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SOME REMARKS ON THE CENTRAL INDEX BASED GROWTH PROPERTIES OF ENTIRE FUNCTION

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Abstract. In complex analysis, several research works have been done using the concepts of different growth indicators of entire functions such as order, lower order etc. In past decades, the study of the growth properties regarding entire function has usually been done using the concepts of maximum modulus and maximum term. On the other hand, Biswas (J. Korean Soc. Math. Educ. Ser. B Pure Appl. Math., 25(3) (2018), pp. 193-201 and Uzbek Math. J., 2018 (2), (2018), pp. 153-160) has initiated to study some growth properties of composite entire functions using the concepts of central index. Considering the idea, here in this paper, we have compared the central index of the composition of two entire functions with their corresponding left and right factors. Mainly, we have established some results related to the growth rates of composite entire functions on the basis of central index using the ideas of $(p,q,t)L$-th order, $(p,q,t)L$-th type and $(p,q,t)L$-th weak type.

1. Introduction, Definitions and Notations

Let $f(z)$ be an entire function defined in the open complex plane $\mathbb{C}$. For entire function $f(z)=\sum\limits_{n=0}^{\infty}a_n z^n$ on $|z|=r$, the maximum modulus function $M_f(r)$, the maximum term function $\mu_f(r)$ and the central index $\nu_f(r)$ are respectively defined as $\max\limits_{|z|=r}|f(z)|$, $\max\limits_{n\geq 0}(|a_n|r^n)$ and $\max\{m:\mu_f(r)=|a_m|r^m\}$. Clearly, central index $\nu_f(r)$ of an entire function $f(z)$ is the greatest exponent m such that $|a_m|r^m=\mu_f(r)$. All the functions $M_f(r)$, $\mu_f(r)$ and $\nu_f(r)$ are real and increasing. Taking another entire function $g(z)$, the ratios $\frac{M_f(r)}{M_g(r)}$ as $r\to+\infty$ is called the comparative growth of $f(z)$ with respect to $g(z)$ in terms of the maximum moduli. Similarly, the ratios $\frac{\mu_f(r)}{\mu_g(r)}$ as $r\to+\infty$ is called the comparative growth of $f(z)$

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with respect to $g(z)$ with the maximum terms. Though the central index $\nu_f(r)$ is much weaker than $M_f(r)$ and $\mu_f(r)$ in some sense, from another point of view, the limiting value of the ratio $\frac{\nu_f(r)}{\nu_g(r)}$ as $r \to +\infty$, is also called the growth of $f(z)$ with respect to $g(z)$ in terms of the central index. Considering these, in this paper, we compare the central index of composition of two entire functions with their corresponding left and right factors.

To start the paper, we first recall the the following definitions:

Definition 1.1. *The order $\rho(f)$ and the lower order $\lambda(f)$ of an entire function $f(z)$ are defined as:*

$$\rho(f) = \limsup_{r\to+\infty} \frac{\log\log M_f(r)}{\log r} \text{ and } \lambda(f) = \liminf_{r\to+\infty} \frac{\log\log M_f(r)}{\log r}.$$

Later, He et al. [8] gave an alternative definition of order and lower order of an entire function $f(z)$ in terms of its central index in the following way:

$$\rho(f) = \limsup_{r\to+\infty} \frac{\log \nu_f(r)}{\log r} \text{ and } \lambda(f) = \liminf_{r\to+\infty} \frac{\log \nu_f(r)}{\log r}.$$

However, Biswas [4, 5] used central index to establish some results in the growth properties of entire functions. After these works, we have investigated some results using the central index in growth of the the composition of two entire functions.

On the other hand, Shen et al. [13] defined the (m,n)-φ order and (m,n)-φ lower order of entire function $f(z)$ which are as follows:

Definition 1.2. [13] *Let $\varphi : [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function and $m \geq n$. The (m,n)-φ order $\rho^{(m,n)}(f,\varphi)$ and (m,n)-φ lower order $\lambda^{(m,n)}(f,\varphi)$ of entire function $f(z)$ are defined as:*

$$\begin{aligned} \rho^{(m,n)}(f,\varphi) &= \limsup_{r\to+\infty} \frac{\log^{[m+1]} M_f(r)}{\log^{[n]} \varphi(r)} \\ \text{and } \lambda^{(m,n)}(f,\varphi) &= \liminf_{r\to+\infty} \frac{\log^{[m+1]} M_f(r)}{\log^{[n]} \varphi(r)}. \end{aligned}$$

If we take $m = p$, $n = 1$ and $\varphi(r) = \log^{[q-1]} r \cdot \exp^{[t+1]} L(r)$, where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e., $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant 'a' i.e., $\lim_{r\to+\infty} \frac{L(ar)}{L(r)} = 1$, then Definition 1.2 turns into the definitions of $(p,q,t)L$-th order and $(p,q,t)L$-th lower order of an entire function $f(z)$ (for details, see [3]) which are as follows:

$$\begin{aligned} \rho^{(p,q,t)L}(f) &= \limsup_{r\to+\infty} \frac{\log^{[p+1]} M_f(r)}{\log^{[q]} r + \exp^{[t]} L(r)} \\ \text{and } \lambda^{(p,q,t)L}(f) &= \liminf_{r\to+\infty} \frac{\log^{[p+1]} M_f(r)}{\log^{[q]} r + \exp^{[t]} L(r)}. \end{aligned}$$

Further, Shen et al. [13] also established the equivalence of the definition of (m,n)-φ order of entire function in terms of maximum modulus and central index under some conditions. For details about it, one may see [13]. In view of Lemma 3.4 of [13] and Definition 1.2, one may write the following Definition.

