)(2+a-b+m)}\,{}_2F_1\left[\begin{array}{c} a+2,\ b+2\ ; \\ \\ 3+a-b+m\ ; \end{array} -1\right] $$

$$ -\frac{a(a+1)(a+2)b(b+1)(b+2)}{cdg(1+a-b+m)(2+a-b+m)(3+a-b+m)}\,{}_2F_1\left[\begin{array}{c} a+3,\ b+3\ ; \\ \\ 4+a-b+m\ ; \end{array} -1\right]. \quad (15) $$

Finally applying the summation formula given by Choi-Rathie-Malani (9) on the right hand side of the equation (15), we arrive at the result (12).

Theorem 2.3. *The following theorem holds true*

$$
{}_5F_4\left[\begin{array}{c} a,\ b,\ c+1,\ d+1,\ g+2; \\ \\ 1+a-b+m,\ c,\ d,\ g; \end{array} -1\right] = \frac{\Gamma(1+a-b+m)}{2\Gamma(a)(1-b)_m}\left\{\sum_{r=0}^{m}\binom{m}{r}\frac{(-1)^r\Gamma\left(\frac{r+a}{2}\right)}{\Gamma\left(\frac{r+a+2-2b}{2}\right)}+\right.
$$

$$
+\frac{(dg+cg+2cd+2c+2d+g+2)}{cdg}\sum_{r=0}^{m+1}\binom{m+1}{r}\frac{(-1)^r\Gamma\left(\frac{r+a+1}{2}\right)}{\Gamma\left(\frac{r+a+1-2b}{2}\right)}+\frac{(10+cd+2dg+2cg+}{cdg}\times
$$

$$
\times\frac{+g^2+4c+4d+7g)}{(g+1)}\sum_{r=0}^{m+2}\binom{m+2}{r}\frac{(-1)^r\Gamma\left(\frac{r+a+2}{2}\right)}{\Gamma\left(\frac{r+a-2b}{2}\right)}+\frac{(7+c+d+2g)}{cdg(g+1)}\sum_{r=0}^{m+3}\binom{m+3}{r}\times
$$

$$
\left.\times\frac{(-1)^r\Gamma\left(\frac{r+a+3}{2}\right)}{\Gamma\left(\frac{r+a-1-2b}{2}\right)}+\frac{1}{cdg(g+1)}\sum_{r=0}^{m+4}\binom{m+4}{r}\frac{(-1)^r\Gamma\left(\frac{r+a+4}{2}\right)}{\Gamma\left(\frac{r+a-2-2b}{2}\right)}\right\}, \tag{16}
$$

where $\Re(b) < \left(\frac{-2+m}{2}\right)$; $a, b, c, d, g, 1+a-b+m \in \mathbb{C}\backslash\mathbb{Z}_0^-$, and $m \in \mathbb{N}_0$.

The proof of the Theorem 2.3 can be derived by using the following pochhammer symbol identity and generalized hypergeometric function of one variable.

$$
\frac{(c+1)_r(d+1)_r(g+2)_r}{(c)_r(d)_r(g)_r}
$$

$$
= \left[1+\frac{(dg+cg+2cd+2c+2d+g+2)r}{cdg}+\frac{(10+cd+2dg+2cg+}{cdg}\right.
$$

$$
\left.\frac{+g^2+4c+4d+7g)r(r-1)}{(g+1)}+\frac{(7+c+d+2g)r(r-1)(r-2)}{cdg(g+1)}+\frac{r(r-1)(r-2)(r-3)}{cdg(g+1)}\right], \tag{17}
$$

