Journal of Fractional Calculus and Applications
Vol. 15(2) July. 2024, No. 4
ISSN: 2090-5858.
ISSN 2090-584X (print)
http://jfca.journals.ekb.eg/

NEW RESULTS ON IMPLICIT DELAY FRACTIONAL DIFFERENTIAL EQUATIONS WITH IMPULSIVE INTEGRAL BOUNDARY CONDITIONS

S. POORNIMA, P. KARTHIKEYAN

ABSTRACT. In this paper, we analyse the existence results for implicit fractional differential equations with impulsive delay integral boundary conditions. The sufficient conditions are established to prove the existence results by using the fixed point theorems such as Banach contraction principle and Schaefer's fixed point theorem. An application is illustrated through an example.

Fractional calculus is a natural extension of ordinary calculus, where integrals and derivatives are defined for arbitrary real orders. Since the development of fractional calculus in the 17th century, numerous derivatives have been developed, including Riemann-Liouville, Hadamard, Grunwald-Letnikov, Caputo, and others. There are numerous works devoted to various fractional operators because the selection of an appropriate fractional derivative (or integral) depends on the system under consideration, refer [9, 10, 13]. In [14], Prathumwan et al. has investigated the study of transmission dynamics of streptococcus suis infection mathematical model between Pig and Human under fractional derivative. Tul Ain et al. [17] has studied ABC fractional derivative for the Alcohol drinking model using Two-Scale fractal dimension.

Impulsive differential equations have played an important role in modeling phenomena. Since the last century, some authors have used impulsive differential systems to describe the model, particularly in describing dynamics of populations subject to abrupt changes as well as other phenomena like harvesting, diseases, and so forth. In [16], Renusumrit et al. has discussed about the existence results for impulsive fractional integro-differential equations involving the Atangana-Baleanu-Caputo derivative under integral boundary conditions. Wattanakejorn et al. [19] has examined the implicit fractional relaxation differential equations with impulsive delay boundary conditions. Equations involving fractional derivatives and time

2010 *Mathematics Subject Classification.* 34A08; 47H08; 34A38; 05B07; 34K20.
Key words and phrases. Fractional derivative; Impulsive; Delay; Integral boundary conditions.
Submitted March 16, 2024. Revised March 24, 2024.

delays are known as fractional delay differential equations. Unlike ordinary derivatives, fractional derivatives are non-local in nature and are capable of modeling memory effects whereas time delays express the history of an earlier state. In [11], Krim et al. has discussed Caputo-Hadamard implicit fractional differential equations with delay. When the boundary conditions are specified at two or more points on the independent variable, a boundary value problem is a higher-order differential equation or a group of differential equations. Numerous chemical engineering applications, including diffusion and reaction in catalysts, heat conduction in fins, and transport phenomena in boundary layers, all involve this issue, see [2, 3, 7]. We will employ the Atangana-Baleanu fractional operator in the Caputo sense to analyse the fractional dynamics of the provided model. The fractional derivatives of Atangana-Baleanu are used because of their nonlocal properties, see reference [1, 5, 12, 15, 18]. Gulalai et al. [6] analyzed a nonlinear modified KdV equation under Atangana Baleanu Caputo derivative.
In [7], Gul et al. examined the existence of the following boundary value problems under $\mathfrak{ABC}$ fractional derivative

$$ {}_0^{\mathcal{ABC}}D_t^{\phi}[\varkappa(\mathfrak{t}) - \mathfrak{V}(t,\varkappa(\mathfrak{t}))] = \mathfrak{W}(t,\varkappa(\mathfrak{t}),\ 0<\phi\leq 1,\ t\in[0,\mathcal{T}] = \mathfrak{J}^{'}, $$
$$ \varkappa(0) = \int_0^{\varrho} \frac{(\varrho-\nu)^{\phi-1}}{\Gamma(\phi)}\mathfrak{U}(\nu,\varkappa(\nu))d\nu, $$

where ${}_0^{\mathcal{ABC}}D_t^{\phi}$- is the AB-Caputo fractional derivative of order ϕ, $\mathfrak{V},\mathfrak{U},\mathfrak{W} : \mathfrak{J}^{'}\times\mathscr{R}\to\mathscr{R}$.
In [16], Reunsumrit et al. discussed the existence results for the following problem

$$ {}_0^{\mathcal{ABC}}D_t^{\phi}[\varkappa(t) - \mathfrak{U}(t,\varkappa(t))] = \mathfrak{V}(t,\varkappa(t),\mathfrak{L}\varkappa(t)),\ 0<\phi\leq 1,\ t\in[0,\mathcal{T}] = \mathfrak{J}^{'}, $$
$$ \Delta(\varkappa)\Big|_{t=t_k} = \mathfrak{I}_k(\varkappa(t_k^-)), $$
$$ \varkappa(0) = \int_0^{\varrho} \frac{(\varrho-\nu)^{\phi-1}}{\Gamma(\phi)}\mathfrak{S}(\nu,\varkappa(\nu))d\nu, $$

where ${}_0^{\mathcal{ABC}}D_t^{\phi}$- is the AB-Caputo fractional derivative of order ϕ, $\mathfrak{U},\mathfrak{S} : \mathfrak{J}^{'}\times\mathscr{R}\to\mathscr{R}$ and $\mathfrak{V}, g : \mathfrak{J}^{'}\times\mathscr{R}^2\to\mathscr{R}$ is continuous function. Where $\mathfrak{L}\varkappa(\mathfrak{t}) = \int_0^t g(t,x(t),\phi(\mathfrak{t}))dt$, and $\mathfrak{I}_k : \mathscr{R}\to\mathscr{R},\ k=1,2,...m.$ $0=\mathtt{t}_0<\mathtt{t}_1<\mathtt{t}_2<...<\mathtt{t}_\mathtt{m}=\mathcal{T}$, $\Delta\varkappa|_{\mathtt{t}=\mathtt{t}_\mathtt{k}} = \varkappa(\mathtt{t}_\mathtt{k}^+)-\varkappa(\mathtt{t}_\mathtt{k}^-)$, and $\varkappa(\mathtt{t}_\mathtt{k}^+) = \lim_{\mathtt{h}\to 0^+}\varkappa(\mathtt{t}_\mathtt{k}+\mathtt{h})$ and $\varkappa(\mathtt{t}_\mathtt{k}^-) = \lim_{\mathtt{h}\to 0^-}\varkappa(\mathtt{t}_\mathtt{k}+\mathtt{h})$ indicates the right and left hand limits of $\varkappa(\mathtt{t})$ at $\mathtt{t}=\mathtt{t}_\mathtt{k}$.

Prompted by the above works, consider the impulsive $\mathcal{ABC}$ fractional implicit differential equations with integral boundary conditions of the form:

$$ \begin{cases} {}_0^{\mathcal{ABC}}D_\mathfrak{t}^{\nu}[\mathfrak{x}(\mathfrak{t}) - \mathfrak{g}(\mathfrak{t},\mathfrak{x}_\mathfrak{t})] = \mathfrak{f}(\mathfrak{t},\mathfrak{x}_\mathfrak{t},{}_0^{\mathcal{ABC}}D_\mathfrak{t}^{\nu}),\ \mathfrak{t}\in[0,\mathfrak{T}] = \mathfrak{J},\ 0<\nu\leq 1, \\ \Delta(\mathfrak{x})\Big|_{\mathfrak{t}=\mathfrak{t}_\mathfrak{z}} = \mathfrak{I}_\mathfrak{z}(\mathfrak{x}_{\mathfrak{t}_\mathfrak{z}^-}) \\ \mathfrak{x}(\mathfrak{t}) = \varphi(\mathfrak{t}),\ \mathfrak{t}\in(-\infty,0] \\ \mathfrak{x}(0) = \int_0^{\mathfrak{T}} \frac{(\mathfrak{T}-\ell)^{\nu-1}}{\Gamma(\nu)}\mathfrak{h}(\ell,\mathfrak{x}_\ell)d\ell, \end{cases} \tag{1.1} $$

where ${}_0^{\mathcal{ABC}}D_\mathfrak{t}^{\nu}$- is the $\mathcal{ABC}$ fractional derivative of order ν, $\mathfrak{g},\mathfrak{h} : \mathfrak{J}\times R\to R$ and $\mathfrak{f} : \mathfrak{J}\times R^2\to R$ is continuous function. Where $\mathfrak{I}_\mathfrak{z} : R\to R,\ \mathfrak{z}=1,2,...m.$ $0=\mathfrak{t}_0<\mathfrak{t}_1<\mathfrak{t}_2<...<\mathfrak{t}_n=\mathfrak{T}$, $\Delta\mathfrak{x}|_\mathfrak{t} = \mathfrak{t}_\mathfrak{z} = \mathfrak{x}(\mathfrak{t}_\mathfrak{z}^+)-\mathfrak{x}(\mathfrak{t}_\mathfrak{z}^-)$, $\mathfrak{x}(\mathfrak{t}_\mathfrak{z}^-) = \lim_{\mathfrak{r}\to 0^-}\mathfrak{x}(\mathfrak{t}_\mathfrak{z}+\mathfrak{r})$ and