Definition 1.3. [1] *Let $f(z)$ be an entire function and $\nu_f(r)$ be the central index of $f(z)$, then*

$$\rho^{(p,q,t)L}(f) = \limsup_{r\to+\infty} \frac{\log^{[p]} \nu_f(r)}{\log^{[q]} r + \exp^{[t]} L(r)}$$

and

$$\lambda^{(p,q,t)L}(f) = \liminf_{r\to+\infty} \frac{\log^{[p]} \nu_f(r)}{\log^{[q]} r + \exp^{[t]} L(r)}.$$

In order to compare the relative growth of two entire functions having same non zero finite $(p,q,t)L$-th order, Biswas [3] have introduced the definitions of $(p,q,t)L$-th type (respectively $(p,q,t)L$-th lower type) of entire function having finite positive $(p,q,t)L$-th order in the following manner:

Definition 1.4. [3] *Let $f(z)$ be an entire function with non-zero finite $(p,q,t)L$-th order $\rho^{(p,q,t)L}(f)$. The $(p,q,t)L$-th type denoted by $\sigma^{(p,q,t)L}(f)$ and $(p,q,t)L$-th lower type denoted by $\overline{\sigma}^{(p,q,t)L}(f)$ are respectively defined as follows:*

$$\sigma^{(p,q,t)L}(f) = \limsup_{r\to+\infty} \frac{\log^{[p]} M_f(r)}{[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)]^{\rho^{(p,q,t)L}(f)}}$$

and

$$\overline{\sigma}^{(p,q,t)L}(f) = \liminf_{r\to+\infty} \frac{\log^{[p]} M_f(r)}{[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)]^{\rho^{(p,q,t)L}(f)}}.$$

Considering the above, here in this present paper, we attempt to prove some results related to the growth rates of composite entire functions on the basis of central index using the ideas of $(p,q,t)L$-th order and $(p,q,t)L$-th type of an entire function. In fact, some works in this field using central index have been already explored in [1, 2, 4, 5, 11, 12]. We have used the standard notations using the theory of entire functions which are available in [10].

2. Main results

In this section, first we present some lemmas which will be needed in the sequel.

Lemma 2.1. [7] *Let $f(z)$ and $g(z)$ be any two entire functions with $g(0) = 0$. Also let β satisfy $0 < \beta < 1$ and $c(\beta) = \frac{(1-\beta)^2}{4\beta}$. Then for all sufficiently large values of r,*

$$M_f(c(\beta) M_g(\beta r)) \leq M_{f\circ g}(r) \leq M_f(M_g(r)).$$

Lemma 2.2. *([8], Theorems 1.9 and 1.10, or [9], Satz 4.3 and 4.4]) Let $f(z)$ be any entire function, then*

$$\log \mu_f(r) = \log|a_0| + \int_0^r \frac{\nu_f(t)}{t} dt \text{ where } a_0 \neq 0,$$

and for $r < R$,

$$M_f(r) < \mu_f(r) \left\{ \nu_f(R) + \frac{R}{R-r} \right\}.$$

Now we present the main results of the paper.

Theorem 2.1. *Let $f(z)$ and $g(z)$ be any two entire functions such that $\rho^{(m,n,t)L}(g) < +\infty$ and $0 < \rho^{(m,n,t)L}(f)$. Then for $m > q$ and for any constant E,*

$$\liminf_{r\to+\infty}\frac{\log^{[p+m-q+1]}\nu_{f\circ g}(r)}{\log^{[m]}\nu_f(2r)+\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]}\leq\frac{\rho^{(m,n,t)L}(g)}{\rho^{(m,n,t)L}(f)}$$

where $\exp^{[t]}[L(\exp(\nu_g(2r)\log(e\cdot r)+E))] = o\{\log^{[m]}\nu_f(r)\}$ *as* $r\to+\infty$.