3. Summation Theorems for ${}_4F_3[-1]$

Theorem 3.4. *The following theorem holds true*

$$
{}_4F_3\left[\begin{array}{c} a,\ b,\ c+1,\ d+1; \\ \\ 1+a-b+m,\ c,\ d; \end{array} -1\right] = \frac{\Gamma(1+a-b+m)}{2\Gamma(a)(1-b)_m}\left\{\sum_{r=0}^{m}\binom{m}{r}\frac{(-1)^r\Gamma\left(\frac{r+a}{2}\right)}{\Gamma\left(\frac{r+a+2-2b}{2}\right)}+\right.
$$

$$
\left.+\frac{(c+d+1)}{cd}\sum_{r=0}^{m+1}\binom{m+1}{r}\frac{(-1)^r\Gamma\left(\frac{r+a+1}{2}\right)}{\Gamma\left(\frac{r+a+1-2b}{2}\right)}+\frac{1}{cd}\sum_{r=0}^{m+2}\binom{m+2}{r}\frac{(-1)^r\Gamma\left(\frac{r+a+2}{2}\right)}{\Gamma\left(\frac{r+a-2b}{2}\right)}\right\}, \tag{18}
$$

where $\Re(b) < \left(\frac{m}{2}\right)$; $a, b, c, d, 1+a-b+m \in \mathbb{C}\backslash\mathbb{Z}_0^-$, and $m \in \mathbb{N}_0$.

The proof of the Theorem 3.4 can be obtained by using the following identity and some series rearrangement techniques.

$$
\frac{(c+1)_r(d+1)_r}{(c)_r(d)_r} = \left(1+\frac{(c+d+1)r}{cd}+\frac{r(r-1)}{cd}\right), \tag{19}
$$

Theorem 3.5. *The following theorem holds true*

$$
{}_4F_3\left[\begin{array}{c} a,\ b,\ c+1,\ d+2; \\ \\ 1+a-b+m,\ c,\ d; \end{array} -1\right] = \frac{\Gamma(1+a-b+m)}{2\Gamma(a)(1-b)_m}\left\{\sum_{r=0}^{m}\binom{m}{r}\frac{(-1)^r\Gamma\left(\frac{r+a}{2}\right)}{\Gamma\left(\frac{r+a+2-2b}{2}\right)}+\right.
$$

$$+\frac{(2c+d+2)}{cd}\sum_{r=0}^{m+1}\binom{m+1}{r}\frac{(-1)^r\Gamma\left(\frac{r+a+1}{2}\right)}{\Gamma\left(\frac{r+a+1-2b}{2}\right)}+\frac{(c+2d+4)}{cd(d+1)}\sum_{r=0}^{m+2}\binom{m+2}{r}\frac{(-1)^r\Gamma\left(\frac{r+a+2}{2}\right)}{\Gamma\left(\frac{r+a-2b}{2}\right)}+$$
$$+\frac{1}{cd(d+1)}\sum_{r=0}^{m+3}\binom{m+3}{r}\frac{(-1)^r\Gamma\left(\frac{r+a+3}{2}\right)}{\Gamma\left(\frac{r+a-1-2b}{2}\right)}\Bigg\}, \tag{20}$$

where $\Re(b)<\left(\frac{-1+m}{2}\right)$; $a,b,c,d,1+a-b+m\in\mathbb{C}\backslash\mathbb{Z}_0^-$, and $m\in\mathbb{N}_0$.

The proof of Theorem 3.5 can be accomplished by following the lines of that of Theorem 3.4. The details are omitted.

Theorem 3.6. *The following theorem holds true*

$${}_4F_3\left[\begin{array}{c} a,\ b,\ c+2,\ d+2; \\ \\ 1+a-b+m,\ c,\ d; \end{array} -1\right]=\frac{\Gamma(1+a-b+m)}{2\Gamma(a)(1-b)_m}\left\{\sum_{r=0}^{m}\binom{m}{r}\frac{(-1)^r\Gamma\left(\frac{r+a}{2}\right)}{\Gamma\left(\frac{r+a+2-2b}{2}\right)}+\right.$$
$$+\frac{(4+2c+2d)}{cd}\sum_{r=0}^{m+1}\binom{m+1}{r}\frac{(-1)^r\Gamma\left(\frac{r+a+1}{2}\right)}{\Gamma\left(\frac{r+a+1-2b}{2}\right)}+\frac{(14+c^2+d^2+4cd+9c+9d)}{cd(c+1)(d+1)}\times$$
$$\sum_{r=0}^{m+2}\binom{m+2}{r}\frac{(-1)^r\Gamma\left(\frac{r+a+2}{2}\right)}{\Gamma\left(\frac{r+a-2b}{2}\right)}+\frac{(8+2c+2d)}{cd(c+1)(d+1)}\sum_{r=0}^{m+3}\binom{m+3}{r}\frac{(-1)^r\Gamma\left(\frac{r+a+3}{2}\right)}{\Gamma\left(\frac{r+a-1-2b}{2}\right)}+$$
$$+\frac{1}{cd(c+1)(d+1)}\sum_{r=0}^{m+4}\binom{m+4}{r}\frac{(-1)^r\Gamma\left(\frac{r+a+4}{2}\right)}{\Gamma\left(\frac{r+a-2-2b}{2}\right)}\Bigg\}, \tag{21}$$