$\mathfrak{x}(\mathfrak{t}_{\mathfrak{z}}^{+}) = \lim_{\mathfrak{r}\to 0^{+}} \mathfrak{x}(\mathfrak{t}_{\mathfrak{z}} + \mathfrak{r})$ represent the left and right hand limits of $\mathfrak{x}(\mathfrak{t})$ at $\mathfrak{t} = \mathfrak{t}_{\mathfrak{z}}$. For any $\mathfrak{t} \in \mathfrak{J}$, we represent $\mathfrak{x}_{\mathfrak{t}}$ by

$$\mathfrak{x}_{\mathfrak{t}}(s) = \mathfrak{x}(\mathfrak{t} + s) \text{ and } -\infty < s \leq 0.$$

The contents of this paper is organized as follows. There are some basic definitions and lemmas in Section 2. Section 3 discusses the uniqueness and existence of fractional implicit differential equations. An example is used in Section 4 to illustrate the applications.

1. Preliminaries

Define all continuous real functions in the Banach space by $C(\mathfrak{J}) = C(\mathfrak{J}, R)$ on $\mathfrak{J} := [0, \mathfrak{T}]$ equipped with the norm

$$\|\mathfrak{y}\| := \sup\{|\mathfrak{y}(\mathfrak{t})| : \mathfrak{t} \in [0, \mathfrak{T}]\}.$$

Let the space $(B, \|.\|_B)$ is a seminormed linear space of functions mapping $(-\infty, 0]$ into R, and satisfying the underlying axioms listed below,
(A1) If $\mathfrak{y} : (-\infty, \mathfrak{T}] \to R$ and $\mathfrak{y}_0 \in B$, then there are constants $L, M, H > 0$, such that for any $\mathfrak{t} \in \mathfrak{J}$ the subsequent conditions retain:

- $\mathfrak{y}_{\mathfrak{t}}$ is in B, and $\mathfrak{y}_{\mathfrak{t}}$ is continuous on $[0, \mathfrak{T})\backslash\{\mathfrak{t}_1, \mathfrak{t}_2, ..., \mathfrak{t}_m\}$,
- $\|\mathfrak{y}_{\mathfrak{t}}\|_{\aleph} \leq K\|\mathfrak{y}_1\|_B + M \sup_{s\in[0,\mathfrak{t}]} |\mathfrak{y}(s)|$,
- $\|\mathfrak{y}(\mathfrak{t})\| \leq H\|\mathfrak{y}_{\mathfrak{t}}\|_B$.

(A2) $\mathfrak{y}_{\mathfrak{t}}$ is a $B-$ valued continuous function on $\mathfrak{J}$, for the function $\mathfrak{y}(.)$ in (A1),
(A3) The space B is complete.

Consider the following space

$$\begin{aligned} PC([0, \mathfrak{T}], R) \quad = \quad & \{\mathfrak{y} : [0, \mathfrak{T}] \to R : \mathfrak{y} \in C((\mathfrak{t}_{\mathfrak{z}}, t_{\mathfrak{z}+1}], R),\ \mathfrak{z} = 0, ..., m, \text{and there exist} \\ & \mathfrak{y}(\mathfrak{t}_{\mathfrak{z}}^{-})\text{and } \mathfrak{y}(\mathfrak{t}_{\mathfrak{z}}^{+}),\ \mathfrak{z} = 1, ..., m, \text{with } \mathfrak{y}(\mathfrak{t}_{\mathfrak{z}}^{-}) = \mathfrak{y}(\mathfrak{t}_{\mathfrak{z}}^{-})\}. \end{aligned}$$

Consider the Banach space $PC([0, \mathfrak{T}], R)$ equipped with the norm

$$\|\mathfrak{y}\|_{PC} = \sup_{\mathfrak{t}\in[0,\mathfrak{T}]} |\mathfrak{y}(\mathfrak{t})|.$$

Set

$$B_b = \{\mathfrak{y} : (-\infty, \mathfrak{T}] \to R\backslash\mathfrak{y} \in PC(\mathfrak{J}, R) \cap B\}.$$

And the space of absolutely continuous valued functions $AC(\mathfrak{J})$ from $\mathfrak{J}$ into R, and set

$$AC^m(\mathfrak{J}) = \{\mathfrak{y} : \mathfrak{J} \to R : \mathfrak{y}, \mathfrak{y}^{'}, \mathfrak{y}^{''}, \ldots, \mathfrak{y}^{m-1} \in \mathcal{C} \quad and \quad \mathfrak{y}^{m-1} \in AC(\mathfrak{J})\}.$$

Definition 2.1 [16] Let $\nu \in \mathfrak{h}^1(0, \mathfrak{T})$ with $\nu \in (0, 1]$, the fractional order $\mathcal{ABC}$ derivative is defined as

$${}_{0}^{\mathcal{ABC}}D_{\mathfrak{t}}^{\nu} w(\mathfrak{t}) = \frac{\mathfrak{M}(\nu)}{1-\nu} \int_0^{\mathfrak{T}} \frac{dw}{d\ell} \mathcal{E}_{\nu} \left[\frac{-\nu(t-\ell)}{1-\nu}\right] d\ell,$$

where $\mathfrak{M}(\nu)$ called normalization function and $\mathcal{E}_{\nu} = \sum_{i=0}^{\infty} \frac{\mathfrak{t}^{i\nu}}{(\nu i + 1)}$ is a Mittag-Leffler function.

Definition 2.2 [16] The $\mathcal{ABC}$ fractional integral for w is written as

$${}_{0}^{\mathcal{ABC}}\mathfrak{J}_{\mathfrak{t}}^{\nu} w(\mathfrak{t}) = \frac{1-\nu}{\mathfrak{M}(\nu)} w(\mathfrak{t}) + \frac{\nu}{\mathfrak{M}(\nu)} \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\ell)^{\nu-1}}{\Gamma(\nu)} w(\ell) d\ell,$$

where $\mathfrak{I}^{\nu}$ is the Riemann - Liouville fractional integral.
Lemma 2.3 [16] Consider the following problem

$$\begin{aligned} {}_0^{\mathcal{ABC}}D_{\mathfrak{t}}^{\nu}\mathfrak{x}(\mathfrak{t}) &= \mathfrak{z}(\mathfrak{t}) \\ \mathfrak{x}(0) &= \mathfrak{x}_0. \end{aligned}$$

Then, the solution is given by

$$\mathfrak{x}(0) = \mathfrak{x}_0 + \frac{1-\nu}{\mathfrak{M}(\nu)}\mathfrak{z}(\mathfrak{t}) + \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)}\int_0^t (\mathfrak{t}-\ell)^{\nu-1}\mathfrak{z}(\ell)d\ell.$$

