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MAXIMUM TERM ORIENTED GROWTH ANALYSIS OF COMPOSITE ENTIRE FUNCTIONS FROM THE VIEW POINT OF (α, β, γ)-ORDER

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ABSTRACT. The Fundamental Theorem of Classical Algebra- "If $f(z)$ is a polynomial of degree n with real or complex coefficients, then the equation $f(z) = 0$ has at least one root" is the most renowned value distribution theorem, and consequently every such given polynomial can take any certain value, real or complex. In the value distribution theory, one study how an entire function assumes some values and, on the other hand, what is the influence of taking certain values on a function in some exact approach. Furthermore it deals with various sides of the behavior of entire functions, one of which is the study of their comparative growth. Accordingly, study of comparative growth properties of composite entire functions in terms of their maximum terms are very well known area of research which we attempt in this paper. Here, in this paper, we have discussed maximum terms based some growth properties of composite entire functions with respect to their left or right factor using (α, β, γ)-order and (α, β, γ)-lower order.

1. INTRODUCTION

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ be an entire function defined on $\mathbb{C}$. The maximum modulus function $M(r, f)$ of $f(z)$ on $|z| = r$ is defined as $M(r, f) = \max_{|z|=r} |f(z)|$ and the maximum term denoted as $\mu(r, f)$ is defined as $\mu(r, f) = \max_{n \geq 0} (|a_n| r^n)$. The ratios $\frac{M(r,f)}{M(r,g)}$ and $\frac{\mu(r,f)}{\mu(r,g)}$ as $r \to +\infty$, are respectively called the comparative growth of entire function $f(z)$ with respect to entire function $g(z)$ in form of the maximum moduli and maximum terms. Order and lower order are classical growth indicators of entire and meromorphic

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functions in complex analysis. Several authors have made the close investigations on the growth properties of entire and meromorphic functions in different directions using the concepts of order, iterated p-order ([5] or [6]), (p,q)-th order [3, 4], (p,q)-φ order [8] and achieved many valuable results. We use the standard notations and definitions of the theory of entire functions which are available in [9, 10] and therefore we do not explain those in details. To start our paper, we just recall the following definition:

Definition 1.1. *The order ρ_f and the lower order λ_f of an entire function $f(z)$ are defined as*

$$\rho_f = \limsup_{r\to+\infty} \frac{\log\log M(r,f)}{\log r} \text{ and } \lambda_f = \liminf_{r\to+\infty} \frac{\log\log M(r,f)}{\log r}.$$

Let L be a class of continuous non-negative on $(-\infty,+\infty)$ functions α such that $\alpha(x)=\alpha(x_0)\geq 0$ for $x\leq x_0$ with $\alpha(x)\uparrow+\infty$ as $x_0\leq x\to+\infty$. We say that $\alpha\in L_1$, if $\alpha\in L$ and $\alpha(a+b)\leq\alpha(a)+\alpha(b)+c$ for all $a,b\geq R_0$ and fixed $c\in(0,+\infty)$. Further we say that $\alpha\in L_2$, if $\alpha\in L$ and $\alpha(x+O(1))=(1+o(1))\alpha(x)$ as $x\to+\infty$. Finally, $\alpha\in L_3$, if $\alpha\in L$ and $\alpha(a+b)\leq\alpha(a)+\alpha(b)$ for all $a,b\geq R_0$, i.e., α is subadditive. Clearly $L_3\subset L_1$.

Particularly, when $\alpha\in L_3$, then one can easily verify that $\alpha(mr)\leq m\alpha(r)$, $m\ (\geq 2)$ is an integer. Up to a normalization, subadditivity is implied by concavity. Indeed, if $\alpha(r)$ is concave on $[0,+\infty)$ and satisfies $\alpha(0)\geq 0$, then for $t\in[0,1]$,

$$\begin{aligned}\alpha(tx) &= \alpha(tx+(1-t)\cdot 0)\\ &\geq t\alpha(x)+(1-t)\alpha(0)\geq t\alpha(x),\end{aligned}$$

so that by choosing $t=\frac{a}{a+b}$ or $t=\frac{b}{a+b}$,

$$\begin{aligned}\alpha(a+b) &= \frac{a}{a+b}\alpha(a+b)+\frac{b}{a+b}\alpha(a+b)\\ &\leq \alpha\left(\frac{a}{a+b}(a+b)\right)+\alpha\left(\frac{b}{a+b}(a+b)\right)\\ &= \alpha(a)+\alpha(b),\ a,b\geq 0.\end{aligned}$$