where $\Re(b)<\left(\frac{-2+m}{2}\right)$; $a,b,c,d,1+a-b+m\in\mathbb{C}\backslash\mathbb{Z}_0^-$, and $m\in\mathbb{N}_0$.

The proof of the Theorem 3.6 can be obtained by using the identity (22) and generalized hypergeometric function of one variable.

$$\frac{(c+2)_r(d+2)_r}{(c)_r(d)_r}=\Bigg(1+\frac{(4+2c+2d)r}{cd}+\frac{(14+c^2+d^2+4cd+9c+9d)r(r-1)}{cd(c+1)(d+1)}$$
$$+\frac{(8+2c+2d)r(r-1)(r-2)}{cd(c+1)(d+1)}+\frac{r(r-1)(r-2)(r-3)}{cd(c+1)(d+1)}\Bigg), \tag{22}$$

Theorem 3.7. *The following theorem holds true*

$${}_4F_3\left[\begin{array}{c} a,\ b,\ c+1,\ d+3; \\ \\ 1+a-b+m,\ c,\ d; \end{array} -1\right]=\frac{\Gamma(1+a-b+m)}{2\Gamma(a)(1-b)_m}\left\{\sum_{r=0}^{m}\binom{m}{r}\frac{(-1)^r\Gamma\left(\frac{r+a}{2}\right)}{\Gamma\left(\frac{r+a+2-2b}{2}\right)}+\right.$$
$$+\frac{(3+3c+d)}{cd}\sum_{r=0}^{m+1}\binom{m+1}{r}\frac{(-1)^r\Gamma\left(\frac{r+a+1}{2}\right)}{\Gamma\left(\frac{r+a+1-2b}{2}\right)}+\frac{(9+3c+3d)}{cd(d+1)}\sum_{r=0}^{m+2}\binom{m+2}{r}\times$$
$$\times\frac{(-1)^r\Gamma\left(\frac{r+a+2}{2}\right)}{\Gamma\left(\frac{r+a-2b}{2}\right)}+\frac{(9+c+3d)}{cd(d+1)(d+2)}\sum_{r=0}^{m+3}\binom{m+3}{r}\frac{(-1)^r\Gamma\left(\frac{r+a+3}{2}\right)}{\Gamma\left(\frac{r+a-1-2b}{2}\right)}+$$
$$+\frac{1}{cd(d+1)(d+2)}\sum_{r=0}^{m+4}\binom{m+4}{r}\frac{(-1)^r\Gamma\left(\frac{r+a+4}{2}\right)}{\Gamma\left(\frac{r+a-2-2b}{2}\right)}\Bigg\}, \tag{23}$$

where $\Re(b)<\left(\frac{-2+m}{2}\right)$; $a,b,c,d,1+a-b+m\in\mathbb{C}\backslash\mathbb{Z}_0^-$, and $m\in\mathbb{N}_0$.

