Journal of Fractional Calculus and Applications
Vol. 15(2) July 2024, No.14.
ISSN: 2090-5858.
ISSN 2090-584X (print)
http://jfca.journals.ekb.eg/

A CLASS OF MULTIVALENT MEROMORPHIC FUNCTIONS INVOLVING AN INTEGRAL OPERATOR

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Abstract. In this paper, for analytic and multivalent functions defined in the punched disc $\mathbb{U}^* = \{\vartheta \in \mathbb{C} : 0 < |\vartheta - \delta| < 1\} = \mathbb{U}\backslash\{\delta\}$, δ be a fixed point in $\mathbb{U}$. We define the new class of multivalent meromorphic Bazilevič functions $\mathcal{M}^m_{\delta,p}(\alpha,\beta,\mu,\rho,\gamma)$ associated with the new integral operator $\mathcal{J}^m_{\delta,p}(\mu,\alpha)$, from which one can obtain many other new operators using the principle of Hadamard product (or convolution) by taking different values of its parameters. Let $\mathcal{P}_k(\rho,p)$ be the class of functions $\theta(\vartheta)$ analytic in $\mathbb{U}$ satisfying $\theta(0) = p$ and $\int_0^{2\pi}\left|\frac{\Re\{\theta(\vartheta)\}-\rho}{p-\rho}\right| d\theta \leq k\pi$, where $\vartheta = re^{i\theta}, k \geq 2$ and $0 \leq \rho < p$. Also satisfying the conditions $1 + \frac{\vartheta\mathcal{F}'_{\rho,p}(\vartheta)}{\mathcal{F}_{\rho,p}(\vartheta)} \in \mathcal{P}_k(\rho,p)$ and $\frac{\vartheta\mathcal{F}'_{\rho,p}(\vartheta)}{\mathcal{F}_{\rho,p}(\vartheta)} \in \mathcal{P}_k(\rho,p)\, 0 \leq \rho < p$. These classes generalize the class of convex and starlike multivalent functions of the order ρ in the same way the class $\mathcal{M}^m_{\delta,p}(\alpha,\beta,\mu,\rho,\gamma)$ of functions of bounded boundary rotation generalizes the class of convex and starlike multivalent functions.And we examine several properties of the class $\mathcal{M}^m_{\delta,p}(\alpha,\beta,\mu,\rho,\gamma)$. Using the method for multivalent functions developed by Noor and Muhammad [4] and Aouf and Seoudy [1], we prove our theorems.

1. Introduction

Let $\sum_{\delta,p}$ be the class of functions:

$$\mathcal{F}(\vartheta) = (\vartheta-\delta)^{-p} + \sum_{\epsilon=1}^{\infty} a_{\epsilon-p}(\vartheta-\delta)^{\epsilon-p} \qquad (p \in \mathbb{N} = \{1,2,3,...\}), \tag{1}$$

2010 *Mathematics Subject Classification.* : 30C45.
Key words and phrases. Bazilevič function, multivalent meromorphic functions, hadamard product, integral operator.
Submitted May 18, 2024. Revised May 31, 2024.

which are analytic and multivalent in $\mathbb{U}^* = \{\vartheta \in \mathbb{C} : 0 < |\vartheta - \delta| < 1\} = \mathbb{U}\backslash\{\delta\}$, δ be a fixed point in $\mathbb{U}$. Let $\mathcal{P}_k(\rho, p)$ be the class of functions $\theta(\vartheta)$ analytic in $\mathbb{U}$ satisfying $\theta(0) = p$ and

$$\int_0^{2\pi} \left| \frac{\Re\{\theta(\vartheta)\} - \rho}{p - \rho} \right| d\theta \leq k\pi, \tag{2}$$

where $\vartheta = re^{i\theta}, k \geq 2$ and $0 \leq \rho < p$.

