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SOME APPLICATIONS OF FRACTIONAL DERIVATIVE AND MITTAG-LEFFLER FUNCTIONS

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Abstract. The aim of this paper is to introduce a new subclass $TS(\omega,\sigma,\varsigma)$ of univalent functions with negative coefficients related to fractional derivative and Mittag-Leffler function in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. We obtain basic properties like coefficient inequality, distortion and covering theorem, radii of starlikeness, convexity and close-to-convexity, extreme points, Hadamard product, and closure theorems for functions belonging to our class.

1. Introduction

In recent years, the subject of fractional calculus, as a calculus of integrals and derivatives of any real or complex order, has gained considerable popularity and importance, which is due mainly to its demonstrated applications in the modeling and analysis of applied problems and real-world situations occurring in numerous seemingly diverse and widespread fields of science and engineering. It does indeed also provide several potentially useful tools and techniques for solving dierential and integral equations, and various other problems involving special functions of mathematical physics as well as their extensions and generalizations in one and more variables. In geometric function theory, fractional derivatives and the Mittag-Leffler function play crucial roles in understanding complex functions and their geometric properties. fractional derivatives generalize this concept to non-integer orders. They provide a powerful tool for analyzing complex systems with fractal or non-local behaviors. In geometric function theory, fractional derivatives are used to study the behavior of complex functions on fractal sets or in domains with irregular boundaries. They help characterize the smoothness and regularity of functions in such contexts.

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Fractional derivatives also enable the study of fractional-order differential equations, which have applications in various fields including physics, engineering, and mathematical modeling.
The Mittag-Leffler function, is a generalization of the exponential function. It arises naturally in the theory of fractional calculus and complex analysis. It is defined as a solution to certain types of fractional differential equations and plays a fundamental role in the theory of fractional calculus. In geometric function theory, the Mittag-Leffler function often appears in the context of fractional order differential operators and fractional integral transforms. It provides a natural extension of exponential functions in fractional calculus. The Mittag-Leffler function is also significant in probability theory, where it appears in the study of random processes, particularly in fractional Brownian motion and related stochastic processes. The fractional derivatives and the Mittag-Leffler function offer powerful tools for analyzing complex functions and their geometric properties, particularly in contexts where traditional calculus techniques may not suffice. They provide insights into the behavior of functions on irregular domains. while visualizing fractional derivatives and the Mittag-Leffler function directly on a geometric plane might be challenging, their geometric interpretations involve understanding their effects on functions' behavior over complex or irregular geometries. These interpretations help elucidate how these mathematical concepts contribute to the study of complex functions in geometric function theory.
Fractional calculus is one of the most intensively developing areas of the mathematical analysis. The fractional calculus operators have gone deep across into the realm of the theory of univalent functions. Various operators of fractional calculus have been studied in the literature rather extensively. We find it to be convenient to recall here the following definitions (cf., e.g., [19, 20, 28]).

The study of the Mittag-Leffler function and its various generalizations has become a very popular topic in mathematics and its applications. The recent growing interest in this function is mainly due to its close relation to the Fractional Calculus and especially to fractional problems which come from applications. For a few decades, the special transcendental function known as the Mittag-Leffler function has attracted the increasing attention of researchers because of its key role in treating problems related to integral and differential equations of fractional order. Since its introduction in 1903-1905 by the Swedish mathematician Mittag-Leffler at the beginning of the last century up to the 1990s, this function was seldom considered by mathematicians and applied scientist.

Nowadays it is well recognized that the Mittag-Leffler function plays a fundamental role in Fractional Calculus even if with a single parameter (as originally introduced by Mittag-Leffler) just to be worth of being referred to as the **Queen Function of Fractional Calculus,** see Mainardi and Gorenflo [17]. We find some information on the Mittag-Leffler functions in any treatise on Fractional Calculus but for more details we refer the reader to the surveys of Haubold, Mathai and Saxena [11] and by Van Mieghem [34] and to the treatise by Gorenflo et al.[10], just devoted to Mittag-Leffler functions, related topics and applications.

Recent attention has been drawn to Mittag-Leffer function research, as this kind of function can be widely applied across engineering, chemical and biological sciences, physics and in applied science. Various factors in applying such functions are evident within chaotic, stochastic and dynamic systems, fractional differential equations, and distribution of statistics. The geometric characteristics such as convexity, close-to-convexity and starlikeness, of the functions investigated here have been broadly examined by many authors, and direct applications from such functions can be seen for a number of fractional calculus tools, including significant work by [2, 3, 5, 6, 23, 33, 31]

The Mittag-Leffler function arises naturally in the solution of fractional order differential and integral equations, and especially in the investigations of fractional generalization of kinetic equation, random walks, Levy flights, super-diffusive transport and in the study of complex systems. Several properties of Mittag-Leffler function and generalized Mittag-Leffler function can be found e.g. in [7, 26, 4, 3, 8, 9, 14, 15].