Proof. In view of the first part of Lemma 2.2, we may obtain that

$$\log\mu_f(2r) = \log|a_0| + \int\limits_0^{2r}\frac{\nu_f(t)}{t}dt \geq \log|a_0| + \nu_f(r)\log 2 \ \ \{cf.\ [6]\ \}. \quad (1)$$

Also from Cauchy's inequality, it is clear that

$$\mu_f(r)\leq M_f(r) \ \ \{cf.\ [14]\ \}. \quad (2)$$

So for any constant K, one may obtain from (1) and (2) that

$$\nu_f(r)\log 2\leq\log M_f(2r)+K \ \ \{cf.\ [6]\ \}. \quad (3)$$

Therefore in view of (3) and the second part of Lemma 2.1, we obtain for all sufficiently large values of r that

$$\begin{aligned}\nu_{f\circ g}(r)\log 2 &\leq \log M_{f\circ g}(2r)+K\leq\log M_f(M_g(2r))+K,\\ i.e.,\ \log^{[p]}(\nu_{f\circ g}(r)\log 2) &\leq \log^{[p]}(\log M_f(M_g(2r))+K),\\ i.e.,\ \log^{[p]}\nu_{f\circ g}(r) &\leq \log^{[p+1]}M_f(M_g(2r))+O(1), \quad (4)\end{aligned}$$

$$i.e.,\ \log^{[p]}\nu_{f\circ g}(r)\leq$$
$$(\rho^{(p,q,t)L}(f)+\varepsilon)[\log^{[q]}M_g(2r)+\exp^{[t]}L(M_g(2r))]+O(1). \quad (5)$$

Further for any constant E, one may get from Lemma 2.2, that

$$\log M_g(r)<\nu_g(r)\log r+\log\nu_g(2r)+E \ \ \{cf.\ [6]\ \}.$$

Therefore from above we obtain that

$$\begin{aligned}\log M_g(r) &< \nu_g(2r)\log r+\nu_g(2r)+E,\\ i.e.,\ \log M_g(r) &< \nu_g(2r)(1+\log r)+E,\\ i.e.,\ M_g(2r) &< \exp[\nu_g(4r)\log(e\cdot 2r)+E]. \quad (6)\end{aligned}$$

From (5) and (6) for all sufficiently large values of r,we have

$$\begin{aligned}\log^{[p]}\nu_{f\circ g}(r) \leq\ & (\rho^{(p,q,t)L}(f)+\varepsilon)\cdot\log^{[q]}M_g(2r)+(\rho^{(p,q,t)L}(f)+\varepsilon)\\ &\times[\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]]+O(1),\end{aligned}$$

$$i.e., \log^{[p]}\nu_{f\circ g}(r)\leq$$
$$(\rho^{(p,q,t)L}(f)+\varepsilon)\cdot\log^{[q]}M_g(2r)$$
$$\times\Big[1+\frac{(\rho^{(p,q,t)L}(f)+\varepsilon)\cdot[\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]]+O(1)}{(\rho^{(p,q,t)L}(f)+\varepsilon)\cdot\log^{[q]}M_g(2r)}\Big],$$

$$\begin{aligned}i.e., \log^{[p]}\nu_{f\circ g}(r) \leq\ & (\rho^{(p,q,t)L}(f)+\varepsilon)\cdot\log^{[q]}M_g(2r)\\ &\times\Big[1+\frac{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]+O(1)}{\log^{[q]}M_g(2r)}\Big],\end{aligned}$$

$$i.e., \log^{[p+1]} \nu_{f\circ g}(r) \leq \log(\rho^{(p,q,t)L}(f)+\varepsilon) + \log^{[q+1]} M_g(2r) + \log\Big[1+\frac{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]+O(1)}{\log^{[q]} M_g(2r)}\Big].$$

Taking $\log\left(1+\frac{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]+O(1)}{\log^{[q]} M_g(2r)}\right)$ $\sim \frac{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]+O(1)}{\log^{[q]} M_g(2r)}$, we get for all sufficiently large values of r,

$$\log^{[p+1]} \nu_{f\circ g}(r) \leq \log^{[q+1]} M_g(2r) + \log(\rho^{(p,q,t)L}(f)+\varepsilon) + \frac{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]+O(1)}{\log^{[q]} M_g(2r)},$$

$$i.e.,\ \log^{[p+1]} \nu_{f\circ g}(r) \leq \log^{[q+1]} M_g(2r) \times \Big[1+\frac{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]}{\log^{[q]} M_g(2r)\cdot \log^{[q+1]} M_g(2r)} + \frac{O(1)+\log^{[q]} M_g(2r)\cdot\log(\rho^{(p,q,t)L}(f)+\varepsilon)}{\log^{[q]} M_g(2r)\cdot \log^{[q+1]} M_g(2r)}\Big],$$

$$i.e.,\ \log^{[p+2]} \nu_{f\circ g}(r) \leq \log^{[q+2]} M_g(2r) + \log\Big[1+\frac{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]}{\prod_{k=q}^{q+1}\log^{[k]} M_g(2r)} + \frac{O(1)+\log^{[q]} M_g(2r)\cdot\log(\rho^{(p,q,t)L}(f)+\varepsilon)}{\prod_{k=q}^{q+1}\log^{[k]} M_g(2r)}\Big].$$

Again using $\log(1+x)\sim x$ for

$$x = \frac{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]}{\prod_{k=q}^{q+1}\log^{[k]} M_g(2r)} + \frac{O(1)+\log^{[q]} M_g(2r)\cdot\log(\rho^{(p,q,t)L}(f)+\varepsilon)}{\prod_{k=q}^{q+1}\log^{[k]} M_g(2r)},$$

we get from above for all sufficiently large positive values of r,

$$\log^{[p+2]} \nu_{f\circ g}(r) \leq \log^{[q+2]} M_g(2r) + \frac{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]}{\prod_{k=q}^{q+1}\log^{[k]} M_g(2r)} + \frac{O(1)+\log^{[q]} M_g(2r)\cdot\log(\rho^{(p,q,t)L}(f)+\varepsilon)}{\prod_{k=q}^{q+1}\log^{[k]} M_g(2r)}.$$