Proof By using the definition 2.2, we get

$$\begin{aligned} \mathfrak{x}(\mathfrak{t}) &= \mathfrak{x}_0 + {}_0^{\mathcal{ABC}}\mathfrak{I}_t^{\nu}\mathfrak{z}(\mathfrak{t}) \\ &= \mathfrak{x}_0 + + \frac{1-\nu}{\mathfrak{M}(\nu)}\mathfrak{z}(\mathfrak{t}) + \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)}\int_0^{\mathfrak{t}} (\mathfrak{t}-\ell)^{\nu-1}\mathfrak{z}(\ell)d\ell. \end{aligned}$$

Theorem 2.4 [16] Let $\mathfrak{Z}$ be a Banach space, and $\aleph : \mathfrak{Z} \to \mathfrak{Z}$ completely continuous operator. If the set $E = \{x \in \mathfrak{Z} : x = \lambda\aleph x, \text{for some } \lambda \in (0,1)\}$ is bounded, then $\aleph$ has fixed points.
Lemma 2.5 Consider the boundary value problem with nonlinear integral boundary conditions, if $\mathfrak{z} \in L(\mathfrak{J})$,

$$\begin{aligned} {}_0^{\mathcal{ABC}}D_{\mathfrak{t}}^{\nu}\mathfrak{x}(\mathfrak{t}) &= \mathfrak{z}(\mathfrak{t}),\ o < \nu < 1,\ \mathfrak{t} \in \mathfrak{J} \\ \mathfrak{x}(0) &= \int_0^{\mathfrak{T}} \frac{(\mathfrak{t}-\ell)^{\nu-1}}{\Gamma(\nu)}\mathfrak{h}(\ell, \mathfrak{x}(\ell))d\ell \end{aligned}$$

then the solution $\mathfrak{x} \in \mathfrak{A}C(\mathfrak{J})$ is given by

$$\mathfrak{x}(\mathfrak{t}) = \int_0^{\mathfrak{T}} \frac{(\mathfrak{t}-\ell)^{\nu-1}}{\Gamma(\nu)}\mathfrak{h}(\ell, \mathfrak{x}(\ell))d\ell + \frac{(1-\nu)}{\mathfrak{M}(\nu)}\mathfrak{z}(\mathfrak{t}) + \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)}\int_0^{\mathfrak{t}} (\mathfrak{t}-\ell)^{\nu-1}\mathfrak{z}(\ell)d\ell. \tag{2.1}$$

Proof By Lemma 2.3, we can get the result (2.1) directly by replacing $\mathfrak{x}_0$ into the boundary condition.
Lemma 2.6 Consider the nonlinear integral boundary value problem

$$\begin{aligned} {}_0^{\mathcal{ABC}}D_{\mathfrak{t}}^{\nu}[\mathfrak{x}(\mathfrak{t}) - \mathfrak{g}(\mathfrak{t}, \mathfrak{x}_{\mathfrak{t}})] &= \mathfrak{f}(\mathfrak{t}),\ \mathfrak{t} \in [0, \mathfrak{T}] = \mathfrak{J},\ 0 < \nu \leq 1, \\ \Delta(\mathfrak{x})\Big|_{\mathfrak{t}=\mathfrak{t}_{\mathfrak{z}}} &= \mathfrak{I}_{\mathfrak{z}}(\mathfrak{x}_{\mathfrak{t}_{\mathfrak{z}}^-}) \\ \mathfrak{x}(\mathfrak{t}) &= \varphi(\mathfrak{t}),\ \mathfrak{t} \in (-\infty, 0] \\ \mathfrak{x}(0) &= \int_0^{\mathfrak{T}} \frac{(\mathfrak{T}-\ell)^{\nu-1}}{\Gamma(\nu)}\mathfrak{h}(\ell, \mathfrak{x}_{\ell})d\ell, \end{aligned} \tag{2.2}$$

then the solution of the problem (2.2) is

$$\mathfrak{x}(\mathfrak{t}) = \begin{cases} \varphi(\mathfrak{t}), \ \mathfrak{t} \in (-\infty, 0] \\ \mathfrak{g}(\mathfrak{t}, \mathfrak{x}_\mathfrak{t}) + \int_0^{\mathfrak{T}} \frac{(\mathfrak{T}-\ell)^{\nu-1}}{\Gamma(\nu)} \mathfrak{h}(\ell, \mathfrak{x}_\ell) d\ell + \frac{(1-\nu)}{\mathfrak{M}(\nu)} \mathfrak{f}(\mathfrak{t}) \\ + \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\ell)^{\nu-1} \mathfrak{f}(\ell) d\ell, \ \text{if } \mathfrak{t} \in [0, \mathfrak{t}_1], \\ \mathfrak{g}(\mathfrak{t}, \mathfrak{x}(\mathfrak{t})) + \int_0^{\mathfrak{T}} \frac{(\mathfrak{T}-\ell)^{\nu-1}}{\Gamma(\nu)} \mathfrak{h}(\ell, \mathfrak{x}_\ell) d\ell + \frac{(1-\nu)}{\mathfrak{M}(\nu)} \mathfrak{f}(\mathfrak{t}) \\ + \sum_{i=1}^{\mathfrak{z}} \frac{(1-\nu)}{\mathfrak{M}(\nu)} \mathfrak{f}(\mathfrak{t}_i) + \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)} \sum_{i=1}^{\mathfrak{z}} \int_{\mathfrak{t}_{i-1}}^{\mathfrak{t}_i} (\mathfrak{t}_i - \ell)^{\nu-1} f(\ell) d\ell \\ + \frac{\nu}{\Gamma(\nu)\mathfrak{M}(\nu)} \int_{\mathfrak{t}_\mathfrak{z}}^{\mathfrak{t}} (\mathfrak{t}-\ell)^{\nu-1} f(\ell) d\ell + \sum_{i=1}^{\mathfrak{z}} \mathfrak{I}_i(\mathfrak{x}(\mathfrak{t}_i^-)), \ \text{if } \mathfrak{t} \in (\mathfrak{t}_\mathfrak{z}, \mathfrak{t}_{\mathfrak{z}+1}]. \end{cases} \tag{2.3}$$

Proof Assume $\mathfrak{t}$ satisfies (2.2).
If $\mathfrak{t} \in [0, \mathfrak{t}_1]$

$${}_0^{\mathcal{ABC}} D_\mathfrak{t}^\nu [\mathfrak{x}(\mathfrak{t}) - \mathfrak{g}(\mathfrak{t}, \mathfrak{x}_\mathfrak{t})] = \mathfrak{f}(\mathfrak{t}).$$

Lemma (2.6) implies

$$\begin{aligned} \mathfrak{x}(\mathfrak{t}) - \mathfrak{g}(\mathfrak{t}, \mathfrak{x}_\mathfrak{t}) &= \int_0^{\mathfrak{T}} \frac{(\mathfrak{T}-\ell)^{\nu-1}}{\Gamma(\nu)} \mathfrak{h}(\ell, \mathfrak{x}_\ell) d\ell + {}_0^{\mathcal{ABC}} \mathfrak{I}_\mathfrak{t}^\nu \mathfrak{f}(\mathfrak{t}) \\ &= \int_0^{\mathfrak{T}} \frac{(\mathfrak{T}-\ell)^{\nu-1}}{\Gamma(\nu)} \mathfrak{h}(\ell, \mathfrak{x}_\ell) d\ell + \frac{(1-\nu)}{\mathfrak{M}(\nu)} \mathfrak{f}(\mathfrak{t}) + \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\ell)^{\nu-1} \mathfrak{f}(\ell) d\ell. \end{aligned}$$