Let A signify the class of all functions $u(z)$ of the type

$$u(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions and satisfy the following usual normalization condition $u(0) = u'(0) - 1 = 0$. We denote by S the subclass of A consisting of functions $u(z)$ which are all univalent in $\mathbb{U}$. A function $u \in A$ is a starlike function of the order $\varsigma, 0 \leq \varsigma < 1$, if it satisfy

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \varsigma, z \in \mathbb{U}. \tag{2}$$

We denote this class with $S^*(\varsigma)$.

A function $u \in A$ is a convex function of the order $\varsigma, 0 \leq \varsigma < 1$, if it fulfil

$$\Re\left\{1 + \frac{zu''(z)}{u'(z)}\right\} > \varsigma, z \in \mathbb{U}. \tag{3}$$

We denote this class with $K(\varsigma)$.

Note that $S^*(0) = S^*$ and $K(0) = K$ are the usual classes of starlike and convex functions in $\mathbb{U}$ respectively.

Let T denote the class of functions analytic in $\mathbb{U}$ that are of the form

$$u(z) = z - \sum_{n=2}^{\infty} a_n z^n, \;\; a_n \geq 0 \;\; z \in \mathbb{U} \tag{4}$$

and let $T^*(\varsigma) = T \cap S^*(\varsigma), \;\; C(\varsigma) = T \cap K(\varsigma)$. The class $T^*(\varsigma)$ and allied classes possess some interesting properties and have been extensively studied by Silverman [27].

Many basically equivalent definitions of fractional computation have been given in literature ((cf.)e.g.,[24] and ([29], p. 45)). We state the following definitions due to Owa and Srivastava [21] which have been used rather frequently in the theory of analytic functions (see also [12, 13]).

Pochhammer symbol $(\alpha)_n$ can be defined as

$$(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1) \;\; \text{if} \;\; n \neq 0$$

and

$$(\alpha)_n = 1 \text{ if } n = 0.$$

The $(\alpha)_n$ can be expressed in terms of the Gamma function as:

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)},\ (n \in \mathbb{N}).$$

In [18], Mittag-Leffler introduced Mittag-Leffler functions

$$\mathcal{H}_\alpha(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha n+1)} z^n,\ (\alpha \in \mathbb{C},\ Re(\alpha)) > 0,$$

and its generalization $\mathcal{H}_{\alpha,\beta}(z)$ introduced by Wiman [35] as

$$\mathcal{H}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha n+\beta)} z^n,\ (\alpha,\beta \in \mathbb{C},\ Re(\alpha),\ Re(\beta)) > 0. \tag{5}$$

Now we define the normalization of Mittag-Leffler function $\mathcal{M}_{\alpha,\beta}(z)$ as follows:

$$\begin{aligned} \mathcal{M}_{\alpha,\beta}(z) &= z\Gamma(\beta)\mathcal{H}_{\alpha,\beta}(z) \\ &= z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} z^n, \end{aligned} \tag{6}$$

where, $z \in U$, $(Re\ \alpha > 0, \beta \in \mathbb{C} \setminus \{0,-1,-2,\cdots\})$. In [30], Srivastava and Owa gave definitions for fractional derivative operator and fractional integral operator in the complex z- plane C in terms of the RiemannLiouville fractional calculus, as follows: The fractional integral of order δ is defined for a function $u(z)$, by

$$I_z^\delta u(z) \equiv I_z^{-\delta} u(z) = \frac{1}{\Gamma(\delta)} \int_0^z (z-t)^{\delta-1} u(t) d(t),\ (\delta > 0).$$

The fractional derivative operator D_z of order δ is defined by

$$D_z^\delta u(z) = D_z I_z^{1-\delta} u(z) = \frac{1}{\Gamma(1-\delta)} D_z \int_0^z \frac{u(t)}{(z-t)^\delta} dt,\ (0 \le \delta < 1).$$

where, the function $u(z)$ is analytic in the simply-connected region of the complex z-plane C containg the origin, and the multiplicity of $(z-t)^{-\delta}$ is removed by requiring $log(z-t)$ to be real when $(z-t) > 0$.
Let $\delta > 0$ and m be the smallest integer, and the extended fractional derivative of $u(z)$ of order δ is defined as:

$$D_z^\delta u(z) = D_z^m I_z^{m-\delta} u(z),\ \delta \ge 0,\ n > -1 \tag{7}$$

provided that it exists. We find from (7) that is

$$D_z^\delta z^n = \frac{\Gamma(n+1)}{\Gamma(n+1-\delta)} z^{n-\delta},\ (0 \le \delta < 1,\ n > -1)$$

and

$$I_z^\delta z^n = \frac{\Gamma(n+1)}{\Gamma(n+1-\delta)} z^{n+\delta},\ (0 < \delta,\ n > -1).$$