Continuing this process, we get

$$\begin{aligned}\log^{[p+m-q+1]}\nu_{f\circ g}(r) &\leq \log^{[q+m-q+1]}M_g(2r)\\ &+\frac{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]}{\prod_{k=q}^{q+m-q}\log^{[k]}M_g(2r)}\\ &+\frac{O(1)+\log^{[q]}M_g(2r)\cdot\log(\rho^{(p,q,t)L}(f)+\varepsilon)}{\prod_{k=q}^{q+m-q}\log^{[k]}M_g(2r)},\end{aligned}$$

$$\begin{aligned}i.e., \log^{[p+m-q+1]}\nu_{f\circ g}(r) &\leq \log^{[m+1]}M_g(2r)\\ &+\frac{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]}{\prod_{k=q}^{m}\log^{[k]}M_g(2r)}\\ &+\frac{O(1)+\log^{[q]}M_g(2r)\cdot\log(\rho^{(p,q,t)L}(f)+\varepsilon)}{\prod_{k=q}^{m}\log^{[k]}M_g(2r)},\end{aligned}$$

$$\begin{aligned}i.e.,\ \log^{[p+m-q+1]}\nu_{f\circ g}(r) &\leq (\rho^{(m,n,t)L}(g)+\varepsilon)[\log^{[n]}2r+\exp^{[t]}L(2r)]\\ &+\frac{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]}{\prod_{k=q}^{m}\log^{[k]}M_g(2r)}\\ &+\frac{O(1)+\log^{[q]}M_g(2r)\cdot\log(\rho^{(p,q,t)L}(f)+\varepsilon)}{\prod_{k=q}^{m}\log^{[k]}M_g(2r)} \quad (7)\end{aligned}$$

Again we have for a sequence of values of r tending to infinity that

$$\log^{[m]}\nu_f(r)\geq(\rho^{(m,n,t)L}(f)-\varepsilon)[\log^{[n]}r+\exp^{[t]}L(r)],$$

$$i.e.,\ \log^{[n]}r+\exp^{[t]}L(r)\leq\frac{\log^{[m]}\nu_f(r)}{(\rho^{(m,n,t)L}(f)-\varepsilon)}. \quad (8)$$

Hence from (7) and (8), it follows for a sequence of values of r tending to infinity that

$$\begin{aligned}\log^{[p+m-q+1]}\nu_{f\circ g}(r) &\leq \left(\frac{\rho^{(m,n,t)L}(g)+\varepsilon}{\rho^{(m,n,t)L}(f)-\varepsilon}\right)\cdot\log^{[m]}\nu_f(2r)\\ &+\frac{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]}{\prod_{k=q}^{m}\log^{[k]}M_g(2r)}\\ &+\frac{O(1)+\log^{[q]}M_g(2r)\cdot\log(\rho^{(p,q,t)L}(f)+\varepsilon)}{\prod_{k=q}^{m}\log^{[k]}M_g(2r)},\end{aligned}$$

$$i.e,\ \frac{\log^{[p+m-q+1]}\nu_{f\circ g}(r)}{\log^{[m]}\nu_f(2r)+\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]}\leq$$

$$\frac{\frac{\rho^{(m,n,t)L}(g)+\varepsilon}{\rho^{(m,n,t)L}(f)-\varepsilon}}{1+\frac{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]}{\log^{[m]}\nu_f(2r)}}$$

$$+\frac{1+\frac{O(1)+\log^{[q]}M_g(2r)\cdot\log(\rho^{(p,q,t)L}(f)+\varepsilon)}{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]}}{[1+\frac{\log^{[m]}\nu_f(2r)}{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]}]\cdot\prod\limits_{k=q}^{m}\log^{[k]}M_g(2r)},$$

$$i.e,\ \frac{\log^{[p+m-q+1]}\nu_{f\circ g}(r)}{\log^{[m]}\nu_f(2r)+\exp^{[t]}[L(\exp(\nu_g\,(4r)\log\,(e\cdot 2r)+E))]}$$

$$\leq\ \frac{\frac{\rho^{(m,n,t)L}(g)+\varepsilon}{\rho^{(m,n,t)L}(f)-\varepsilon}}{1+\frac{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]}{\log^{[m]}\nu_f(2r)}}$$

$$+\frac{\frac{1}{\prod\limits_{k=q}^{m}\log^{[k]}M_g(2r)}+\frac{O(1)}{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]\cdot\prod\limits_{k=q}^{m}\log^{[k]}M_g(2r)}}{[1+\frac{\log^{[m]}\nu_f(2r)}{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]}]}$$

$$+\frac{\frac{\log(\rho^{(p,q,t)L}(f)+\varepsilon)}{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]\cdot\prod\limits_{k=q+1}^{m}\log^{[k]}M_g(2r)}}{[1+\frac{\log^{[m]}\nu_f(2r)}{\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]}]}.\quad (9)$$