The proof of the Theorem 3.7 can be obtained by using the identity (24) and summation formula given by Choi-Rathie-Malani (9) . The involved details are omitted.

$$\frac{(c+1)_r(d+3)_r}{(c)_r(d)_r}=\Bigg(1+\frac{(3+3c+d)r}{cd}+\frac{(9+3c+3d)r(r-1)}{cd(d+1)}$$

$$+\frac{(9+c+3d)r(r-1)(r-2)}{cd(d+1)(d+2)}+\frac{r(r-1)(r-2)(r-3)}{cd(d+1)(d+2)}\Bigg). \tag{24}$$

4. Summation Theorems for ${}_3F_2[-1]$

Theorem 4.8. *The following theorem holds true*

$$ {}_3F_2\left[\begin{array}{c} a,\ b,\ c+1; \\ 1+a-b+m,\ c; \end{array} -1\right] = \frac{\Gamma(1+a-b+m)}{2\Gamma(a)(1-b)_m}\left\{\sum_{r=0}^{m}\binom{m}{r}\frac{(-1)^r\Gamma\left(\frac{r+a}{2}\right)}{\Gamma\left(\frac{r+a+2-2b}{2}\right)}+\right.$$

$$\left.+\frac{1}{c}\sum_{r=0}^{m+1}\binom{m+1}{r}\frac{(-1)^r\Gamma\left(\frac{r+a+1}{2}\right)}{\Gamma\left(\frac{r+a+1-2b}{2}\right)}\right\}, \tag{25}$$

where $\Re(b) < \left(\frac{1+m}{2}\right)$; $a, b, c, 1+a-b+m \in \mathbb{C}\backslash\mathbb{Z}_0^-$, and $m \in \mathbb{N}_0$.

The proof of Theorem 4.8 would flow along the lines of that of Theorem 3.7 with the help of formula (9).

Theorem 4.9. *The following theorem holds true*

$$ {}_3F_2\left[\begin{array}{c} a,\ b,\ c+2; \\ 1+a-b+m,\ c; \end{array} -1\right] = \frac{\Gamma(1+a-b+m)}{2\Gamma(a)(1-b)_m}\left\{\sum_{r=0}^{m}\binom{m}{r}\frac{(-1)^r\Gamma\left(\frac{r+a}{2}\right)}{\Gamma\left(\frac{r+a+2-2b}{2}\right)}+\right.$$

$$\left.+\frac{2}{c}\sum_{r=0}^{m+1}\binom{m+1}{r}\frac{(-1)^r\Gamma\left(\frac{r+a+1}{2}\right)}{\Gamma\left(\frac{r+a+1-2b}{2}\right)}+\frac{1}{c(c+1)}\sum_{r=0}^{m+2}\binom{m+2}{r}\frac{(-1)^r\Gamma\left(\frac{r+a+2}{2}\right)}{\Gamma\left(\frac{r+a-2b}{2}\right)}\right\}, \tag{26}$$

where $\Re(b) < \left(\frac{m}{2}\right)$; $a, b, c, 1+a-b+m \in \mathbb{C}\backslash\mathbb{Z}_0^-$, and $m \in \mathbb{N}_0$.

Theorem 4.10. *The following theorem holds true*

$$ {}_3F_2\left[\begin{array}{c} a,\ b,\ c+3; \\ 1+a-b+m,\ c; \end{array} -1\right] = \frac{\Gamma(1+a-b+m)}{2\Gamma(a)(1-b)_m}\left\{\sum_{r=0}^{m}\binom{m}{r}\frac{(-1)^r\Gamma\left(\frac{r+a}{2}\right)}{\Gamma\left(\frac{r+a+2-2b}{2}\right)}+\right.$$

$$+\frac{3}{c}\sum_{r=0}^{m+1}\binom{m+1}{r}\frac{(-1)^r\Gamma\left(\frac{r+a+1}{2}\right)}{\Gamma\left(\frac{r+a+1-2b}{2}\right)}+\frac{3}{c(c+1)}\sum_{r=0}^{m+2}\binom{m+2}{r}\frac{(-1)^r\Gamma\left(\frac{r+a+2}{2}\right)}{\Gamma\left(\frac{r+a-2b}{2}\right)}+$$

$$\left.+\frac{1}{c(c+1)(c+2)}\sum_{r=0}^{m+3}\binom{m+3}{r}\frac{(-1)^r\Gamma\left(\frac{r+a+3}{2}\right)}{\Gamma\left(\frac{r+a-1-2b}{2}\right)}\right\}, \tag{27}$$

where $\Re(b) < \left(\frac{-1+m}{2}\right)$; $a, b, c, 1+a-b+m \in \mathbb{C}\backslash\mathbb{Z}_0^-$, and $m \in \mathbb{N}_0$.