Owa and Srivastava [21], defined the differential Integral operator $\Omega_z^\delta : A \to A$ in the term of series:

$$\Omega_z^{\delta}u(z) = \frac{\Gamma(2-\delta)}{\Gamma(2)} z^{\delta} D_z^{\delta}u(z) \tag{8}$$
$$= z + \sum_{n=2}^{\infty} \left(\frac{\Gamma(2-\delta)\Gamma(n+1)}{\Gamma(2)\Gamma(n+1-\delta)} \right) a_n z^n$$

where, $\delta < 2,$ and $z \in U$.)
Here $D_z^{\delta}u(z)$ represents the fractional of u(z) of order δ when $-\infty < \delta < 0$ and a fractional derivative of $u(z)$ of order δ when $0 \leq \delta < 2$. Now, by using the definition of convolution of (6) and (8), we define fractional differential integral operator $D_z^{\delta,\alpha,\beta} : A \longrightarrow A$, associated with normalized Mittag-Leffler function $\mathcal{M}_{\alpha,\beta}(z)$as follows:

$$\mathfrak{D}_z^{\delta,\alpha,\beta}u(z) = z + \sum_{n=2}^{\infty} \Theta(n,\delta,\alpha,\beta) a_n z^n \tag{9}$$

where $\Theta(n,\delta,\alpha,\beta) = \left(\frac{\Gamma(2-\delta)\Gamma(n+1)}{\Gamma(2)\Gamma(n+1-\delta)} \right) \left(\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \right) a_n z^n$
and $(\delta < 2,\ Re\ \alpha > 0,\ \beta \in \mathbb{C} \setminus \{0,-1,-2,\cdots\}),\ z \in U$.
It is noted that

$$\mathfrak{D}_z^{0,0,1}u(z) = u(z).$$

Motivated by work of [1, 22, 25, 32], we define a new subclass of functions belonging to the class A.

Definition 1 For $0 \leq \omega < 1, 0 \leq \sigma < 1, 0 < \varsigma < 1,$ and $0 \leq \vartheta < 1$, we let $TS(\omega,\sigma,\varsigma)$ be the subclass of u consisting of functions of the form (4) and its geometrical condition satisfy

$$\left| \frac{\omega \left((\mathfrak{D}_z^{\delta,\alpha,\beta}u(z))' - \frac{\mathfrak{D}_z^{\delta,\alpha,\beta}u(z)}{z} \right)}{\sigma(\mathfrak{D}_z^{\delta,\alpha,\beta}u(z))' + (1-\omega)\frac{\mathfrak{D}_z^{\delta,\alpha,\beta}u(z)}{z}} \right| < \varsigma,\ z \in \mathbb{U}$$

where $\mathfrak{D}_z^{\delta,\alpha,\beta}u(z)$, is given by (9).

2. Coefficient Inequality

In the following theorem, we obtain a necessary and sufficient condition for function to be in the class $TS(\omega,\sigma,\varsigma)$.

Theorem 1 Let the function u be defined by (4). Then $u \in TS(\omega,\sigma,\varsigma)$ if and only if

$$\sum_{n=2}^{\infty} [\omega(n-1) + \varsigma(n\sigma + 1 - \omega)]\Theta(n,\delta,\alpha,\beta)a_n \leq \varsigma(\sigma + (1-\omega)), \tag{10}$$

where $0 < \varsigma < 1, 0 \leq \omega < 1,$ and $0 \leq \sigma < 1$, The result (10) is sharp for the function

$$u(z) = z - \frac{\varsigma(\sigma + (1-\omega))}{[\omega(n-1) + \varsigma(n\sigma + 1 - \omega)]\Theta(n,\delta,\alpha,\beta)} z^n,\ n \geq 2.$$

Proof. Suppose that the inequality (10) holds true and $|z|=1$. Then we obtain

$$\begin{aligned}
&\left|\omega\left((\mathfrak{D}_z^{\delta,\alpha,\beta}u(z))' - \frac{\mathfrak{D}_z^{\delta,\alpha,\beta}u(z)}{z}\right)\right| - \varsigma\left|\sigma\left(\mathfrak{D}_z^{\delta,\alpha,\beta}u(z))' + (1-\omega)\frac{\mathfrak{D}_z^{\delta,\alpha,\beta}u(z)}{z}\right)\right| \\
&= \left|-\omega\sum_{n=2}^{\infty}(n-1)\Theta(n,\delta,\alpha,\beta)a_n z^{n-1}\right| \\
&\quad - \varsigma\left|\sigma + (1-\omega) - \sum_{n=2}^{\infty}(n\sigma + 1 - \omega)\Theta(n,\delta,\alpha,\beta)a_n z^{n-1}\right| \\
&\leq \sum_{n=2}^{\infty}[\omega(n-1) + \varsigma(n\sigma + 1 - \omega)]\Theta(n,\delta,\alpha,\beta)a_n - \varsigma(\sigma + (1-\omega)) \\
&\leq 0.
\end{aligned}$$