As a non-decreasing, subadditive and unbounded function, $\alpha(r)$ satisfies

$$\alpha(r)\leq\alpha(r+R_0)\leq\alpha(r)+\alpha(R_0)$$

for any $R_0\geq 0$. This yields that $\alpha(r)\sim\alpha(r+R_0)$ as $r\to+\infty$. Throughout the present paper, we take α, α_1, α_2, $\alpha_3\in L_1$, $\beta\in L_2$, $\gamma\in L_3$.

However, Heittokangas et al. [2] have introduced a new concept of φ-order of entire function considering φ as subadditive function. For details, one may see [2]. Later on Belaïdi et al. [1] have extended this concept and have introduced the definitions of (α,β,γ)-order and (α,β,γ)-lower order of an entire function $f(z)$ in terms of maximum moduli in the following way:

Definition 1.2. [1] *The (α,β,γ)-order denoted by $\rho_{(\alpha,\beta,\gamma)}[f]$ and (α,β,γ)-lower order denoted by $\lambda_{(\alpha,\beta,\gamma)}[f]$ of an entire function $f(z)$ are defined as:*

$$\rho_{(\alpha,\beta,\gamma)}[f] = \limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}(M(r,f)))}{\beta(\log(\gamma(r)))}$$
$$\text{and } \lambda_{(\alpha,\beta,\gamma)}[f] = \liminf_{r\to+\infty}\frac{\alpha(\log^{[2]}(M(r,f)))}{\beta(\log(\gamma(r)))}.$$

Remark 1. *Let $\alpha(r)=\log^{[p]} r$ $(p\geq 0)$, $\beta(r)=\log^{[q]} r$ $(q\geq 0)$ and $\gamma(r)=r$, where $\log^{[k]} r=\log(\log^{[k-1]} r)$ $(k\geq 1)$, with convention that $\log^{[0]} r=r$. If $p=0$ and $q=0$, i.e., $\alpha(r)=\beta(r)=r$, the Definition 1.2 coincides with Definition 1.1, when $\alpha(r)=\log^{[p-1]} r$, $(p\geq 1)$, $\beta(r)=r$, we obtain the iterated p-order and iterated lower p-order (see* [6]*), moreover when $\alpha(r)=\log^{[p-1]} r$ and $\beta(r)=\log^{[q-1]} r$, $(p\geq q\geq 1)$, we get the (p,q)-order and lower (p,q)-order (see* [3, 4]*).*

The Definition 1.2 can be alternatively written using maximum term, which is shown in the following proposition:

Proposition 1. *The (α,β,γ)-order and (α,β,γ)-lower order of an entire function $f(z)$, having maximum term $\mu(r,f)$, are defined as:*

$$\rho_{(\alpha,\beta,\gamma)}[f] = \limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}(\mu(r,f)))}{\beta(\log(\gamma(r)))}$$
$$\text{and } \lambda_{(\alpha,\beta,\gamma)}[f] = \liminf_{r\to+\infty}\frac{\alpha(\log^{[2]}(\mu(r,f)))}{\beta(\log(\gamma(r)))}.$$

Proof. By Cauchy's inequality, it is well known that

$$\begin{aligned}
\mu(r,f) &\leq M(r,f)\ \{cf.\ [7]\},\\
\text{i.e., } \log^{[2]}\mu(r,f) &\leq \log^{[2]} M(r,f),\\
\text{i.e., } \alpha(\log^{[2]}\mu(r,f)) &\leq \alpha(\log^{[2]} M(r,f)),\\
\text{i.e., } \frac{\alpha(\log^{[2]}\mu(r,f))}{\beta(\log(\gamma(r)))} &\leq \frac{\alpha(\log^{[2]} M(r,f))}{\beta(\log(\gamma(r)))},\\
\text{i.e., } \limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}\mu(r,f))}{\beta(\log(\gamma(r)))} &\leq \limsup_{r\to+\infty}\frac{\alpha(\log^{[2]} M(r,f))}{\beta(\log(\gamma(r)))},\\
\text{i.e., } \limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}\mu(r,f))}{\beta(\log(\gamma(r)))} &\leq \rho_{(\alpha,\beta,\gamma)}[f]. \qquad (1)
\end{aligned}$$