If $\mathfrak{t} \in (\mathfrak{t}_1, \mathfrak{t}_2]$,

$$\begin{aligned} \mathfrak{x}(\mathfrak{t}) - \mathfrak{g}(\mathfrak{t}, \mathfrak{x}_\mathfrak{t}) &= \mathfrak{x}(\mathfrak{t}_1^+) - \mathfrak{g}(\mathfrak{t}_1, \mathfrak{x}_{\mathfrak{t}_1}) + \frac{(1-\nu)}{\mathfrak{M}(\nu)} \mathfrak{f}(\mathfrak{t}) + \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)} \int_{\mathfrak{f}(\mathfrak{t})\mathfrak{t}_1}^{\mathfrak{t}} (\mathfrak{t}-\ell)^{\nu-1} \mathfrak{f}(\ell) d\ell \\ &= \Delta \mathfrak{x}\Big|_{\mathfrak{t}=\mathfrak{t}_1} + \mathfrak{x}(\mathfrak{t}_1^-) - \mathfrak{g}(\mathfrak{t}_1, \mathfrak{x}_{\mathfrak{t}_1}) + \frac{(1-\nu)}{\mathfrak{M}(\nu)} \mathfrak{f}(\mathfrak{t}) + \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)} \int_{\mathfrak{t}_1}^{\mathfrak{t}} (\mathfrak{t}-\ell)^{\nu-1} \mathfrak{f}(\ell) d\ell \\ &= \mathfrak{I}_1(\mathfrak{x}_{\mathfrak{t}_1^-}) + \Bigg[\int_0^{\mathfrak{T}} \frac{(\mathfrak{T}-\ell)^{\nu-1}}{\Gamma(\nu)} \mathfrak{h}(\ell, \mathfrak{x}_\ell) d\ell + \frac{(1-\nu)}{\mathfrak{M}(\nu)} \mathfrak{f}(\mathfrak{t}_1) \\ &\quad + \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)} \int_0^{\mathfrak{t}_1} (\mathfrak{t}_1-\ell)^{\nu-1} \mathfrak{f}(\ell) d\ell \Bigg] + \frac{(1-\nu)}{\mathfrak{M}(\nu)} \mathfrak{f}(\mathfrak{t}) + \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)} \int_{\mathfrak{t}_1}^{\mathfrak{t}} (\mathfrak{t}-\ell)^{\nu-1} \mathfrak{f}(\ell) d\ell \\ &= \mathfrak{I}_1(\mathfrak{x}_{\mathfrak{t}_1^-}) + \int_0^{\mathfrak{T}} \frac{(\mathfrak{T}-\ell)^{\nu-1}}{\Gamma(\nu)} \mathfrak{h}(\ell, \mathfrak{x}_\ell) d\ell + \frac{(1-\nu)}{\mathfrak{M}(\nu)} \mathfrak{f}(\mathfrak{t}) + \frac{(1-\nu)}{\mathfrak{M}(\nu)} \mathfrak{f}(\mathfrak{t}_1) \\ &+ \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)} \int_0^{\mathfrak{t}_1} (\mathfrak{t}_1-\ell)^{\nu-1} \mathfrak{f}(\ell) d\ell + \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)} \int_{\mathfrak{t}_1}^{\mathfrak{t}} (\mathfrak{t}-\ell)^{\nu-1} \mathfrak{f}(\ell) d\ell. \end{aligned}$$

If $\mathfrak{t} \in (\mathfrak{t}_2, \mathfrak{t}_3]$,

$$\begin{aligned}
\mathfrak{x}(\mathfrak{t}) - \mathfrak{g}(\mathfrak{t}, \mathfrak{x}_\mathfrak{t}) &= \mathfrak{x}(\mathfrak{t}_2^+) - \mathfrak{g}(\mathfrak{t}_2, \mathfrak{x}_{t_2}) + \frac{(1-\nu)}{\mathfrak{M}(\nu)}\mathfrak{f}(\mathfrak{t}) + \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)}\int_{\mathfrak{t}_2}^{\mathfrak{t}} (\mathfrak{t}-\ell)^{\nu-1}\mathfrak{f}(\ell)d\ell \\
&= \Delta\mathfrak{x}\Big|_{\mathfrak{t}=\mathfrak{t}_2} + \mathfrak{x}(\mathfrak{t}_2^-) - \mathfrak{g}(\mathfrak{t}_2, \mathfrak{x}_{\mathfrak{t}_2}) + \frac{(1-\nu)}{\mathfrak{M}(\nu)}\mathfrak{f}(\mathfrak{t}) + \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)}\int_{\mathfrak{t}_2}^{\mathfrak{t}} (\mathfrak{t}-\ell)^{\nu-1}\mathfrak{f}(\mathfrak{t})(\ell)d\ell \\
&= \mathfrak{I}_2(\mathfrak{x}_{\mathfrak{t}_2^-}) + \Bigg[\int_0^{\mathfrak{T}} \frac{(\mathfrak{T}-\ell)^{\nu-1}}{\Gamma(\nu)}\mathfrak{h}(\ell, \mathfrak{x}_\ell)d\ell + \mathfrak{I}_1(\mathfrak{x}_{\mathfrak{t}_1^-}) + \frac{(1-\nu)}{\mathfrak{M}(\nu)}\mathfrak{f}(\mathfrak{t}_2) + \frac{(1-\nu)}{\mathfrak{M}(\nu)}\mathfrak{f}(\mathfrak{t}_1) \\
&\quad + \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)}\int_0^{\mathfrak{t}_1} (\mathfrak{t}_1-\ell)^{\nu-1}\mathfrak{f}(\ell)d\ell + + \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)}\int_{\mathfrak{t}_1}^{\mathfrak{t}_2} (\mathfrak{t}_2-\ell)^{\nu-1}\mathfrak{f}(\ell)d\ell\Bigg] \\
&+ \frac{(1-\nu)}{\mathfrak{M}(\nu)}\mathfrak{f}(\mathfrak{t}) + \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)}\int_{\mathfrak{t}_2}^{\mathfrak{t}} (\mathfrak{t}-\ell)^{\nu-1}\mathfrak{f}(\ell)d\ell \\
&= \int_0^{\mathfrak{T}} \frac{(\mathfrak{T}-\ell)^{\nu-1}}{\Gamma(\nu)}\mathfrak{h}(\ell, \mathfrak{x}_\ell)d\ell + \Big[\mathfrak{I}_1(\mathfrak{x}_{\mathfrak{t}_1^-}) + \mathfrak{I}_2(\mathfrak{x}_{\mathfrak{t}_2^-})\Big] + \frac{(1-\nu)}{\mathfrak{M}(\nu)}\mathfrak{f}(\mathfrak{t}) \\
&+ \frac{(1-\nu)}{\mathfrak{M}(\nu)}\left[\mathfrak{f}(\mathfrak{t}_1) + \mathfrak{f}(\mathfrak{t}_2)\right] + \Bigg[\frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)}\int_0^{\mathfrak{t}_1} (\mathfrak{t}_1-\ell)^{\nu-1}\mathfrak{f}(\ell)d\ell \\
&\quad + \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)}\int_{\mathfrak{t}_1}^{\mathfrak{t}_2} (\mathfrak{t}_2-\ell)^{\nu-1}\mathfrak{f}(\ell)d\ell\Bigg] + \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)}\int_{\mathfrak{t}_2}^{\mathfrak{t}} (\mathfrak{t}-\ell)^{\nu-1}\mathfrak{f}(\ell)d\ell.
\end{aligned}$$

Repeating this process in these ways, the solution $\mathfrak{x}(\mathfrak{t})$, for $\mathfrak{t} \in (\mathfrak{t}_\mathfrak{z}, \mathfrak{t}_{\mathfrak{z}+1}]$, where $\mathfrak{z} = 1, ..., m$, can be written as

$$\begin{aligned}
\mathfrak{x}(\mathfrak{t}) &= \mathfrak{g}(\mathfrak{t}, \mathfrak{x}(\mathfrak{t})) + \int_0^{\mathfrak{T}} \frac{(\mathfrak{T}-\ell)^{\nu-1}}{\Gamma(\nu)}\mathfrak{h}(\ell, \mathfrak{x}_\ell)d\ell + \frac{(1-\nu)}{\mathfrak{M}(\nu)}\mathfrak{f}(\mathfrak{t}) + \sum_{i=1}^{\mathfrak{z}} \frac{(1-\nu)}{\mathfrak{M}(\nu)}\mathfrak{f}(\mathfrak{t}_i) \\
&+ \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)}\sum_{i=1}^{\mathfrak{z}}\int_{\mathfrak{t}_{i-1}}^{\mathfrak{t}_i} (\mathfrak{t}_i-\ell)^{\nu-1}f(\ell)d\ell + \frac{\nu}{\Gamma(\nu)\mathfrak{M}(\nu)}\int_{\mathfrak{t}_\mathfrak{z}}^{\mathfrak{t}} (\mathfrak{t}-\ell)^{\nu-1}f(\ell)d\ell \\
&+ \sum_{i=1}^{\mathfrak{z}} \mathfrak{I}_i(\mathfrak{x}(\mathfrak{t}_i^-)).
\end{aligned}$$

2. Main Results

The following hypothesis are need to prove the main results.