Hence, by maximum modulus principle,$u \in TS(\omega,\sigma,\varsigma)$. Now assume that $u \in TS(\omega,\sigma,\varsigma)$ so that

$$\left|\frac{\omega\left((\mathfrak{D}_z^{\delta,\alpha,\beta}u(z))' - \frac{\mathfrak{D}_z^{\delta,\alpha,\beta}u(z)}{z}\right)}{\sigma(\mathfrak{D}_z^{\delta,\alpha,\beta}u(z))' + (1-\omega)\frac{\mathfrak{D}_z^{\delta,\alpha,\beta}u(z)}{z}}\right| < \varsigma,\ z \in \mathbb{U}$$

Hence

$$\left|\omega\left((\mathfrak{D}_z^{\delta,\alpha,\beta}u(z))' - \frac{\mathfrak{D}_z^{\delta,\alpha,\beta}u(z)}{z}\right)\right| < \varsigma\left|\sigma\left(\mathfrak{D}_z^{\delta,\alpha,\beta}u(z))' + (1-\omega)\frac{\mathfrak{D}_z^{\delta,\alpha,\beta}u(z)}{z}\right)\right|.$$

Therefore, we get

$$\begin{aligned}
&\left|-\sum_{n=2}^{\infty}\omega(n-1)\Theta(n,\delta,\alpha,\beta)a_n z^{n-1}\right| \\
&< \varsigma\left|\sigma + (1-\omega) - \sum_{n=2}^{\infty}(n\sigma + 1 - \omega)\Theta(n,\delta,\alpha,\beta)a_n z^{n-1}\right|.
\end{aligned}$$

Thus

$$\sum_{n=2}^{\infty}[\omega(n-1) + \varsigma(n\sigma + 1 - \omega)]\Theta(n,\delta,\alpha,\beta)a_n \leq \varsigma(\sigma + (1-\omega))$$

and this completes the proof.

Corollary 1 Let the function $u \in TS(\omega,\sigma,\varsigma)$.Then

$$a_n \leq \frac{\varsigma(\sigma + (1-\omega))}{[\omega(n-1) + \varsigma(n\sigma + 1 - \omega)]\Theta(n,\delta,\alpha,\beta)}z^n,\ n \geq 2.$$

3. Distortion and Covering Theorem

We introduce the growth and distortion theorems for the functions in the class $TS(\omega,\sigma,\varsigma)$

Theorem 2 Let the function $u \in TS(\omega,\sigma,\varsigma)$. Then

$$\begin{aligned}
&|z| - \frac{\varsigma(\sigma + (1-\omega))}{\Theta(2,\delta,\alpha,\beta)[\omega + \varsigma(2\sigma + 1 - \omega)]}|z|^2 \leq |u(z)| \\
&\qquad \leq |z| + \frac{\varsigma(\sigma + (1-\omega))}{\Theta(2,\delta,\alpha,\beta)[\omega + \varsigma(2\sigma + 1 - \omega)]}|z|^2.
\end{aligned}$$

The result is sharp and attained

$$u(z) = z - \frac{\varsigma(\sigma + (1-\omega))}{\Theta(2,\delta,\alpha,\beta)[\omega + \varsigma(2\sigma + 1 - \omega)]} z^2.$$

Proof.

$$|u(z)| = \left| z - \sum_{n=2}^{\infty} a_n z^n \right| \leq |z| + \sum_{n=2}^{\infty} a_n |z|^n$$
$$\leq |z| + |z|^2 \sum_{n=2}^{\infty} a_n.$$

By Theorem 2, we get

$$\sum_{n=2}^{\infty} a_n \leq \frac{\varsigma(\sigma + (1-\omega))}{[\omega + \varsigma(2\sigma + 1 - \omega)]\Theta(n,\delta,\alpha,\beta)}. \tag{11}$$

Thus

$$|u(z)| \leq |z| + \frac{\varsigma(\sigma + (1-\omega))}{\Theta(2,\delta,\alpha,\beta)[\omega + \varsigma(2\sigma + 1 - \omega)]} |z|^2.$$