Also for $0\leq r<R$,

$$M(r,f)\leq\frac{R}{R-r}\mu(R,f)\ \{cf.\ [7]\}.$$

Taking $R=2r$, we get

$$\begin{aligned}
M(r,f) &\leq 2\mu(2r,f),\\
\text{i.e., } \log^{[2]} M(r,f) &\leq \log^{[2]}\mu(2r,f)+O(1),\\
\text{i.e., } \alpha(\log^{[2]} M(r,f)) &\leq \alpha(\log^{[2]}\mu(2r,f))+O(1),
\end{aligned}$$

Since $\gamma(2r) \leq 2\gamma(r)$, so from above it follows that

$$\begin{aligned}
\limsup_{r\to+\infty}\frac{\alpha(\log^{[2]} M(r,f))}{\beta\left(\log(\gamma(r))\right)} &\leq \limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}\mu(2r,f))+O(1)}{\beta\left(\log(\frac{1}{2}\gamma(2r))\right)}\\
&= \limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}\mu(2r,f))+O(1)}{(1+o(1))\beta\left(\log(\gamma(2r))\right)}\\
&= \limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}\mu(r,f))}{\beta\left(\log(\gamma(r))\right)},\\
\text{i.e., } \rho_{(\alpha,\beta,\gamma)}[f] &\leq \limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}\mu(r,f))}{\beta\left(\log(\gamma(r))\right)}. \qquad (2)
\end{aligned}$$

From (1) and (2), we have

$$\rho_{(\alpha,\beta,\gamma)}[f] = \limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}\mu(r,f))}{\beta\left(\log(\gamma(r))\right)}.$$

Using similar technique one can easily prove that

$$\lambda_{(\alpha,\beta,\gamma)}[f] = \liminf_{r\to+\infty}\frac{\alpha(\log^{[2]}(\mu(r,f)))}{\beta\left(\log(\gamma(r))\right)}.$$

□

In this paper, we study some growth properties of the composite entire functions on the basis of (α,β,γ)-order and (α,β,γ)-lower order relating to maximum term.

2. Main results

In this section, the main results of the paper are presented.

Theorem 2.1. *Let $f(z)$ and $g(z)$ be two entire functions such that $0 < \lambda_{(\alpha,\beta,\gamma)}[f] \leq \rho_{(\alpha,\beta,\gamma)}[f] < +\infty$ and $\lambda_{(\alpha,\beta,\gamma)}[f\circ g] = +\infty$. Then*

$$\lim_{r\to+\infty}\frac{\alpha(\log^{[2]}(\mu(r,f\circ g)))}{\alpha(\log^{[2]}(\mu(r,f)))} = +\infty.$$

Proof. If possible, let the conclusion of the theorem does not hold. Then we can find a constant $\Delta > 0$ such that for a sequence of values of r tending to infinity

$$\alpha(\log^{[2]}(\mu(r,f\circ g))) \leq \Delta\cdot\alpha(\log^{[2]}(\mu(r,f))). \qquad (3)$$

Again from the first part of Proposition 1, it follows for all sufficiently large values of r that

$$\alpha(\log^{[2]}(\mu(r,f))) \leq (\rho_{(\alpha,\beta,\gamma)}[f]+\epsilon)\beta(\log(\gamma(r))). \qquad (4)$$

From (3) and (4), for a sequence of values of r tending to $+\infty$, we have

$$\alpha(\log^{[2]}(\mu(r,f\circ g))) \leq \Delta(\rho_{(\alpha,\beta,\gamma)}[f]+\epsilon)\beta(\log(\gamma(r))),$$

$$\text{i.e., } \frac{\alpha(\log^{[2]}(\mu(r,f\circ g)))}{\beta(\log(\gamma(r)))} \leq \Delta(\rho_{(\alpha,\beta,\gamma)}[f]+\epsilon),$$

$$\text{i.e., } \liminf_{r\to+\infty}\frac{\alpha(\log^{[2]}(\mu(r,f\circ g)))}{\beta(\log(\gamma(r)))} < +\infty.$$

Using the second part of Proposition 1, we have

$$\lambda_{(\alpha,\beta,\gamma)}[f\circ g] < +\infty.$$

This is a contradiction.