Padmanabhan and Parvatham [5] presented the class $\mathcal{P}_k(\rho, p)$. Pinchuk [6] defined a class $\mathcal{P}_k(0, 1) = \mathcal{P}_k$ for $\rho = 0$, $p = 1$. Also we observe that $\mathcal{P}_2(\rho, 1) = \mathcal{P}(\rho)$, the class of functions with positive real parts greater than ρ and $\mathcal{P}_2(0, 1) = \mathcal{P}$, the class of functions with positive real part. From (2), we have $\theta(\vartheta) \in \mathcal{P}_k(\rho, p)$ if and only if there exists $\theta_1, \theta_2 \in \mathcal{P}(\rho, p)$ such that

$$\theta(\vartheta) = \left(\frac{k}{4} + \frac{1}{2}\right)\theta_1(\vartheta) - \left(\frac{k}{4} - \frac{1}{2}\right)\theta_2(\vartheta) \quad (\vartheta \in \mathbb{U}). \tag{3}$$

As is well known, the class $\mathcal{P}_k(\rho, p)$ is a convex set (see [3] at $p = 1$).

For functions $\mathcal{F}(\vartheta) \in \sum_{\delta,p}$ given by (1) and $\mathcal{G}(\vartheta) \in \sum_{\delta,p}$ given by

$$\mathcal{G}(\vartheta) = (\vartheta - \delta)^{-p} + \sum_{\epsilon=1}^{\infty} a_{\epsilon-p}(\vartheta - \delta)^{\epsilon-p} \qquad (p \in \mathbb{N}), \tag{4}$$

their Hadamard product (or convolution) is

$$(\mathcal{F} * \mathcal{G})(\vartheta) = (\vartheta - \delta)^{-p} + \sum_{\epsilon=1}^{\infty} a_{\epsilon-p} b_{\epsilon-p}(\vartheta - \delta)^{\epsilon-p} = (\mathcal{G} * \mathcal{F})(\vartheta). \tag{5}$$

We define the following operator $\mathcal{J}^m_{\delta,p}(\mu, \alpha)$. For $\mathcal{F} \in \sum_{\delta,P}$, μ, $\alpha \geq 0$, $p \in \mathbb{N}$, δ be a fixed point in BbbU and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ by:

$$\begin{aligned}
\mathcal{J}^0_{\delta,p}(\mu,\alpha)\mathcal{F}(\vartheta) &= \mathcal{F}(\vartheta), \\
\mathcal{J}^1_{\delta,p}(\mu,\alpha)\mathcal{F}(\vartheta) &= \frac{(p+\alpha)}{\mu}(\vartheta-\delta)_{\delta}^{-\left(p+\frac{p+\alpha}{\mu}\right)\vartheta}(\vartheta-\delta)^{p+\frac{p+\alpha}{\mu}-1}\mathcal{F}(t)\ dt = \mathcal{J}_{\delta,p}(\mu,\alpha)\mathcal{F}(\vartheta) \\
&= (\vartheta-\delta)^{-p} + \sum_{\epsilon=1}^{\infty}\left(\frac{p+\alpha}{p+\mu(k+p)+\alpha}\right)a_{\epsilon-p}(\vartheta-\delta)^{\epsilon-p}, \\
\mathcal{J}^2_{\delta,p}(\mu,\alpha)\mathcal{F}(\vartheta) &= \frac{(p+\alpha)}{\mu}(\vartheta-\delta)_{\delta}^{-\left(p+\frac{p+\alpha}{\mu}\right)\vartheta}(\vartheta-\delta)^{p+\frac{p+\alpha}{\mu}-1}\mathcal{J}^1_{\delta}(\mu,\alpha)\mathcal{F}(\vartheta)\ dt \\
&= (\vartheta-\delta)^{-p} + \sum_{\epsilon=1}^{\infty}\left(\frac{p+\alpha}{p+\mu(k+p)+\alpha}\right)^2 a_{\epsilon-p}(\vartheta-\delta)^{\epsilon-p},
\end{aligned} \tag{6}$$

and

$$\begin{aligned}
\mathcal{J}^m_{\delta,p}(\mu,\alpha)\mathcal{F}(\vartheta) &= \mathcal{J}_{\delta,p}(\mu,\alpha)\mathcal{F}(\vartheta)\left(\mathcal{J}^{m-1}_{\delta,p}(\mu,\alpha)\mathcal{F}(\vartheta)\right) \\
&= (\vartheta-\delta)^{-p} + \sum_{\epsilon=1}^{\infty}\left(\frac{p+\alpha}{p+\mu(k+p)+\alpha}\right)^m a_{\epsilon-p}(\vartheta-\delta)^{\epsilon-p}.
\end{aligned} \tag{7}$$