Since $\exp^{[t]}[L(\exp(\nu_g\,(2r)\log\,(e\cdot r)+E))]=o\{\log^{[m]}\nu_f(r)\}$ as $r\to+\infty$ and $\varepsilon(>0)$ is arbitrary, we obtain from (9) that

$$\liminf_{r\to+\infty}\frac{\log^{[p+m-q+1]}\nu_{f\circ g}(r)}{\log^{[m]}\nu_f(2r)+\exp^{[t]}[L(\exp[(\nu_g\,(4r)\log\,(e\cdot 2r)+E)]}\leq\frac{\rho^{(m,n,t)L}(g)}{\rho^{(m,n,t)L}(f)}.$$

Thus the theorem is established. □

Now one can easily state the following theorems without their proofs as those can be carried out in the line of Theorem 2.1.

Theorem 2.2. *Let $f(z)$ and $g(z)$ be any two entire functions such that $\lambda^{(m,n,t)L}(g)<+\infty$ and $0<\lambda^{(m,n,t)L}(f)$. Then for $m>q$ and for any constant E,*

$$\liminf_{r\to+\infty}\frac{\log^{[p+m-q+1]}\nu_{f\circ g}(r)}{\log^{[m]}\nu_f(2r)+\exp^{[t]}[L(\exp(\nu_g\,(4r)\log\,(e\cdot 2r)+E))]}\leq\frac{\lambda^{(m,n,t)L}(g)}{\lambda^{(m,n,t)L}(f)},$$

where $\exp^{[t]}[L(\exp(\nu_g\,(2r)\log\,(e\cdot r)+E))]=o\{\log^{[m]}\nu_f(r)\}$ as $r\to+\infty$.

Theorem 2.3. *Let $f(z)$ and $g(z)$ be any two entire functions such that $0<\lambda^{(m,n,t)L}(g)\leq\rho^{(m,n,t)L}(g)<+\infty$. Then for $m>q$ and for any constant E,*

$$\liminf_{r\to+\infty}\frac{\log^{[p+m-q+1]}\nu_{f\circ g}(r)}{\log^{[m]}\nu_g(2r)+\exp^{[t]}[L(\exp[(\nu_g\,(4r)\log\,(e\cdot 2r)+E)]}\leq 1,$$

where $\exp^{[t]}[L(\exp(\nu_g\,(2r)\log\,(e\cdot r)+E))]=o\{\log^{[m]}\nu_g(r)\}$ as $r\to+\infty$.

Theorem 2.4. *Let $f(z)$ and $g(z)$ be any two entire functions such that $\lambda^{(p,q,t)L}(g)>0$, $0<\lambda^{(p,q,t)L}(f)<+\infty$ and $\sigma^{(m,n,t)L}(g)<+\infty$ where $m<q$. Then*

$$\liminf_{r\to+\infty}\frac{\log^{[p]}\nu_{f\circ g}(r)}{\log^{[p]}\nu_g(\exp^{[q]}[\log^{[n-1]}2r\cdot\exp^{[t+1]}L(2r)]^{\rho^{(m,n,t)L}(g)})}$$

$$\leq \frac{\lambda^{(p,q,t)L}(f)\cdot\sigma^{(m,n,t)L}(g)}{\lambda^{(p,q,t)L}(g)},$$

where $\exp^{[t]}[L(\exp[\nu_g(2r)\log(e\cdot r)+E])]=o([\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\alpha})$ *as* $r\to+\infty$*, for some* $\alpha<\rho^{(m,n,t)L}(g)$ *and any constant* E*.*

Proof. Since $0<\lambda^{(p,q,t)L}(f)<+\infty$, then it follows from (4)and (6) for a sequence of values of r tending to infinity that

$$\log^{[p]}\nu_{f\circ g}(r) \leq (\lambda^{(p,q,t)L}(f)+\varepsilon)\Big[\log^{[q]}M_g(2r) + \exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]\Big]+O(1),$$

$$i.e.,\ \log^{[p]}\nu_{f\circ g}(r)\leq(\lambda^{(p,q,t)L}(f)+\varepsilon)\Big[\log^{[m]}M_g(2r) + \exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]\Big]+O(1),$$

$$i.e.,\ \log^{[p]}\nu_{f\circ g}(r)\leq(\lambda^{(p,q,t)L}(f)+\varepsilon) \times\Big[(\sigma^{(m,n,t)L}(g)+\varepsilon)[\log^{[n-1]}2r\cdot\exp^{[t+1]}L(2r)]^{\rho^{(m,n,t)L}(g)} + \exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]\Big]+O(1)\ . \tag{10}$$

Also, we obtain for all sufficiently large values of r that

$$\log^{[p]}\nu_g(\exp^{[q]}[\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho^{(m,n,t)L}(g)}) \geq(\lambda^{(p,q,t)L}(g)-\varepsilon)[\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho^{(m,n,t)L}(g)} +(\lambda^{(p,q,t)L}(g)-\varepsilon)\exp^{[t]}[L(\exp^{[q]}[\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho^{(m,n,t)L}(g)})],$$

$$i.e.,\ \log^{[p]}\nu_g(\exp^{[q]}[\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho^{(m,n,t)L}(g)}) > (\lambda^{(p,q,t)L}(g)-\varepsilon)[\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho^{(m,n,t)L}(g)}.$$