Thus the theorem follows. $\square$

Remark 2. *If we take "$0 < \lambda_{(\alpha,\beta,\gamma)}[g] \leq \rho_{(\alpha,\beta,\gamma)}[g] < +\infty$" instead of "$0 < \lambda_{(\alpha,\beta,\gamma)}[f] \leq \rho_{(\alpha,\beta,\gamma)}[f] < +\infty$" and other conditions remain same, the conclusion of Theorem 2.1 remains true with "$\alpha(\log^{[2]}(\mu(r,g)))$" in place of "$\alpha(\log^{[2]}(\mu(r,f)))$" in the denominator.*

Remark 3. *Theorem 2.1 and Remark 2 are also valid with "limit superior" instead of "limit" if "$\lambda_{(\alpha,\beta,\gamma)}[f\circ g] = +\infty$" is replaced by "$\rho_{(\alpha,\beta,\gamma)}[f\circ g] = +\infty$" and the other conditions remain the same.*

Theorem 2.2. *Let $f(z)$ and $g(z)$ be two entire functions such that $0 < \lambda_{(\alpha_1,\beta,\gamma)}[f\circ g] \leq \rho_{(\alpha_1,\beta,\gamma)}[f\circ g] < +\infty$ and $0 < \lambda_{(\alpha,\beta,\gamma)}[f] \leq \rho_{(\alpha,\beta,\gamma)}[f] < +\infty$. Then*

$$\begin{aligned}\frac{\lambda_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\rho_{(\alpha_2,\beta,\gamma)}[f]} &\leq \liminf_{r\to+\infty}\frac{\alpha_1\left(\log^{[2]}(\mu(r,f\circ g))\right)}{\alpha_2\left(\log^{[2]}(\mu(r,f))\right)}\\ &\leq \min\left\{\frac{\lambda_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\lambda_{(\alpha_2,\beta,\gamma)}[f]},\frac{\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\rho_{(\alpha_2,\beta,\gamma)}[f]}\right\}\\ &\leq \max\left\{\frac{\lambda_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\lambda_{(\alpha_2,\beta,\gamma)}[f]},\frac{\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\rho_{(\alpha_2,\beta,\gamma)}[f]}\right\}\\ &\leq \limsup_{r\to+\infty}\frac{\alpha_1\left(\log^{[2]}(\mu(r,f\circ g))\right)}{\alpha_2\left(\log^{[2]}(\mu(r,f))\right)} \leq \frac{\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\lambda_{(\alpha_2,\beta,\gamma)}[f]}.\end{aligned}$$

Proof. Using Proposition 1, we have from the definitions of $\lambda_{(\alpha_1,\beta,\gamma)}[f\circ g]$, $\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]$, $\lambda_{(\alpha_2,\beta,\gamma)}[f]$ and $\rho_{(\alpha_2,\beta,\gamma)}[f]$, for arbitrary positive ε and for all sufficiently large values of r that

$$\alpha_1\left(\log^{[2]}(\mu(r,f\circ g))\right) \geqslant \left(\lambda_{(\alpha_1,\beta,\gamma)}[f\circ g]-\varepsilon\right)\beta(\log(\gamma(r))), \tag{5}$$

$$\alpha_1\left(\log^{[2]}(\mu(r,f\circ g))\right) \leq \left(\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]+\varepsilon\right)\beta(\log(\gamma(r))), \tag{6}$$

$$\alpha_2\left(\log^{[2]}(\mu(r,f))\right) \geqslant \left(\lambda_{(\alpha_2,\beta,\gamma)}[f]-\varepsilon\right)\beta(\log(\gamma(r))), \tag{7}$$

$$\text{and } \alpha_2\left(\log^{[2]}(\mu(r,f))\right) \leq \left(\rho_{(\alpha_2,\beta,\gamma)}[f]+\varepsilon\right)\beta(\log(\gamma(r))). \tag{8}$$