It follows that

$$(\vartheta-\delta)\mu(\mathcal{J}^{m+1}_{\delta,p}(\mu,\alpha)\mathcal{F}(\vartheta))' = (p+\alpha)\mathcal{J}^m_{\delta,p}(\mu,\alpha)\mathcal{F}(\vartheta) - [\alpha + p(1+\mu)]\,\mathcal{J}^{m+1}_{\delta,p}(\mu,\alpha)\mathcal{F}(\vartheta),\ \mu \neq 0. \tag{8}$$

Note that: At $\delta = 0$, $\mathcal{J}^m_{0,p}(\mu,\alpha)\mathcal{F}(\vartheta) = \mathcal{J}^m_p(\mu,\alpha)\mathcal{F}(\vartheta)$.

Definition 1.1 *A function $\mathcal{F}(\vartheta) \in \mathcal{M}_{\delta,p}^{m}(\alpha,\beta,\mu,\rho,\gamma)$ if it satisfies:*

$$\left[(1-\beta)\left((\vartheta-\delta)^{p}\mathcal{J}_{\delta,p}^{m+1}(\mu,\alpha)\mathcal{F}(\vartheta)\right)^{\gamma}+\beta\left(\frac{\mathcal{J}_{\delta,p}^{m}(\mu,\alpha)\mathcal{F}(\vartheta)}{\mathcal{J}_{\delta,p}^{m+1}(\mu,\alpha)\mathcal{F}(\vartheta)}\right)\left((\vartheta-\delta)^{p}\mathcal{J}_{\delta,p}^{m+1}(\mu,\alpha)\mathcal{F}(\vartheta)\right)^{\gamma}\right]\in\mathcal{P}_{k}(\rho) \tag{9}$$

$$(m>0,\ \mu,\alpha\geq 0,\ k\geq 2,\ \beta\geq 0,\ \delta>0,\ 0\leq\rho<p,\ p\in\mathbb{N};\ \vartheta\in\mathbb{U}).$$

We examine several properties of the class $\mathcal{M}_{\delta,p}^{m}(\alpha,\beta,\mu,\rho,\gamma)$.

2. Main Results

Let $m>0,\ \mu,\alpha\geq 0,\ k\geq 2,\ \beta\geq 0,\ \delta\geq 0,\ 0\leq\rho<p,\ p\in\mathbb{N},\ \vartheta\in\mathbb{U}$ and $\mathcal{F},\mathcal{G}\in\sum_{\delta,p},\ \delta$ be a fixed point in $\mathbb{U}$.

To validate our results we require the subsequent lemma.

Lemma 2.1 [2]. *Let $u=u_1+iu_2,\ v=v_1+iv_2$ and $\Phi(u,v)$ be a function satisfying:* (i) *$\Phi(u,v)$ is continuous in a domain $\mathbb{D}\in\mathbb{C}^2$.*

(ii) $(0,1)\in\mathbb{D}$ *and* $\Phi(1,0)>0$.

(iii) $\Re\{\Phi(iu_2,v_1)\}>0$ *whenever* $(iu_2,v_1)\in\mathbb{D}$ *and* $v_1\leq-\frac{1}{2}\left(1+u_2^2\right)$.

If $\theta(\vartheta)=1+c_\epsilon\vartheta^\epsilon+c_{\epsilon+1}\vartheta^{\epsilon+1}+\ldots$ is analytic in $\mathbb{U}$ such that $\left(\theta(\vartheta),(\vartheta-\delta)\theta'(\vartheta)\right)\in$ $\mathbb{D}$

and $\Re\left\{\Phi\left(\theta(\vartheta),(\vartheta-\delta)\theta'(\vartheta)\right)\right\}>0$ for $\vartheta\in\mathbb{U}$, then $\Re\{\theta(\vartheta)\}>0$ in $\mathbb{U}$.