(A1) For the constants $a_g > 0$, for any $\mathfrak{x}, \mathfrak{y} \in B_b$

$$|\mathfrak{g}(\mathfrak{t}, \mathfrak{x}(\mathfrak{t})) - \mathfrak{g}(\mathfrak{t}, \mathfrak{y}(\mathfrak{t}))| \leq a_g\|\mathfrak{x}(\mathfrak{t}) - \mathfrak{y}(\mathfrak{t})\|_{PC}.$$

(A2) For constants a_f, b_f, for any $\mathfrak{x}_1, \mathfrak{y}_1 \in B_b,\ \mathfrak{x}_2, \mathfrak{y}_2 \in R$

$$|\mathfrak{f}(\mathfrak{t}, \mathfrak{x}_1(\mathfrak{t}), \mathfrak{x}_2(\mathfrak{t})) - \mathfrak{f}(\mathfrak{t}, \mathfrak{y}_1(\mathfrak{t}), \mathfrak{y}_2(\mathfrak{t}))| \leq a_f\|\mathfrak{x}_1(\mathfrak{t}) - \mathfrak{y}_1(\mathfrak{t})\|_{PC} + b_f|\mathfrak{x}_2(\mathfrak{t}) - \mathfrak{y}_2(\mathfrak{t})|.$$

(A3) For the constants $a_i > 0$, for any $\mathfrak{x}, \mathfrak{y} \in B_b$

$$|\mathfrak{I}_\mathfrak{z}\mathfrak{x}(\mathfrak{t}) - \mathfrak{I}_k\mathfrak{y}(\mathfrak{t})| \leq a_i(\|\mathfrak{x}(\mathfrak{t}) - \mathfrak{y}(\mathfrak{t})\|_{PC}.$$

(A4) For the constants $a_h > 0$, for any $\mathfrak{x}, \mathfrak{y} \in B_b$

$$|\mathfrak{h}(\mathfrak{t}, \mathfrak{x}(\mathfrak{t})) - \mathfrak{h}(\mathfrak{t}, \mathfrak{y}(\mathfrak{t}))| \leq a_h\|\mathfrak{x}(\mathfrak{t}) - \mathfrak{y}(\mathfrak{t})\|_{PC}.$$

(A5) There exists constants $a_1 > 0$ and $0 < a_2 < 1$ such that

$$|\mathfrak{f}(\mathfrak{t}, \mathfrak{x}(\mathfrak{t}), \mathfrak{y}(\mathfrak{t}))| \leq a_1\|\mathfrak{x}\|_{PC} + a_2|\mathfrak{y}|.$$

for $\mathfrak{t} \in \mathfrak{J}$, $\mathfrak{x} \in B_b$ and $\mathfrak{y} \in R$.
(A6) There exists constants $n_1, n_2 > 0$ such that

$$|\mathfrak{I}_{\mathfrak{z}}(\mathfrak{x})| \leq n_1 \|\mathfrak{x}\|_{PC} + n_2$$

for each $\mathfrak{x} \in B_b$.
(A7) There exists constants $d_1, d_2 > 0$ such that

$$|\mathfrak{g}(\mathfrak{t}, \mathfrak{x}(\mathfrak{t}))| \leq d_1 \|\mathfrak{x}\|_{PC} + d_2$$

for each $\mathfrak{x} \in B_b$.
Theorem 3.1 Assume the hypothesis (A1) - (A4) holds, then the problem (1.1) has a unique solution if

$$\Theta = M\left[a_g + \left[\frac{1-\nu}{\mathfrak{M}(\nu)} + \nu\frac{T^\nu}{\mathfrak{M}(\nu)\Gamma(\nu+1)}\right](m+1)\frac{a_f}{1-b_f} + ma_i\right] < 1.$$

Proof Consider the operator $P : B_b \to B_b$ by

$$P\mathfrak{x}(\mathfrak{t}) = \begin{cases} \varphi(\mathfrak{t}); \mathfrak{t} \in (-\infty, 0] \\ \mathfrak{g}(\mathfrak{t}, \mathfrak{x}_\mathfrak{t}) + \int_0^{\mathfrak{T}} \frac{(\mathfrak{T}-\ell)^{\nu-1}}{\Gamma(\nu)}\mathfrak{h}(\ell, \mathfrak{x}_\ell)d\ell + \frac{(1-\nu)}{\mathfrak{M}(\nu)}\mathfrak{f}(\mathfrak{t}) \\ +\sum_{i=1}^{\mathfrak{z}} \frac{(1-\nu)}{\mathfrak{M}(\nu)}\mathfrak{f}(\mathfrak{t}_i) + \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)}\sum_{i=1}^{\mathfrak{z}}\int_{\mathfrak{t}_{i-1}}^{\mathfrak{t}_i}(\mathfrak{t}_i-\ell)^{\nu-1}f(\ell)d\ell \\ +\frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)}\int_{\mathfrak{t}_\mathfrak{z}}^{\mathfrak{t}}(\mathfrak{t}-\ell)^{\nu-1}f(\ell)d\ell + \sum_{i=1}^{\mathfrak{z}}\mathfrak{I}_i(\mathfrak{x}(t_i^-)),\ \mathfrak{t} \in \mathfrak{J}, \end{cases} \tag{3.1}$$

where $\mathfrak{f}(\mathfrak{t}) \in C(\mathfrak{J}, R)$ be such that

$$\mathfrak{F}(\mathfrak{t}) = \mathfrak{f}(\mathfrak{t}, \mathfrak{x}_\mathfrak{t}, {}_0^{\mathcal{ABC}}D_\mathfrak{t}^\nu \mathfrak{x}(\mathfrak{t})).$$

Let $\mathfrak{x}(.) : (-\infty, \mathfrak{T}] \to R$ be a function indicated by

$$\mathfrak{x}(\mathfrak{t}) = \begin{cases} \phi(\mathfrak{t}); & \mathfrak{t} \in (-\infty, 0], \\ \int_0^{\mathfrak{T}} \frac{(\mathfrak{T}-\ell)^{\nu-1}}{\Gamma(\nu)}\mathfrak{h}(\ell, \mathfrak{x}_\ell)d\ell; & \mathfrak{t} \in \mathfrak{J}. \end{cases}$$

Then $\mathfrak{x}_0 = \phi$, For each $z \in C(\mathfrak{J})$, with $z(0) = 0$, we denote by the function $\bar{z}$ is defined by

$$\bar{z} = \begin{cases} 0; & \mathfrak{t} \in (-\infty, 0], \\ z(\mathfrak{t}); & \mathfrak{t} \in \mathfrak{J}. \end{cases}$$

If $\mathfrak{u}(.)$ satisfies the integral equation

$$\begin{aligned} \mathfrak{u}(\mathfrak{t}) = {} & \mathfrak{g}(\mathfrak{t}, \mathfrak{x}_\mathfrak{t}) + \int_0^{\mathfrak{T}} \frac{(\mathfrak{T}-\ell)^{\nu-1}}{\Gamma(\nu)}\mathfrak{h}(\ell, \mathfrak{x}_\ell)d\ell + \frac{(1-\nu)}{\mathfrak{M}(\nu)}\mathfrak{f}(\mathfrak{t}) \\ & + \sum_{i=1}^{\mathfrak{z}} \frac{(1-\nu)}{\mathfrak{M}(\nu)}\mathfrak{f}(\mathfrak{t}_i) + \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)}\sum_{i=1}^{\mathfrak{z}}\int_{\mathfrak{t}_{i-1}}^{\mathfrak{t}_i}(\mathfrak{t}_i-\ell)^{\nu-1}f(\ell)d\ell \\ & + \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)}\int_{t_\mathfrak{z}}^{\mathfrak{t}}(\mathfrak{t}-\ell)^{\nu-1}\mathfrak{f}(\ell)d\ell + \sum_{i=1}^{\mathfrak{z}}\mathfrak{I}_i(\mathfrak{x}(t_i^-)). \end{aligned}$$