Also

$$|u(z)| \geq |z| - \sum_{n=2}^{\infty} a_n |z|^n$$
$$\geq |z| - |z|^2 \sum_{n=2}^{\infty} a_n$$
$$\geq |z| - \frac{\varsigma(\sigma + (1-\omega))}{\Theta(2,\delta,\alpha,\beta)[\omega + \varsigma(2\sigma + 1 - \omega)]} |z|^2.$$

Theorem 3 Let $u \in TS(\omega, \sigma, \varsigma)$. Then

$1 - \frac{2\varsigma(\sigma+(1-\omega))}{\Theta(2,\delta,\alpha,\beta)[\omega+\varsigma(2\sigma+1-\omega)]}|z| \leq |u'(z)| \leq 1 + \frac{2\varsigma(\sigma+(1-\omega))}{\Theta(2,\delta,\alpha,\beta)[\omega+\varsigma(2\sigma+1-\omega)]}|z|$

with equality for

$$u(z) = z - \frac{2\varsigma(\sigma + (1-\omega))}{\Theta(2,\delta,\alpha,\beta)[\omega + \varsigma(2\sigma + 1 - \omega)]} z^2.$$

Proof. Notice that

$$\Theta(2,\delta,\alpha,\beta)[\omega + \varsigma(2\sigma + 1 - \omega)] \sum_{n=2}^{\infty} n a_n$$
$$\leq \sum_{n=2}^{\infty} n[\omega(n-1) + \varsigma(n\sigma + 1 - \omega)]\Theta(n,\delta,\alpha,\beta) a_n$$
$$\leq \varsigma(\sigma + (1-\omega)), \tag{12}$$

from Theorem 2. Thus

$$\begin{aligned}|u'(z)| &= \left|1-\sum_{n=2}^{\infty} na_n z^{n-1}\right| \\ &\leq 1+\sum_{n=2}^{\infty} na_n |z|^{n-1} \\ &\leq 1+|z|\sum_{n=2}^{\infty} na_n \\ &\leq 1+|z|\frac{2\varsigma(\sigma+(1-\omega))}{\Theta(2,\delta,\alpha,\beta)[\omega+\varsigma(2\sigma+1-\omega)]}. \qquad (13)\end{aligned}$$

On the other hand

$$\begin{aligned}|u'(z)| &= \left|1-\sum_{n=2}^{\infty} na_n z^{n-1}\right| \\ &\geq 1-\sum_{n=2}^{\infty} na_n |z|^{n-1} \\ &\geq 1-|z|\sum_{n=2}^{\infty} na_n \\ &\geq 1-|z|\frac{2\varsigma(\sigma+(1-\omega))}{\Theta(2,\delta,\alpha,\beta)[\omega+\varsigma(2\sigma+1-\omega)]}. \qquad (14)\end{aligned}$$

Combining (13) and (14), we get the result.

4. Radii of Starlikeness, Convexity and Close-to-Convexity

In the following theorems, we obtain the radii of starlikeness, convexity and close-to-convexity for the class $TS(\omega,\sigma,\varsigma)$.

Theorem 4 Let $u \in TS(\omega,\sigma,\varsigma)$. Then u is starlike in $|z| < R_1$ of order ρ, $0 \leq \rho < 1$, where

$$R_1 = \inf_n \left\{ \frac{(1-\rho)(\omega(n-1)+\varsigma(n\sigma+1-\omega))\Theta(n,\delta,\alpha,\beta)}{(n-\rho)\varsigma(\sigma+(1-\omega))} \right\}^{\frac{1}{n-1}}, \ n \geq 2. \qquad (15)$$

Proof. u is starlike of order $\rho, 0 \leq \rho < 1$ if

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \rho.$$

Thus it is enough to show that

$$\left|\frac{zu'(z)}{u(z)}-1\right| = \left|\frac{-\sum\limits_{n=2}^{\infty}(n-1)a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty}a_n z^{n-1}}\right| \leq \frac{\sum\limits_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}a_n|z|^{n-1}}.$$

Thus

$$\left|\frac{zu'(z)}{u(z)}-1\right| \leq 1-\rho \ if \ \sum_{n=2}^{\infty}\frac{(n-\rho)}{(1-\rho)}a_n|z|^{n-1} \leq 1. \qquad (16)$$

Hence by Theorem 2, (16) will be true if

$$\frac{n-\rho}{1-\rho}|z|^{n-1} \leq \frac{(\omega(n-1)+\varsigma(n\sigma+1-\omega))\Theta(n,\delta,\alpha,\beta)}{\varsigma(\sigma+(1-\omega)}$$

or if

$$|z| \leq \left[\frac{(1-\rho)(\omega(n-1)+\varsigma(n\sigma+1-\omega))\Theta(n,\delta,\alpha,\beta)}{(n-\rho)\varsigma(\sigma+(1-\omega))}\right]^{\frac{1}{n-1}}, n \geq 2. \quad (17)$$

The theorem follows easily from (17).