Using the method for multivalent functions developed by Noor and Muhammad [4] and Aouf and Seoudy [1], we prove the following theorems.

Theorem 2.1 *If $\mathcal{F}(\vartheta)\in\mathcal{M}_{\delta,p}^{m}(\alpha,\beta,\mu,\rho,\gamma)$, then*

$$\left((\vartheta-\delta)^{p}\mathcal{J}_{\delta,p}^{m+1}(\mu,\alpha)\mathcal{F}(\vartheta)\right)^{\gamma}\in\mathcal{P}_{k}(\eta), \tag{10}$$

where η is given by

$$\eta=\frac{2\gamma\rho(\alpha+p)+\lambda\beta}{2\gamma(\alpha+p)+\lambda\beta}. \tag{11}$$

Proof. Let

$$\left((\vartheta-\delta)^{p}\mathcal{J}_{\delta,p}^{m+1}(\mu,\alpha)\mathcal{F}(\vartheta)\right)^{\delta}=\mathcal{H}(\vartheta)=(1-\eta)\theta(\vartheta)+\eta \tag{12}$$

$$=\left(\frac{k}{4}+\frac{1}{2}\right)\{(1-\eta)\theta_1(\vartheta)+\eta\}-\left(\frac{k}{4}-\frac{1}{2}\right)\{(1-\eta)\theta_2(\vartheta)+\eta\},$$

where $\theta_i(\vartheta)\ (i=1,2)$ are analytic in $\mathbb{U}$ with $\theta_i(0)=1\ (i=1,2)$, and $\theta(\vartheta)$ is given by (3). Differentiating (12) with respect to ϑ and using (8), we obtain

$$\left[(1-\beta)\left((\vartheta-\delta)^{p}\mathcal{J}_{\delta,p}^{m+1}(\mu,\alpha)\mathcal{F}(\vartheta)\right)^{\gamma}+\beta\left(\frac{\mathcal{J}_{\delta,p}^{m}(\mu,\alpha)\mathcal{F}(\vartheta)}{\mathcal{J}_{\delta,p}^{m+1}(\mu,\alpha)\mathcal{F}(\vartheta)}\right)\left((\vartheta-\delta)^{p}\mathcal{J}_{\delta,p}^{m+1}(\mu,\alpha)\mathcal{F}(\vartheta)\right)^{\gamma}\right]$$

$$=\left\{(1-\eta)\theta(\vartheta)+\eta+\frac{\lambda\beta(1-\eta)(\vartheta-\delta)\theta'(\vartheta)}{\gamma(\alpha+p)}\right\}\in\mathcal{P}_k(\rho)\quad(\vartheta\in\mathbb{U}),$$

which implies that

$$\frac{1}{1-\rho}\left\{\eta-\rho+(1-\eta)\theta_i(\vartheta)+\frac{\lambda\beta(1-\eta)(\vartheta-\delta)\theta_i'(\vartheta)}{\gamma(\alpha+p)}\right\}\in\mathcal{P}\quad(\vartheta\in\mathbb{U};i=1,2).$$

Let $\Phi(u,v)$ be such that $u=\theta_i(\vartheta),\ v=(\vartheta-\delta)\theta_i'(\vartheta)$, that is

$$\Phi(u,v)=\eta-\rho+(1-\eta)u+\frac{\lambda\beta(1-\eta)v}{\gamma(\alpha+p)}.$$

Thus, Lemma 2.1$'$s first two requirements are met.To confirm (iii), we have

$$\begin{aligned}\Re\left\{\Phi\left(iu_2,v_1\right)\right\} &= \eta-\rho+\Re\left\{\frac{\lambda\beta\left(1-\eta\right)v_1}{\gamma\left(\alpha+p\right)}\right\}\\ &\le \eta-\rho-\frac{\lambda\beta\left(1-\eta\right)\left(1+u_2^2\right)}{2\gamma\left(\alpha+p\right)}\\ &= \frac{\mathcal{A}+\mathcal{B}u_2^2}{2\mathcal{C}},\end{aligned}$$

where

$$\begin{aligned}\mathcal{A} &= 2\gamma\left(\alpha+p\right)\left(\eta-\rho\right)-\lambda\beta\left(1-\eta\right),\\ \mathcal{B} &= -\lambda\beta\left(1-\eta\right),\\ \mathcal{C} &= 2\gamma\left(\alpha+p\right).\end{aligned}$$