We can disintegrate $\mathfrak{u}(.)$ as $\mathfrak{u}(\mathfrak{t}) = \bar{z}(\mathfrak{t}) + \mathfrak{x}(\mathfrak{t})$; for $\mathfrak{t} \in \mathfrak{J}$, which shows that $\mathfrak{u}_\mathfrak{t} = \bar{z}_\mathfrak{t} + \mathfrak{x}_\mathfrak{t}$ $\forall \mathfrak{t} \in \mathfrak{J}$, and $z(.)$ fulfills

$$\begin{aligned} z(\mathfrak{t}) &= \mathfrak{g}(\mathfrak{t}, z_\mathfrak{t}) + \frac{(1-\nu)}{\mathfrak{M}(\nu)}\mathfrak{f}(\mathfrak{t}) + \sum_{i=1}^{\mathfrak{z}} \frac{(1-\nu)}{\mathfrak{M}(\nu)}\mathfrak{f}(\mathfrak{t}_i) \\ &+ \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)} \sum_{i=1}^{\mathfrak{z}} \int_{\mathfrak{t}_{i-1}}^{\mathfrak{t}_i} (\mathfrak{t}_i - \ell)^{\nu-1}\mathfrak{f}(\ell)d\ell \\ &+ \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)} \int_{\mathfrak{t}_\mathfrak{z}}^{\mathfrak{t}} (\mathfrak{t} - \ell)^{\nu-1}\mathfrak{f}(\ell)d\ell + \sum_{i=1}^{\mathfrak{z}} \mathfrak{I}_i(z(\mathfrak{t}_i^-)), \end{aligned}$$

where

$$\mathfrak{f}(\mathfrak{t}) = \mathfrak{f}(\mathfrak{t}, \bar{z}_\mathfrak{t} + \mathfrak{x}_\mathfrak{t}, \mathfrak{f}(\mathfrak{t})).$$

Consider

$$C_0 = \{z \in C(\mathfrak{J});\ z_0 = 0\}.$$

The norm $\|.\|_\mathfrak{T}$ in C_0 is denoted by

$$\|z\|_\mathfrak{T} = \|z_0\|_{B_b} + \sup_{\mathfrak{t}\in\mathfrak{J}} |\mathfrak{u}(\mathfrak{t})| = \sup_{\mathfrak{t}\in\mathfrak{J}} |\mathfrak{u}(\mathfrak{t})|;\ \mathfrak{u} \in C_0.$$

C_0 is a Banach space with norm $\|.\|_\mathfrak{T}$.
Define the operator $P_1 : C_0 \to C_0$

$$\begin{aligned} P_1 z(\mathfrak{t}) &= \mathfrak{g}(\mathfrak{t}, z_\mathfrak{t}) + \frac{(1-\nu)}{\mathfrak{M}(\nu)}\mathfrak{F}(\mathfrak{t}) \\ &+ \sum_{i=1}^{\mathfrak{z}} \frac{(1-\nu)}{\mathfrak{M}(\nu)}\mathfrak{f}(\mathfrak{t}_i) + \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)} \sum_{i=1}^{\mathfrak{z}} \int_{\mathfrak{t}_{i-1}}^{\mathfrak{t}_i} (\mathfrak{t}_i - \ell)^{\nu-1}\mathfrak{f}(\ell)d\ell \\ &+ \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)} \int_{\mathfrak{t}_\mathfrak{z}}^{\mathfrak{t}} (\mathfrak{t} - \ell)^{\nu-1}\mathfrak{f}(\ell)d\ell + \sum_{i=1}^{\mathfrak{z}} \mathfrak{I}_i(z(\mathfrak{t}_i^-)), \end{aligned}$$

where

$$\mathfrak{f}(\mathfrak{t}) = \mathfrak{f}(\mathfrak{t}, \bar{z}_\mathfrak{t} + \mathfrak{x}_\mathfrak{t}, \mathfrak{f}(\mathfrak{t})),\ \mathfrak{t} \in \mathfrak{J}.$$

Thus, the operator P has a fixed point is identical to P_1 has a fixed point. Now, let's establish that P_1 has a fixed point. We shall prove that $P_1 : C_0 \to C_0$ is a contraction map.

Take $z, z^{'} \in C_0$, then $\forall\ \mathfrak{t} \in \mathfrak{J}$,

$$\begin{aligned}
\|P_1(z)(\mathfrak{t}) - P_1(z^{'})(\mathfrak{t})\| \leq \sup_{\mathfrak{t}\in\mathfrak{J}} \Bigg| & \mathfrak{g}(\mathfrak{t}, z_{\mathfrak{t}}) + \frac{(1-\nu)}{\mathfrak{M}(\nu)}\mathfrak{f}(\mathfrak{t}) + \sum_{i=1}^{\mathfrak{z}} \frac{(1-\nu)}{\mathfrak{M}(\nu)}\mathfrak{f}(\mathfrak{t}_i) \\
& + \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)} \sum_{i=1}^{\mathfrak{z}} \int_{\mathfrak{t}_{i-1}}^{\mathfrak{t}_i} (\mathfrak{t}_i - \ell)^{\nu-1}\mathfrak{f}(\ell)d\ell \\
& + \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)} \int_{\mathfrak{t}_{\mathfrak{z}}}^{\mathfrak{t}} (\mathfrak{t}-\ell)^{\nu-1}\mathfrak{f}(\ell)d\ell + \sum_{i=1}^{\mathfrak{z}} \mathfrak{I}_i(z(\mathfrak{t}_i^-)) \\
& - \Bigg\{ \mathfrak{g}(\mathfrak{t}, z^{'}_{(\mathfrak{t}}) + \frac{(1-\nu)}{\mathfrak{M}(\nu)}\mathfrak{f}^{'}(\mathfrak{t}) + \sum_{i=1}^{\mathfrak{z}} \frac{(1-\nu)}{\mathfrak{M}(\nu)}\mathfrak{f}^{'}(\mathfrak{t}_i) \\
& + \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)} \sum_{i=1}^{\mathfrak{z}} \int_{\mathfrak{t}_{i-1}}^{\mathfrak{t}_i} (\mathfrak{t}_i - \ell)^{\nu-1}\mathfrak{f}^{'}(\ell)d\ell \\
& + \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)} \int_{\mathfrak{t}_{\mathfrak{z}}}^{\mathfrak{t}} (\mathfrak{t}-\ell)^{\nu-1}\mathfrak{f}^{'}(\ell)d\ell + \sum_{i=1}^{\mathfrak{z}} \mathfrak{I}_i(z^{'}(\mathfrak{t}_i^-)) \Bigg\} \Bigg| \\
& \leq \sup_{\mathfrak{t}\in\mathfrak{J}} |\mathfrak{g}(\mathfrak{t}, z_{\mathfrak{t}}) - \mathfrak{g}(\mathfrak{t}, z^{'}_{\mathfrak{t}})| \\
& + \frac{(1-\nu)}{\mathfrak{M}(\nu)}|\mathfrak{f}(\mathfrak{t}) - \mathfrak{f}^{'}(\mathfrak{t})| + \sum_{i=1}^{\mathfrak{z}} \frac{(1-\nu)}{\mathfrak{M}(\nu)}|\mathfrak{f}^{'}((\mathfrak{t}_i) - \mathfrak{f}^{'}(\mathfrak{t}_i)| \\
& + \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)} \sum_{i=1}^{\mathfrak{z}} \int_{\mathfrak{t}_{i-1}}^{\mathfrak{t}_i} (\mathfrak{t}_i - \ell)^{\nu-1}|\mathfrak{f}(\ell) - \mathfrak{f}^{'}(\ell)|d\ell \\
& + \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)} \int_{\mathfrak{t}_{\mathfrak{z}}}^{\mathfrak{t}} (\mathfrak{t}-\ell)^{\nu-1}|\mathfrak{f}(\ell) - \mathfrak{f}^{'}(\ell)|d\ell + \sum_{i=1}^{\mathfrak{z}} |\mathfrak{I}_i(z(\mathfrak{t}_i^-)) - \mathfrak{I}_i(z^{'}(\mathfrak{t}_i^-))|,
\end{aligned}$$