Again for a sequence of values of r tending to infinity,

$$\alpha_1\left(\log^{[2]}(\mu(r,f\circ g))\right) \leq \left(\lambda_{(\alpha_1,\beta,\gamma)}[f\circ g]+\varepsilon\right)\beta(\log(\gamma(r))), \tag{9}$$

$$\alpha_1\left(\log^{[2]}(\mu(r,f\circ g))\right) \geqslant \left(\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]-\varepsilon\right)\beta(\log(\gamma(r))), \tag{10}$$

$$\alpha_2\left(\log^{[2]}(\mu(r,f))\right) \leq \left(\lambda_{(\alpha_2,\beta,\gamma)}[f]+\varepsilon\right)\beta(\log(\gamma(r))), \tag{11}$$

$$\text{and } \alpha_2\left(\log^{[2]}(\mu(r,f))\right) \geqslant \left(\rho_{(\alpha_2,\beta,\gamma)}[f]-\varepsilon\right)\beta(\log(\gamma(r))). \tag{12}$$

Now from (5) and (8) it follows for all sufficiently large values of r that

$$\frac{\alpha_1\left(\log^{[2]}(\mu(r,f\circ g))\right)}{\alpha_2\left(\log^{[2]}(\mu(r,f))\right)} \geqslant \frac{\lambda_{(\alpha_1,\beta,\gamma)}[f\circ g]-\varepsilon}{\rho_{(\alpha_2,\beta,\gamma)}[f]+\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to+\infty}\frac{\alpha_1\left(\log^{[2]}(\mu(r,f\circ g))\right)}{\alpha_2\left(\log^{[2]}(\mu(r,f))\right)} \geqslant \frac{\lambda_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\rho_{(\alpha_2,\beta,\gamma)}[f]}. \tag{13}$$

Combining (7) and (9) , we have for a sequence of values of r tending to infinity that

$$\frac{\alpha_1\left(\log^{[2]}(\mu(r,f\circ g))\right)}{\alpha_2\left(\log^{[2]}(\mu(r,f))\right)} \leq \frac{\lambda_{(\alpha_1,\beta,\gamma)}[f\circ g]+\varepsilon}{\lambda_{(\alpha_2,\beta,\gamma)}[f]-\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\liminf_{r\to+\infty}\frac{\alpha_1\left(\log^{[2]}(\mu(r,f\circ g))\right)}{\alpha_2\left(\log^{[2]}(\mu(r,f))\right)} \leq \frac{\lambda_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\lambda_{(\alpha_2,\beta,\gamma)}[f]}. \tag{14}$$

Again from (5) and (11), for a sequence of values of r tending to infinity, we get

$$\frac{\alpha_1\left(\log^{[2]}(\mu(r,f\circ g))\right)}{\alpha_2\left(\log^{[2]}(\mu(r,f))\right)} \geq \frac{\lambda_{(\alpha_1,\beta,\gamma)}[f\circ g]-\varepsilon}{\lambda_{(\alpha_2,\beta,\gamma)}[f]+\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to+\infty}\frac{\alpha_1\left(\log^{[2]}(\mu(r,f\circ g))\right)}{\alpha_2\left(\log^{[2]}(\mu(r,f))\right)} \geq \frac{\lambda_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\lambda_{(\alpha_2,\beta,\gamma)}[f]}. \tag{15}$$

Also, it follows from (6) and (7) , for all sufficiently large values of r that

$$\frac{\alpha_1\left(\log^{[2]}(\mu(r,f\circ g))\right)}{\alpha_2\left(\log^{[2]}(\mu(r,f))\right)} \leq \frac{\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]+\varepsilon}{\lambda_{(\alpha_2,\beta,\gamma)}[f]-\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to+\infty}\frac{\alpha_1\left(\log^{[2]}(\mu(r,f\circ g))\right)}{\alpha_2\left(\log^{[2]}(\mu(r,f))\right)} \leq \frac{\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\lambda_{(\alpha_2,\beta,\gamma)}[f]}. \tag{16}$$

Now from (6) and (12) , it follows for a sequence of values of r tending to infinity that

$$\frac{\alpha_1\left(\log^{[2]}(\mu(r,f\circ g))\right)}{\alpha_2\left(\log^{[2]}(\mu(r,f))\right)} \leq \frac{\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]+\varepsilon}{\rho_{(\alpha_2,\beta,\gamma)}[f]-\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to+\infty}\frac{\alpha_1\left(\log^{[2]}(\mu(r,f\circ g))\right)}{\alpha_2\left(\log^{[2]}(\mu(r,f))\right)}\leq\frac{\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\rho_{(\alpha_2,\beta,\gamma)}[f]}. \tag{17}$$