We note that $\Re\left\{\Phi\left(iu_2,v_1\right)\right\}<0$ if and only if $\mathcal{A}=0,\mathcal{B}<0$. From (11), we have $0\le\eta<1$, $\mathcal{A}=0$ and $\mathcal{B}<0$. Thus applying Lemma 2.1, we have $\theta_i\left(\vartheta\right)\in\mathcal{P}\ (i=1,2)$ and consequently $\theta\left(\vartheta\right)\in\mathcal{P}_k\left(\eta\right)$ for $\vartheta\in\mathbb{U}$.

Theorem 2.1 *If* $\mathcal{F}\left(\vartheta\right)\in\mathcal{M}_{\delta,p}^m\left(\alpha,\beta,\mu,\rho,\gamma\right)$, *then*

$$\left((\vartheta-\delta)^p\mathcal{J}_{\delta,p}^{m+1}(\mu,\alpha)\mathcal{F}\left(\vartheta\right)\right)^{\frac{\gamma}{2}}\in\mathcal{P}_k\left(\xi\right),\tag{13}$$

where ξ *is given by*

$$\xi=\frac{\beta\lambda+\sqrt{\beta^2\mu^2+4\rho\gamma\left(\alpha+p\right)\left[\gamma\left(\alpha+p\right)+\beta\lambda\right]}}{2\left[\gamma\left(\alpha+p\right)+\beta\lambda\right]}.\tag{14}$$

Proof. Let $\mathcal{F}\left(\vartheta\right)\in\mathcal{M}_{\delta,p}^m\left(\alpha,\beta,\mu,\rho,\gamma\right)$ and

$$\left((\vartheta-\delta)^p\mathcal{J}_{\delta,p}^{m+1}(\mu,\alpha)\mathcal{F}\left(\vartheta\right)\right)^{\gamma}=\mathcal{G}(\vartheta)=\left[\left(1-\xi\right)\theta\left(\vartheta\right)+\xi\right]^2\tag{15}$$

$$=\left(\frac{k}{4}+\frac{1}{2}\right)\left[\left(1-\xi\right)\theta_1\left(\vartheta\right)+\xi\right]^2-\left(\frac{k}{4}-\frac{1}{2}\right)\left[\left(1-\xi\right)\theta_2\left(\vartheta\right)+\xi\right]^2,$$

where $\theta_i\left(\vartheta\right)\ (i=1,2)$ are analytic in $\mathbb{U}$ with $\theta_i\left(0\right)=1\ (i=1,2)$ and $\theta(\vartheta)$ is given by (3). Differentiating (15) with respect to ϑ and using (8), we obtain

$$\begin{aligned}&\left[\left(1-\beta\right)\left((\vartheta-\delta)^p\mathcal{J}_{\delta,p}^{m+1}(\mu,\alpha)\mathcal{F}\left(\vartheta\right)\right)^{\gamma}+\beta\left(\frac{\mathcal{J}_{\delta,p}^{m}(\mu,\alpha)\mathcal{F}(\vartheta)}{\mathcal{J}_{\delta,p}^{m+1}(\mu,\alpha)\mathcal{F}(\vartheta)}\right)\left((\vartheta-\delta)^p\mathcal{J}_{\delta,p}^{m+1}(\mu,\alpha)\mathcal{F}\left(\vartheta\right)\right)^{\gamma}\right]\\ =&\left\{\left[\left(1-\xi\right)\theta\left(\vartheta\right)+\xi\right]^2+\left[\left(1-\xi\right)\theta\left(\vartheta\right)+\xi\right]\frac{2\beta\lambda\left(1-\xi\right)\left(\vartheta-\delta\right)\theta'\left(\vartheta\right)}{\gamma\left(\alpha+p\right)}\right\}\in\mathcal{P}_k\left(\rho\right)\quad\left(\vartheta\in\mathbb{U}\right),\end{aligned}$$