Theorem 4.11. *The following theorem holds true*

$$ {}_3F_2\left[\begin{array}{c} a,\ b,\ c+4; \\ 1+a-b+m,\ c; \end{array} -1\right] = \frac{\Gamma(1+a-b+m)}{2\Gamma(a)(1-b)_m}\left\{\sum_{r=0}^{m}\binom{m}{r}\frac{(-1)^r\Gamma\left(\frac{r+a}{2}\right)}{\Gamma\left(\frac{r+a+2-2b}{2}\right)}+\right.$$

$$+\frac{4}{c}\sum_{r=0}^{m+1}\binom{m+1}{r}\frac{(-1)^r\Gamma\left(\frac{r+a+1}{2}\right)}{\Gamma\left(\frac{r+a+1-2b}{2}\right)}+\frac{6}{c(c+1)}\sum_{r=0}^{m+2}\binom{m+2}{r}\frac{(-1)^r\Gamma\left(\frac{r+a+2}{2}\right)}{\Gamma\left(\frac{r+a-2b}{2}\right)}+$$

$$+\frac{4}{c(c+1)(c+2)}\sum_{r=0}^{m+3}\binom{m+3}{r}\frac{(-1)^r\Gamma\left(\frac{r+a+3}{2}\right)}{\Gamma\left(\frac{r+a-1-2b}{2}\right)}+$$

$$\left.+\frac{1}{c(c+1)(c+2)(c+3)}\sum_{r=0}^{m+4}\binom{m+4}{r}\frac{(-1)^r\Gamma\left(\frac{r+a+4}{2}\right)}{\Gamma\left(\frac{r+a-2-2b}{2}\right)}\right\}, \tag{28}$$

where $\Re(b) < \left(\frac{-2+m}{2}\right)$; $a, b, c, 1+a-b+m \in \mathbb{C}\backslash\mathbb{Z}_0^-$, and $m \in \mathbb{N}_0$.

The proof of Theorem 4.9, Theorem 4.10 and Theorem 4.11 can be derived by using the following Pochhammer symbol identities respectively

$$\frac{(c+2)_r}{(c)_r} = \left(1 + \frac{2r}{c} + \frac{r(r-1)}{c(c+1)}\right). \tag{29}$$

$$\frac{(c+3)_r}{(c)_r} = \left(1 + \frac{3r}{c} + \frac{3r(r-1)}{c(c+1)} + \frac{r(r-1)(r-2)}{c(c+1)(c+2)}\right). \tag{30}$$

$$\frac{(c+4)_r}{(c)_r} = \left(1 + \frac{4r}{c} + \frac{6r(r-1)}{c(c+1)} + \frac{4r(r-1)(r-2)}{c(c+1)(c+2)} + \frac{r(r-1)(r-2)(r-3)}{c(c+1)(c+2)(c+3)}\right). \tag{31}$$

5. Conclusion

The vast popularity and immense usefulness of the hypergeometric function and the generalized hypergeometric functions of one variable have inspired and stimulated a large number of researchers to introduce and investigate hypergeometric functions.

In the present paper we have derived some extensions and generalizations of Kümmer's first classical summation theorem (4) for ${}_6F_5[-1]$, ${}_5F_4[-1]$, ${}_4F_3[-1]$ and ${}_3F_2[-1]$, where certain numerator and denominator parameters differ by a positive integer as claimed in the above theorems. We wish to point out that all the formulas developed in this paper have been tested numerically with the aid of *Mathematica*, a general system of doing mathematics through computer. We conclude this paper with the note that these summation formulas will be of interest and will help in advance research in this important area of classical Special functions.

Conflicts of interests: The authors declare that there are no conflicts of interests.

Acknowledgments: The authors are very grateful to the anonymous referees for their constructive and encouraging comments which improved this paper.

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