Now from (10) and above it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]}\nu_{f\circ g}(r)}{\log^{[p]}\nu_g(\exp^{[q]}[\log^{[n-1]}2r\cdot\exp^{[t+1]}L(2r)]^{\rho^{(m,n,t)L}(g)})} \leq\frac{(\lambda^{(p,q,t)L}(f)+\varepsilon)[(\sigma^{(m,n,t)L}(g)+\varepsilon)[\log^{[n-1]}2r\cdot\exp^{[t+1]}L(2r)]^{\rho^{(m,n,t)L}(g)}}{(\lambda^{(p,q,t)L}(g)-\varepsilon)[\log^{[n-1]}2r\cdot\exp^{[t+1]}L(2r)]^{\rho^{(m,n,t)L}(g)}} +\frac{(\lambda^{(p,q,t)L}(f)+\varepsilon)\exp^{[t]}[L(\exp(\nu_g(4r)\log(e\cdot 2r)+E))]}{(\lambda^{(p,q,t)L}(g)-\varepsilon)[\log^{[n-1]}2r\cdot\exp^{[t+1]}L(2r)]^{\rho^{(m,n,t)L}(g)}} +\frac{O(1)}{(\lambda^{(p,q,t)L}(g)-\varepsilon)[\log^{[n-1]}2r\cdot\exp^{[t+1]}L(2r)]^{\rho^{(m,n,t)L}(g)}}. \tag{11}$$

As $\alpha<\rho^{(m,n,t)L}(g)$ and

$$\exp^{[t]}[L(\exp[\nu_g(2r)\log(e\cdot r)+E])]=o([\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\alpha})$$

as $r\to+\infty$, we obtain that

$$\lim_{r\to+\infty}\frac{\exp^{[t]}[L(\exp[(\nu_g(4r)\log(e\cdot 2r)+E)]}{[\log^{[n-1]}2r\cdot\exp^{[t+1]}L(2r)]^{\rho^{(m,n,t)L}(g)}}=0\ . \tag{12}$$

Since $\varepsilon(>0)$ is arbitrary, it follows from (11) and (12) that

$$\liminf_{r\to+\infty}\frac{\log^{[p]}\nu_{f\circ g}(r)}{\log^{[p]}\nu_g(\exp^{[q]}[\log^{[n-1]}2r\cdot\exp^{[t+1]}L(2r)]^{\rho^{(m,n,t)L}(g)})}\leq\frac{\lambda^{(p,q,t)L}(f)\cdot\sigma^{(m,n,t)L}(g)}{\lambda^{(p,q,t)L}(g)}.$$

Thus the theorem follows. □

In the line of Theorem 2.4, the following theorems can be easily carried out and therefore their proofs are omitted:

Theorem 2.5. *Let $f(z)$ and $g(z)$ be any two entire functions such that $\rho^{(p,q,t)L}(g)>0$, $0<\rho^{(p,q,t)L}(f)<+\infty$ and $\sigma^{(m,n,t)L}(g)<+\infty$ where $m<q$. Then*

$$\liminf_{r\to+\infty}\frac{\log^{[p]}\nu_{f\circ g}(r)}{\log^{[p]}\nu_g(\exp^{[q]}[\log^{[n-1]}2r\cdot\exp^{[t+1]}L(2r)]^{\rho^{(m,n,t)L}(g)})}\leq\frac{\rho^{(p,q,t)L}(f)\cdot\sigma^{(m,n,t)L}(g)}{\rho^{(p,q,t)L}(g)},$$

where $\exp^{[t]}[L(\exp[\nu_g(2r)\log(e\cdot r)+E])]=o([\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\alpha})$ as $r\to+\infty$, for some $\alpha<\rho^{(m,n,t)L}(g)$ and any constant E.

Theorem 2.6. *Let $f(z)$ and $g(z)$ be any two entire functions such that $0<\lambda^{(p,q,t)L}(f)\leq\rho^{(p,q,t)L}(f)<+\infty$ and $\sigma^{(m,n,t)L}(g)<+\infty$ where $m<q$. Then*

$$\liminf_{r\to+\infty}\frac{\log^{[p]}\nu_{f\circ g}(r)}{\log^{[p]}\nu_f(\exp^{[q]}[\log^{[n-1]}2r\cdot\exp^{[t+1]}L(2r)]^{\rho^{(m,n,t)L}(g)})}\leq\sigma^{(m,n,t)L}(g),$$

where $\exp^{[t]}[L(\exp[\nu_g(2r)\log(e\cdot r)+E])]=o([\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\alpha})$ as $r\to+\infty$, for some $\alpha<\rho^{(m,n,t)L}(g)$ and any constant E.