Theorem 5 Let $u \in TS(\omega,\sigma,\varsigma)$. Then u is convex in $|z| < R_2$ of order $\rho, 0 \leq \rho < 1$, where

$$R_2 = \inf_n \left\{\frac{(1-\rho)(\omega(n-1)+\varsigma(n\sigma+1-\omega))\Theta(n,\delta,\alpha,\beta)}{n(n-\rho)\varsigma(\sigma+(1-\omega))}\right\}^{\frac{1}{n-1}}, \ n \geq 2. \quad (18)$$

Proof. u is convex of order $\rho, 0 \leq \rho < 1$ if

$$\Re\left\{1+\frac{zu''(z)}{u'(z)}\right\} > \rho.$$

Thus it is enough to show that

$$\left|\frac{zu''(z)}{u'(z)}\right| = \left|\frac{-\sum\limits_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty} na_n z^{n-1}}\right| \leq \frac{\sum\limits_{n=2}^{\infty} n(n-1)a_n|z|^{n-1}}{1-\sum\limits_{n=2}^{\infty} na_n|z|^{n-1}}.$$

Thus

$$\left|\frac{zu''(z)}{u'(z)}\right| \leq 1-\rho \ if \ \sum_{n=2}^{\infty}\frac{n(n-\rho)}{(1-\rho)}a_n|z|^{n-1} \leq 1. \quad (19)$$

Hence by Theorem 2, (19) will be true if

$$\frac{n(n-\rho)}{1-\rho}|z|^{n-1} \leq \frac{(\omega(n-1)+\varsigma(n\sigma+1-\omega))\Theta(n,\delta,\alpha,\beta)}{\varsigma(\sigma+(1-\omega)}$$

or if

$$|z| \leq \left[\frac{(1-\rho)(\omega(n-1)+\varsigma(n\sigma+1-\omega))\Theta(n,\delta,\alpha,\beta)}{n(n-\rho)\varsigma(\sigma+(1-\omega))}\right]^{\frac{1}{n-1}}, n \geq 2. \quad (20)$$

The theorem follows easily from (20).

Theorem 6 Let $u \in TS(\omega,\sigma,\varsigma)$. Then u is close-to-convex in $|z| < R_3$ of order $\rho, \ \ 0 \leq \rho < 1$, where

$$R_3 = \inf_n \left\{\frac{(1-\rho)(\omega(n-1)+\varsigma(n\sigma+1-\omega))\Theta(n,\delta,\alpha,\beta)}{n\varsigma(\sigma+(1-\omega))}\right\}^{\frac{1}{n-1}}, \ n \geq 2. \quad (21)$$

Proof. u is close-to-convex of order $\rho, 0 \leq \rho < 1$ if

$$\Re\{u'(z)\} > \rho.$$

Thus it is enough to show that

$$|u'(z)-1| = \left|-\sum_{n=2}^{\infty} na_n z^{n-1}\right| \leq \sum_{n=2}^{\infty} na_n|z|^{n-1}.$$

Thus

$$|u'(z)-1|\leq 1-\rho \; if \; \sum_{n=2}^{\infty}\frac{n}{(1-\rho)}a_n|z|^{n-1}\leq 1. \tag{22}$$

Hence by Theorem 2, (22) will be true if

$$\frac{n}{1-\rho}|z|^{n-1}\leq \frac{(\omega(n-1)+\varsigma(n\sigma+1-\omega))\Theta(n,\delta,\alpha,\beta)}{\varsigma(\sigma+(1-\omega)}$$

or if

$$|z|\leq\left[\frac{(1-\rho)(\omega(n-1)+\varsigma(n\sigma+1-\omega))\Theta(n,\delta,\alpha,\beta)}{n\varsigma(\sigma+(1-\omega))}\right]^{\frac{1}{n-1}}, n\geq 2. \tag{23}$$

The theorem follows easily from (23).

5. Extreme Points

In the following theorem, we obtain extreme points for the class $TS(\omega,\sigma,\varsigma)$.