where $\mathfrak{f}, \mathfrak{f}^{'} \in C(\mathfrak{J}, R)$ be such that

$$\mathfrak{f}(\mathfrak{t}) = \mathfrak{f}((\mathfrak{t}, \bar{z}_{\mathfrak{t}} + \mathfrak{x}_{\mathfrak{t}}, \mathfrak{f}(\mathfrak{t}))$$

and

$$\mathfrak{f}^{'}(\mathfrak{t}) = \mathfrak{f}^{'}(\mathfrak{t}, \bar{z}^{'}_{\mathfrak{t}} + \mathfrak{x}_{\mathfrak{t}}, \mathfrak{f}^{'}(\mathfrak{t})).$$

Since, for each $\mathfrak{t} \in J$, we have

$$\begin{aligned}
|\mathfrak{f}(\mathfrak{t}) - \mathfrak{f}^{'}(\mathfrak{t})| &= |\mathfrak{f}(\mathfrak{t}, \bar{z}_{\mathfrak{t}} + \mathfrak{x}_{\mathfrak{t}}, \mathfrak{f}(\mathfrak{t})) - \mathfrak{f}^{'}(\mathfrak{t}, \bar{z}^{'}_{\mathfrak{t}} + \mathfrak{x}_{\mathfrak{t}}, \mathfrak{f}^{'}(\mathfrak{t}))| \\
&\leq a_f \|\bar{z}_{\mathfrak{t}} + \mathfrak{x}_{\mathfrak{t}} - \bar{z}^{'}_{\mathfrak{t}} - \mathfrak{x}_{\mathfrak{t}}\|_{PC} + b_f |\mathfrak{f}((\mathfrak{t})) - \mathfrak{f}^{'}(\mathfrak{t})| \\
|\mathfrak{f}(\mathfrak{t}) - \mathfrak{f}^{'}(\mathfrak{t})| &\leq \frac{a_f}{1-b_f} \|\bar{z}_t - \bar{z}^{'}_{\mathfrak{t}}\|_{PC}
\end{aligned}$$

$$\begin{aligned}
\|P_1(z)(\mathfrak{t})-P_1(z^{'})(\mathfrak{t})\| &\leq a_g\|\bar{z}_\mathfrak{t}-\bar{z}_\mathfrak{t}^{'}\|_{PC}+\frac{1-\nu}{\mathfrak{M}(\nu)}\frac{a_f}{1-b_f}\|\bar{z}_\mathfrak{t}-\bar{z}_\mathfrak{t}^{'}\|_{PC}\\
&+m\frac{1-\nu}{\mathfrak{M}(\nu)}\frac{a_f}{1-b_f}\|\bar{z}_\mathfrak{t}-\bar{z}_\mathfrak{t}^{'}\|_{PC}+\frac{\nu\mathfrak{T}^\nu}{\mathfrak{M}(\nu)\Gamma(\nu+1)}m\frac{a_f}{1-b_f}\|\bar{z}_\mathfrak{t}-\bar{z}_\mathfrak{t}^{'}\|_{PC}\\
&+\frac{\nu\mathfrak{T}^\nu}{\mathfrak{M}(\nu)\Gamma(\nu+1)}\frac{a_f}{1-b_f}\|\bar{z}_\mathfrak{t}-\bar{z}_\mathfrak{t}^{'}\|_{PC}+ma_i\|\bar{z}_\mathfrak{t}-\bar{z}_\mathfrak{t}^{'}\|_{PC}\\
&\leq\left\{a_g+\left[\frac{1-\nu}{\mathfrak{M}(\nu)}+\frac{\nu\mathfrak{T}^\nu}{\mathfrak{M}(\nu)\Gamma(\nu+1)}\right](m+1)\frac{a_f}{1-b_f}+ma_i\right\}\|\bar{z}_\mathfrak{t}-\bar{z}_\mathfrak{t}^{'}\|_{PC}\\
&\leq M\left\{a_g+\left[\frac{1-\nu}{\mathfrak{M}(\nu)}+\frac{\nu\mathfrak{T}^\nu}{\mathfrak{M}(\nu)\Gamma(\nu+1)}\right](m+1)\frac{a_f}{1-b_f}+ma_i\right\}\sup_{\mathfrak{t}\in\mathfrak{J}}\|\bar{z}_\mathfrak{t}-\bar{z}_\mathfrak{t}^{'}\|_{B_b}\\
&\leq M\left\{a_g+\left[\frac{1-\nu}{\mathfrak{M}(\nu)}+\frac{\nu\mathfrak{T}^\nu}{\mathfrak{M}(\nu)\Gamma(\nu+1)}\right](m+1)\frac{a_f}{1-b_f}+ma_i\right\}\sup_{\mathfrak{t}\in\mathfrak{J}}\|\bar{z}_\mathfrak{t}-\bar{z}'_t\|_{\mathfrak{T}}.
\end{aligned}$$

Hence we obtain

$$\|P_1(\bar{z})(\mathfrak{t})-P_1(\bar{z^{'}})(\mathfrak{t})\|\leq\Theta\|\bar{z}_\mathfrak{t}-\bar{z}_\mathfrak{t}^{'}\|_{\mathfrak{T}}. \tag{3.2}$$

Therefore, P_1 is a contraction and (1.1) has unique solution.

Theorem 3.2 Assume the hypothesis (A1) - (A7) holds, then the problem (1.1) has at least one solution.

Proof We consider the operator $P_1 : C_0 \to C_0$ defined (previously), for each given $R > 0$, we define the ball Denote the ball

$$\mathcal{B}_\mathcal{R}=\{\mathfrak{x}\in C_0,\ \|\mathfrak{x}\|_\mathfrak{T}\leq R\}.$$

Step 1. P_1 is continuous.
Let the sequence $\{z_n\}$ such that $z_n \to z$ in C_0.
For each $\mathfrak{t} \in \mathfrak{J}$, we have

$$\begin{aligned}
&\|P_1(z_n)(\mathfrak{t})-P_1(z)(\mathfrak{t})\|\\
&\leq\sup_{t\in J}|\mathfrak{g}(\mathfrak{t},z_{nt})-\mathfrak{g}(\mathfrak{t},z_\mathfrak{t})|\\
&+\frac{(1-\nu)}{\mathfrak{M}(\nu)}|f_n(\mathfrak{t})-\mathfrak{f}(\mathfrak{t})|+\sum_{i=1}^{\mathfrak{z}}\frac{(1-\nu)}{\mathfrak{M}(\nu)}|\mathfrak{f}_n(\mathfrak{t}_i)-\mathfrak{f}(\mathfrak{t}_i)|\\
&+\frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)}\sum_{i=1}^{\mathfrak{z}}\int_{\mathfrak{t}_{i-1}}^{\mathfrak{t}_i}(\mathfrak{t}_i-\ell)^{\nu-1}|\mathfrak{f}_n(\ell)-\mathfrak{f}(\ell)|d\ell\\
&+\frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)}\int_{\mathfrak{t}_\mathfrak{z}}^{\mathfrak{t}}(\mathfrak{t}-\ell)^{\nu-1}|\mathfrak{f}_n(\ell)-\mathfrak{f}(\ell)|d\ell+\sum_{i=1}^{\mathfrak{z}}|\mathfrak{I}_i(z_n(\mathfrak{t}_i^-))-\mathfrak{I}_i(z(\mathfrak{t}_i^-))|
\end{aligned}$$

where $\mathfrak{f}_n, \mathfrak{f} \in C(J, R)$ be such that

$$\mathfrak{f}_n(\mathfrak{t})=\mathfrak{f}(\mathfrak{t},\bar{z}_{n\mathfrak{t}}+\mathfrak{x}_\mathfrak{t},\mathfrak{f}_n(\mathfrak{t}))$$

and

$$\mathfrak{f}(\mathfrak{t})=\mathfrak{f}(\mathfrak{t},\bar{z}_\mathfrak{t}+\mathfrak{x}_\mathfrak{t},\mathfrak{f}(\mathfrak{t}))$$