Combining (8) and (10), we get for a sequence of values of r tending to infinity that

$$\frac{\alpha_1\left(\log^{[2]}(\mu(r,f\circ g))\right)}{\alpha_2\left(\log^{[2]}(\mu(r,f))\right)}\geqslant\frac{\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]-\varepsilon}{\rho_{(\alpha_2,\beta,\gamma)}[f]+\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\limsup_{r\to+\infty}\frac{\alpha_1\left(\log^{[2]}(\mu(r,f\circ g))\right)}{\alpha_2\left(\log^{[2]}(\mu(r,f))\right)}\geqslant\frac{\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\rho_{(\alpha_2,\beta,\gamma)}[f]}. \tag{18}$$

Thus the theorem follows from (13), (14), (15), (16), (17) and (18). □

Remark 4. *If we take "$0<\lambda_{(\alpha_3,\beta,\gamma)}[g]\leq\rho_{(\alpha_3,\beta,\gamma)}[g]<+\infty$" instead of "$0<\lambda_{(\alpha_2,\beta,\gamma)}[f]\leq\rho_{(\alpha_2,\beta,\gamma)}[f]<+\infty$" and other conditions remain same, the conclusion of Theorem 2.2 remains true with "$\lambda_{(\alpha_3,\beta,\gamma)}[g]$", "$\rho_{(\alpha_3,\beta,\gamma)}[g]$" and "$\alpha_3\left(\log^{[2]}(\mu(r,g))\right)$" in place of "$\lambda_{(\alpha_2,\beta,\gamma)}[f]$", "$\rho_{(\alpha_2,\beta,\gamma)}[f]$" and "$\alpha_2\left(\log^{[2]}(\mu(r,f))\right)$" respectively in the denominators.*

3. Conclusion

Belaïdi et al. [1] have introduced the idea of (α,β,γ)-order of entire function in terms of maximum modulus, by which some existing growth indicators have been extended. They have also proved some results in the field of differential equation. In this paper, we have given the equivalent definition of (α,β,γ)-order using maximum term and have generalized some previous results. The study may be an ample scope for further research.

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References

[1] B. Belaïdi and T. Biswas, Study of complex oscillation of solutions of a second order linear differential equation with entire coefficients of (α,β,γ)-order, WSEAS Trans. Math., 21**,** 361-370, 2022, doi: 10.37394/23206.2022.21.43.

[2] J. Heittokangas, J. Wang, Z. T. Wen and H. Yu, Meromorphic functions of finite φ-order and linear q-difference equations, J. Difference Equ. Appl., 27**,** 9, 1280-1309, 2021, doi: 10.1080/10236198.2021.1982919.

[3] O. P. Juneja, G. P. Kapoor and S. K. Bajpai, On the (p,q)-order and lower (p,q)-order of an entire function, J. Reine Angew. Math., 282**,** 53-67, 1976.

[4] O. P. Juneja, G. P. Kapoor and S. K. Bajpai, On the (p,q)-type and lower (p,q)-type of an entire function, J. Reine Angew. Math., 290**,** 180-190, 1977.

[5] L. Kinnunen, Linear differential equations with solutions of finite iterated order, Southeast Asian Bull. Math., 22, 4, 385-405, 1998.

[6] D. Sato, On the rate of growth of entire functions of fast growth, Bull. Amer. Math. Soc., 69, 411-414, 1963.

[7] A. P. Singh and M. S. Baloria, On the maximum modulus and maximum term of composition of entire functions, Indian J. Pure Appl. Math., 22**,** 12, 1019-1026, 1991.

[8] X. Shen, J. Tu and H. Y. Xu, Complex oscillation of a second-order linear differential equation with entire coefficients of $[p,q]$-φ order, Adv. Difference Equ., 2014 : 200, 2014, 14 pages, doi: 10.1186/1687-1847-2014-200.
[9] G. Valiron, Lectures on the general theory of integral functions, Chelsea Publishing Company, New York, 1949.
[10] L. Yang, Value distribution theory, Springer-Verlag, Berlin, 1993.

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