Theorem 7 Let $u_1(z)=z$ and

$$u_n(z)=z-\frac{\varsigma(\sigma+(1-\omega))}{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\Theta(n,\delta,\alpha,\beta)}z^n,\ \text{for } n=2,3,\cdots.$$

Then $u\in TS(\omega,\sigma,\varsigma)$ if and only if it can be expressed in the form

$$u(z)=\sum_{n=1}^{\infty}\theta_n u_n(z),\ where\ \theta_n\geq 0\ and\ \sum_{n=1}^{\infty}\theta_n=1.$$

Proof. Assume that $u(z)=\sum\limits_{n=1}^{\infty}\theta_n u_n(z)$, hence we get

$$u(z)=z-\sum_{n=2}^{\infty}\frac{\varsigma(\sigma+(1-\omega))\theta_n}{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\Theta(n,\delta,\alpha,\beta)}z^n.$$

Now, $u\in TS(\omega,\sigma,\varsigma)$, since

$$\begin{aligned}&\sum_{n=2}^{\infty}\frac{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\Theta(n,\delta,\alpha,\beta)}{\varsigma(\sigma+(1-\omega))}\\ &\times\frac{\varsigma(\sigma+(1-\omega))\theta_n}{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\Theta(n,\delta,\alpha,\beta)}\\ =&\sum_{n=2}^{\infty}\theta_n=1-\theta_1\leq 1.\end{aligned}$$

Conversely, suppose $u\in TS(\omega,\sigma,\varsigma)$. Then we show that u can be written in the form $\sum\limits_{n=1}^{\infty}\theta_n u_n(z)$.

Now $u\in TS(\omega,\sigma,\varsigma)$ implies from Theorem 2

$$a_n\leq\frac{\varsigma(\sigma+(1-\omega))}{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\Theta(n,\delta,\alpha,\beta)}.$$

Setting $\theta_n=\frac{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\Theta(n,\delta,\alpha,\beta)}{\varsigma(\sigma+(1-\omega))}a_n, n=2,3,\cdots$

and $\theta_1=1-\sum\limits_{n=2}^{\infty}\theta_n$, we obtain $u(z)=\sum\limits_{n=1}^{\infty}\theta_n u_n(z)$.

6. Hadamard product

In the following theorem, we obtain the convolution result for functions belongs to the class $TS(\omega,\sigma,\varsigma)$.

Theorem 8 Let $u, g \in TS(\omega,\sigma,\varsigma,\vartheta)$. Then $u*g \in TS(\omega,\sigma,\zeta,\vartheta)$ for

$$u(z)=z-\sum_{n=2}^{\infty}a_nz^n, g(z)=z-\sum_{n=2}^{\infty}b_nz^n \text{ and } (u*g)(z)=z-\sum_{n=2}^{\infty}a_nb_nz^n,$$

where

$$\zeta \geq \frac{\varsigma^2(\sigma+(1-\omega))\omega(n-1)}{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]^2\Theta(n,\delta,\alpha,\beta)-\varsigma^2(\sigma+(1-\omega))(n\sigma+1-\omega)}.$$

Proof. $u \in TS(\omega,\sigma,\varsigma)$ and so

$$\sum_{n=2}^{\infty}\frac{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\Theta(n,\delta,\alpha,\beta)}{\varsigma(\sigma+(1-\omega))}a_n \leq 1, \tag{24}$$

and

$$\sum_{n=2}^{\infty}\frac{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\Theta(n,\delta,\alpha,\beta)}{\varsigma(\sigma+(1-\omega))}b_n \leq 1. \tag{25}$$

We have to find the smallest number ζ such that

$$\sum_{n=2}^{\infty}\frac{[\omega(n-1)+\zeta(n\sigma+1-\omega)]\Theta(n,\delta,\alpha,\beta)}{\zeta(\sigma+(1-\omega))}a_nb_n \leq 1. \tag{26}$$

By Cauchy-Schwarz inequality

$$\sum_{n=2}^{\infty}\frac{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\Theta(n,\delta,\alpha,\beta)}{\varsigma(\sigma+(1-\omega))}\sqrt{a_nb_n} \leq 1. \tag{27}$$

Therefore it is enough to show that

$$\frac{[\omega(n-1)+\zeta(n\sigma+1-\omega)]\Theta(n,\delta,\alpha,\beta)}{\zeta(\sigma+(1-\omega))}a_nb_n \leq \frac{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\Theta(n,\delta,\alpha,\beta)}{\varsigma(\sigma+(1-\omega))}\sqrt{a_nb_n}.$$

That is

$$\sqrt{a_nb_n} \leq \frac{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\zeta}{[\omega(n-1)+\zeta(n\sigma+1-\omega)]\varsigma}. \tag{28}$$

From (27)

$$\sqrt{a_nb_n} \leq \frac{\varsigma(\sigma+(1-\omega))}{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\Theta(n,\delta,\alpha,\beta)}.$$

Thus it is enough to show that

$$\frac{\varsigma(\sigma+(1-\omega))}{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\Theta(n,\delta,\alpha,\beta)} \leq \frac{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\zeta}{[\omega(n-1)+\zeta(n\sigma+1-\omega)]\varsigma},$$

which simplifies to

$$\zeta \geq \frac{\varsigma^2(\sigma+(1-\omega))\omega(n-1)}{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]^2\Theta(n,\delta,\alpha,\beta)-\varsigma^2(\sigma+(1-\omega))(n\sigma+1-\omega)}.$$

7. Closure Theorems

We shall prove the following closure theorems for the class $TS(\omega,\sigma,\varsigma)$.