which implies that

$$\frac{1}{1-\rho}\left\{\left[\left(1-\xi\right)\theta\left(\vartheta\right)+\xi\right]^2+\left[\left(1-\xi\right)\theta\left(\vartheta\right)+\xi\right]\frac{2\beta\lambda\left(1-\xi\right)\left(\vartheta-\delta\right)\theta'\left(\vartheta\right)}{\gamma\left(\alpha+p\right)}-\rho\right\}\in\mathcal{P}\quad\left(i=1,2\right).$$

Let $\Phi\left(u,v\right)$ be such that $u=\theta_i(\vartheta)$, $v=(\vartheta-\delta)\theta_i'\left(\vartheta\right)$, that is

$$\Phi\left(u,v\right)=\left[\left(1-\xi\right)u+\xi\right]^2+\left[\left(1-\xi\right)u+\xi\right]\frac{2\beta\lambda\left(1-\xi\right)v}{\gamma\left(\alpha+p\right)}-\rho.$$

So, the conditions (i) and (ii) of Lemma 2.1 are satisfied. To verify (iii), we have

$$\begin{aligned}\Re\left\{\Phi\left(iu_2,v_1\right)\right\} &= \xi^2-(1-\xi)^2u_2^2+\frac{2\beta\lambda\xi\left(1-\xi\right)v_1}{\gamma\left(\alpha+p\right)}-\rho \\ &\leq \xi^2-\rho-(1-\xi)^2u_2^2-\frac{\beta\lambda\xi\left(1-\xi\right)\left(1+u_2^2\right)}{\gamma\left(\alpha+p\right)} \\ &= \frac{\mathcal{A}+\mathcal{B}u_2^2}{\mathcal{C}},\end{aligned}$$

where

$$\begin{aligned}\mathcal{A} &= \xi^2\gamma\left(\alpha+p\right)-\beta\lambda\xi\left(1-\xi\right)-\rho\gamma\left(\alpha+p\right), \\ \mathcal{B} &= -\left[\gamma\left(\alpha+p\right)(1-\xi)^2+\beta\lambda\xi\left(1-\xi\right)\right], \\ \mathcal{C} &= \gamma\left(\alpha+p\right).\end{aligned}$$

We note that $\Re\left\{\Phi\left(iu_2,v_1\right)\right\}<0$ if and only if $\mathcal{A}=0,\mathcal{B}<0$. From (14), we have $0\leq\xi<1$, $\mathcal{A}=0$ and $\mathcal{B}<0$. Thus applying Lemma 2.1,we have $\theta_i\left(\vartheta\right)\in\mathcal{P}\left(i=1,2\right)$ and consequently calG$(\vartheta)\in\mathcal{P}_k\left(\xi\right)$ for $\vartheta\in\mathbb{U}$.

Remark 2.1. Let $\delta=0$, in Theorem 2.1 and 2.3, we get the matching outcomes for the operator $\mathcal{J}_p^m(\mu,\alpha)$.

3. Conclusions

This study presents the definition of the new class of Multivalent meromorphic Bazilevič functions $\mathcal{M}_{\delta,p}^m\left(\alpha,\beta,\mu,\rho,\gamma\right)$ related to the new integral operator $\mathcal{J}_{\delta,p}^m(\mu,\alpha)$, from which several intriguing results are obtained.

Acknowledgements

Not applicable.

Author Contributions

A. O. Mostafa and S. M. Madian : Conceptualization, methodology, resources, review and editing, supervision.

Z. M. Saleh, A. O. Mostafa and S. M. Madian: validation, formal analysis, investigation.

Z. M. Saleh : data curation , writing—original draft preparation.

Funding

Not applicable

Availability of data and material

During the current study the data sets are derived arithmetically.

Declarations

Competing interests

The authors don't have competing for any interests.

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