Theorem 2.7. *Let $f(z)$ and $g(z)$ be any two entire functions such that $0<\lambda^{(p,q,t)L}(f)\leq\rho^{(p,q,t)L}(f)<+\infty$ and $\overline{\sigma}^{(m,n,t)L}(g)<+\infty$ where $m<q$. Then*

$$\liminf_{r\to+\infty}\frac{\log^{[p]}\nu_{f\circ g}(r)}{\log^{[p]}\nu_f(\exp^{[q]}[\log^{[n-1]}2r\cdot\exp^{[t+1]}L(2r)]^{\rho^{(m,n,t)L}(g)})}\leq\frac{\rho^{(p,q,t)L}(f)\cdot\overline{\sigma}^{(m,n,t)L}(g)}{\lambda^{(p,q,t)L}(f)},$$

where $\exp^{[t]}[L(\exp[\nu_g(2r)\log(e\cdot r)+E])]=o([\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\alpha})$ as $r\to+\infty$, for some $\alpha<\rho^{(m,n,t)L}(g)$ and any constant E.

Theorem 2.8. *Let $f(z)$ and $g(z)$ be any two entire functions such that $\lambda^{(p,q,t)L}(g)>0$, $0<\rho^{(p,q,t)L}(f)<+\infty$ and $\overline{\sigma}^{(m,n,t)L}(g)<+\infty$ where $m<q$. Then*

$$\liminf_{r\to+\infty}\frac{\log^{[p]}\nu_{f\circ g}(r)}{\log^{[p]}\nu_g(\exp^{[q]}[\log^{[n-1]}2r\cdot\exp^{[t+1]}L(2r)]^{\rho^{(m,n,t)L}(g)})}\leq\frac{\rho^{(p,q,t)L}(f)\cdot\overline{\sigma}^{(m,n,t)L}(g)}{\lambda^{(p,q,t)L}(g)},$$

where $\exp^{[t]}[L(\exp[\nu_g(2r)\log(e\cdot r)+E])]=o([\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\alpha})$ as $r\to+\infty$, for some $\alpha<\rho^{(m,n,t)L}(g)$ and any constant E.

Theorem 2.9. *Let $f(z)$ and $g(z)$ be any two entire functions such that $\lambda^{(p,q,t)L}(g) > 0$, $0 < \lambda^{(p,q,t)L}(f) < +\infty$ and $\overline{\tau}^{(m,n,t)L}(g) < +\infty$ where $m < q$. Then*

$$\liminf_{r\to+\infty} \frac{\log^{[p]} \nu_{f\circ g}(r)}{\log^{[p]} \nu_g(\exp^{[q]}[\log^{[n-1]} 2r \cdot \exp^{[t+1]} L(2r)]^{\lambda^{(m,n,t)L}(g)})} \leq \frac{\lambda^{(p,q,t)L}(f) \cdot \overline{\tau}^{(m,n,t)L}(g)}{\lambda^{(p,q,t)L}(g)},$$

where $\exp^{[t]}[L(\exp[\nu_g(2r)\log(e\cdot r)+E])] = o([\log^{[n-1]} r \cdot \exp^{[t+1]} L(r)]^{\alpha})$ *as* $r \to +\infty$, *for some* $\alpha < \lambda^{(m,n,t)L}(g)$ *and any constant* E.

Theorem 2.10. *Let $f(z)$ and $g(z)$ be any two entire functions such that $\rho^{(p,q,t)L}(g) > 0$, $0 < \rho^{(p,q,t)L}(f) < +\infty$ and $\overline{\tau}^{(m,n,t)L}(g) < +\infty$ where $m < q$. Then*

$$\liminf_{r\to+\infty} \frac{\log^{[p]} \nu_{f\circ g}(r)}{\log^{[p]} \nu_g(\exp^{[q]}[\log^{[n-1]} 2r \cdot \exp^{[t+1]} L(2r)]^{\lambda^{(m,n,t)L}(g)})} \leq \frac{\rho^{(p,q,t)L}(f) \cdot \overline{\tau}^{(m,n,t)L}(g)}{\rho^{(p,q,t)L}(g)},$$

where $\exp^{[t]}[L(\exp[\nu_g(2r)\log(e\cdot r)+E])] = o([\log^{[n-1]} r \cdot \exp^{[t+1]} L(r)]^{\alpha})$ *as* $r \to +\infty$, *for some* $\alpha < \lambda^{(m,n,t)L}(g)$ *and any constant* E.

Theorem 2.11. *Let $f(z)$ and $g(z)$ be any two entire functions such that $\lambda^{(p,q,t)L}(g) > 0$, $0 < \rho^{(p,q,t)L}(f) < +\infty$ and $\overline{\tau}^{(m,n,t)L}(g) < +\infty$ where $m < q$. Then*

$$\limsup_{r\to+\infty} \frac{\log^{[p]} \nu_{f\circ g}(r)}{\log^{[p]} \nu_g(\exp^{[q]}[\log^{[n-1]} r \cdot \exp^{[t+1]} L(r)]^{\lambda^{(m,n,t)L}(g)})} \leq \frac{\rho^{(p,q,t)L}(f) \cdot \overline{\tau}^{(m,n,t)L}(g)}{\lambda^{(p,q,t)L}(g)},$$

where $\exp^{[t]}[L(\exp[\nu_g(2r)\log(e\cdot r)+E])] = o([\log^{[n-1]} r \cdot \exp^{[t+1]} L(r)]^{\alpha})$ *as* $r \to +\infty$, *for some* $\alpha < \lambda^{(m,n,t)L}(g)$ *and any constant* E.