Theorem 9 Let $u_j \in TS(\omega,\sigma,\varsigma), j = 1,2,\ldots,s$. Then

$$g(z) = \sum_{j=1}^{s} c_j u_j(z) \in TS(\omega,\sigma,\varsigma)$$

For $u_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n$, where $\sum_{j=1}^{s} c_j = 1$.

Proof.

$$\begin{aligned} g(z) &= \sum_{j=1}^{s} c_j u_j(z) \\ &= z - \sum_{n=2}^{\infty}\sum_{j=1}^{s} c_j a_{n,j} z^n \\ &= z - \sum_{n=2}^{\infty} e_n z^n, \end{aligned}$$

where $e_n = \sum_{j=1}^{s} c_j a_{n,j}$. Thus $g(z) \in TS(\omega,\sigma,\varsigma)$ if

$$\sum_{n=2}^{\infty} \frac{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\Theta(n,\delta,\alpha,\beta)}{\varsigma(\sigma+(1-\omega))} e_n \leq 1,$$

that is, if

$$\begin{aligned} &\sum_{n=2}^{\infty}\sum_{j=1}^{s} \frac{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\Theta(n,\delta,\alpha,\beta)}{\varsigma(\sigma+(1-\omega))} c_j a_{n,j} \\ =&\sum_{j=1}^{s} c_j \sum_{n=2}^{\infty} \frac{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\Theta(n,\delta,\alpha,\beta)}{\varsigma(\sigma+(1-\omega))} a_{n,j} \\ \leq&\sum_{j=1}^{s} c_j = 1. \end{aligned}$$

Theorem 10 Let $u, g \in TS(\omega,\sigma,\varsigma)$. Then

$$h(z) = z - \sum_{n=2}^{\infty} (a_n^2 + b_n^2) z^n \in TS(\omega,\sigma,\varsigma), where$$

$$\zeta \geq \frac{2\omega(n-1)\varsigma^2(\sigma+(1-\omega))}{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]^2\Theta(n,\delta,\alpha,\beta) - 2\varsigma^2(\sigma+(1-\omega))(n\sigma+1-\omega)}.$$

Proof. Since $u, g \in TS(\omega,\sigma,\varsigma)$, so Theorem2 yields

$$\sum_{n=2}^{\infty} \left[\frac{(\omega(n-1)+\varsigma(n\sigma+1-\omega))\Theta(n,\delta,\alpha,\beta)}{\varsigma(\sigma+(1-\omega))} a_n\right]^2 \leq 1$$

and

$$\sum_{n=2}^{\infty} \left[\frac{(\omega(n-1)+\varsigma(n\sigma+1-\omega))\Theta(n,\delta,\alpha,\beta)}{\varsigma(\sigma+(1-\omega))} b_n\right]^2 \leq 1.$$

We obtain from the last two inequalities

$$\sum_{n=2}^{\infty}\frac{1}{2}\left[\frac{(\omega(n-1)+\varsigma(n\sigma+1-\omega))\Theta(n,\delta,\alpha,\beta)}{\varsigma(\sigma+(1-\omega))}\right]^2(a_n^2+b_n^2)\leq 1. \tag{29}$$

But $h(z)\in TS(\omega,\sigma,\zeta,q,m)$, if and only if

$$\sum_{n=2}^{\infty}\frac{[\omega(n-1)+\zeta(n\sigma+1-\omega)]\Theta(n,\delta,\alpha,\beta)}{\zeta(\sigma+(1-\omega))}(a_n^2+b_n^2)\leq 1, \tag{30}$$

where $0<\zeta<1$, however (29) implies (30) if

$$\frac{[\omega(n-1)+\zeta(n\sigma+1-\omega)]\Theta(n,\delta,\alpha,\beta)}{\zeta(\sigma+(1-\omega))} \leq \frac{1}{2}\left[\frac{(\omega(n-1)+\varsigma(n\sigma+1-\omega))\Theta(n,\delta,\alpha,\beta)}{\varsigma(\sigma+(1-\omega))}\right]^2.$$

Simplifying, we get

$$\zeta\geq\frac{2\omega(n-1)\varsigma^2(\sigma+(1-\omega))}{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]^2\Theta(n,\delta,\alpha,\beta)-2\varsigma^2(\sigma+(1-\omega))(n\sigma+1-\omega)}.$$

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