Theorem 2.12. *Let $f(z)$ and $g(z)$ be any two entire functions such that $0 < \lambda^{(p,q,t)L}(f) < \rho^{(p,q,t)L}(f) < +\infty$ and $\overline{\tau}^{(m,n,t)L}(g) < +\infty$ where $m < q$. Then*

$$\limsup_{r\to+\infty} \frac{\log^{[p]} \nu_{f\circ g}(r)}{\log^{[p]} \nu_f(\exp^{[q]}[\log^{[n-1]} 2r \cdot \exp^{[t+1]} L(2r)]^{\lambda^{(m,n,t)L}(g)})} \leq \frac{\rho^{(p,q,t)L}(f) \cdot \overline{\tau}^{(m,n,t)L}(g)}{\lambda^{(p,q,t)L}(f)},$$

where $\exp^{[t]}[L(\exp[\nu_g(2r)\log(e\cdot r)+E])] = o([\log^{[n-1]} r \cdot \exp^{[t+1]} L(r)]^{\alpha})$ *as* $r \to +\infty$, *for some* $\alpha < \lambda^{(m,n,t)L}(g)$ *and any constant* E.

Theorem 2.13. *Let $f(z)$ and $g(z)$ be any two entire functions such that $0 < \lambda^{(p,q,t)L}(f) \leq \rho^{(p,q,t)L}(f) < +\infty$ and $\overline{\tau}^{(m,n,t)L}(g) < +\infty$ where $m < q$. Then*

$$\liminf_{r\to+\infty} \frac{\log^{[p]} \nu_{f\circ g}(r)}{\log^{[p]} \nu_f(\exp^{[q]}[\log^{[n-1]} 2r \cdot \exp^{[t+1]} L(2r)]^{\lambda^{(m,n,t)L}(g)})} \leq \overline{\tau}^{(m,n,t)L}(g),$$

where $\exp^{[t]}[L(\exp[\nu_g(2r)\log(e\cdot r)+E])] = o([\log^{[n-1]} r \cdot \exp^{[t+1]} L(r)]^{\alpha})$ *as* $r \to +\infty$, *for some* $\alpha < \lambda^{(m,n,t)L}(g)$ *and any constant* E.

Theorem 2.14. *Let $f(z)$ and $g(z)$ be any two entire functions such that $\lambda^{(p,q,t)L}(g) > 0$, $0 < \rho^{(p,q,t)L}(f) < +\infty$ and $\tau^{(m,n,t)L}(g) < +\infty$ where $m < q$. Then*

$$\liminf_{r\to+\infty} \frac{\log^{[p]} \nu_{f\circ g}(r)}{\log^{[p]} \nu_g(\exp^{[q]}[\log^{[n-1]} 2r \cdot \exp^{[t+1]} L(2r)]^{\lambda^{(m,n,t)L}(g)})} \leq \frac{\rho^{(p,q,t)L}(f) \cdot \tau^{(m,n,t)L}(g)}{\lambda^{(p,q,t)L}(g)},$$

where $\exp^{[t]}[L(\exp[\nu_g(2r)\log(e\cdot r) + E])] = o([\log^{[n-1]} r \cdot \exp^{[t+1]} L(r)]^{\alpha})$ *as* $r \to +\infty$*, for some* $\alpha < \lambda^{(m,n,t)L}(g)$ *and any constant* E.

Theorem 2.15. *Let $f(z)$ and $g(z)$ be any two entire functions such that $0 < \lambda^{(p,q,t)L}(f) < \rho^{(p,q,t)L}(f) < +\infty$ and $\tau^{(m,n,t)L}(g) < +\infty$ where $m < q$. Then*

$$\liminf_{r\to+\infty} \frac{\log^{[p]} \nu_{f\circ g}(r)}{\log^{[p]} \nu_f(\exp^{[q]}[\log^{[n-1]} 2r \cdot \exp^{[t+1]} L(2r)]^{\lambda^{(m,n,t)L}(g)})} \leq \frac{\rho^{(p,q,t)L}(f) \cdot \tau^{(m,n,t)L}(g)}{\lambda^{(p,q,t)L}(f)},$$

where $\exp^{[t]}[L(\exp[\nu_g(2r)\log(e\cdot r) + E])] = o([\log^{[n-1]} r \cdot \exp^{[t+1]} L(r)]^{\alpha})$ *as* $r \to +\infty$*, for some* $\alpha < \lambda^{(m,n,t)L}(g)$ *and any constant* E.

3. Conclusion

The main aim of this paper is to establish several results regarding the growth of composite entire functions in terms of their central index. The study will provide a scope for further research in different growth measurements. The interested researchers may be motivated from this idea and they can try to investigate the growth results of entire function regarding the generalized order using the central index.

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