Here, $\mathfrak{f}, \mathfrak{f}_n$ are continuous and $\|z_n - z\|_\mathfrak{T} \to 0$ as $n \to \infty$ then by the Lebesgue dominated convergence theorem

$$\|P_1(z_n)-P_1(z)\|_\mathfrak{T}\to 0 \text{ as } n\to\infty.$$

Hence, P_1 is Continuous.
Step 2: $P_1(B_R)$ is bounded. Let $z \in B_R$, for each $\mathfrak{t} \in \mathfrak{J}$, we have

$$\begin{aligned}|\mathfrak{f}(\mathfrak{t})| &= |\mathfrak{f}(\mathfrak{t}, \bar{z}_\mathfrak{t} + \mathfrak{x}(\mathfrak{t}), \mathfrak{f}(\mathfrak{t}))| \\ &\leq a_1\|\bar{z}_\mathfrak{t} + \mathfrak{x}(\mathfrak{t})\| + a_2|\mathfrak{f}(\mathfrak{t})| \\ &\leq a_1[\|\bar{z}_\mathfrak{t}\| + \|\mathfrak{x}(\mathfrak{t})\|] + a_2|\mathfrak{f}(\mathfrak{t})| \\ &\leq a_1 MR + a_1 K\|\phi\| + a_2\|\mathfrak{f}(\mathfrak{t})\|_\infty\end{aligned}$$

then

$$\|\mathfrak{f}(\mathfrak{t})\|_\infty \leq \frac{a_1 MR + a_1 K\|\phi\|}{1 - a_2} := \chi$$

Thus,

$$\begin{aligned}|P_1 z(\mathfrak{t})| &= |\mathfrak{g}(\mathfrak{t}, z_\mathfrak{t})| + \frac{(1-\nu)}{\mathfrak{M}(\nu)}|\mathfrak{f}(\mathfrak{t})| + \sum_{i=1}^{\mathfrak{z}} \frac{(1-\nu)}{\mathfrak{M}(\nu)}|\mathfrak{f}(\mathfrak{t}_i)| \\ &+ \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)} \sum_{i=1}^{\mathfrak{z}} \int_{\mathfrak{t}_{i-1}}^{\mathfrak{t}_i} (\mathfrak{t}_i - \ell)^{\nu-1}|\mathfrak{f}(\ell)|d\ell \\ &+ \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)} \int_{\mathfrak{t}_\mathfrak{z}}^{\mathfrak{t}} (\mathfrak{t} - \ell)^{\nu-1}|\mathfrak{f}(\ell)|d\ell + \sum_{i=1}^{\mathfrak{z}} |\mathfrak{I}_i(z(\mathfrak{t}_i^-))|, \\ &\leq d_1\|z\| + d_2 + \frac{(1-\nu)}{\mathfrak{M}(\nu)}\chi + m\frac{(1-\nu)}{\mathfrak{M}(\nu)}\chi + \frac{m\mathfrak{T}^\nu}{\mathfrak{M}(\nu)\Gamma(\nu)}\chi + \frac{\mathfrak{T}^\nu}{\mathfrak{M}(\nu)\Gamma(\nu)}\chi + m(n_1\|z(\mathfrak{t}_i^-)\| + n_2) \\ &\leq d_1 R + d_2 + \left[\frac{(1-\nu)}{\mathfrak{M}(\nu)} + \frac{\mathfrak{T}^\nu}{\mathfrak{M}(\nu)\Gamma(\nu)}\right](m+1)\chi + m(n_1 R + n_2) := l_1.\end{aligned}$$

Hence,

$$\|P_1(z)\|_\mathfrak{T} \leq l_1.$$

Consequently, P_1 maps bounded sets into bounded sets in C_0.
Step 3: $P_1(B_R)$ is equicontinuous.
Let $\mathfrak{t}_{\mathfrak{z}-1}, \mathfrak{t}_\mathfrak{z} \in (0, \mathfrak{T}],\ \mathfrak{t}_{\mathfrak{z}-1} < \mathfrak{t}_\mathfrak{z}$, and $\mathfrak{z} \in B_R$. Then

$$\begin{aligned}|P_1(\mathfrak{x})(\mathfrak{t}_\mathfrak{z}) - P_1(\mathfrak{x})(\mathfrak{t}_{\mathfrak{z}-1})| &= \frac{(1-\nu)}{\mathfrak{M}(\nu)}\mathfrak{f}(\mathfrak{t}_\mathfrak{z}) + \sum_{i=1}^{\mathfrak{z}} \frac{(1-\nu)}{\mathfrak{M}(\nu)}\mathfrak{f}(\mathfrak{t}_\mathfrak{z}) + \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)} \sum_{i=1}^{\mathfrak{z}} \int_{\mathfrak{t}_{i-1}}^{\mathfrak{t}_i} (\mathfrak{t}_\mathfrak{z} - \ell)^{\nu-1}\mathfrak{f}(\ell)d\ell \\ &+ \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)} \int_{\mathfrak{t}_\mathfrak{z}}^{t} (\mathfrak{t}_\mathfrak{z} - \ell)^{\nu-1}\mathfrak{f}(\ell)d\ell + \sum_{i=1}^{\mathfrak{z}} \mathfrak{I}_i(\mathfrak{x}(\mathfrak{t}_\mathfrak{z}^-)) \\ &- \frac{(1-\nu)}{\mathfrak{M}(\nu)}\mathfrak{f}(\mathfrak{t}_{\mathfrak{z}-1}) - \sum_{i=1}^{\mathfrak{z}} \frac{(1-\nu)}{\mathfrak{M}(\nu)}\mathfrak{f}(\mathfrak{t}_{\mathfrak{z}-1}) - \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)} \sum_{i=1}^{\mathfrak{z}} \int_{\mathfrak{t}_{i-1}}^{\mathfrak{t}_i} (\mathfrak{t}_{\mathfrak{z}-1} - \ell)^{\nu-1}\mathfrak{f}(\ell)d\ell \\ &- \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)} \int_{\mathfrak{t}_\mathfrak{z}}^{t} (\mathfrak{t}_{\mathfrak{z}-1} - \ell)^{\nu-1}\mathfrak{f}(\ell)d\ell - \sum_{i=1}^{\mathfrak{z}} \mathfrak{I}_i(\mathfrak{x}(\mathfrak{t}_{\mathfrak{z}-1}^-))\Big| \\ &\leq \frac{(1-\nu)}{\mathfrak{M}(\nu)}|\mathfrak{f}(\mathfrak{t}_\mathfrak{z}) - \mathfrak{f}(\mathfrak{t}_{\mathfrak{z}-1})| + \sum_{i=1}^{\mathfrak{z}} \frac{(1-\nu)}{\mathfrak{M}(\nu)}|\mathfrak{f}(\mathfrak{t}_\mathfrak{z}) - \mathfrak{f}(\mathfrak{t}_{\mathfrak{z}-1})| + \frac{(m+1)}{\mathfrak{M}((\nu)\Gamma(\nu))}(\mathfrak{t}_\mathfrak{z}^\nu - \mathfrak{t}_{\mathfrak{z}-1}^\nu) \\ &+ \sum_{i=1}^{\mathfrak{z}} |\mathfrak{I}_i(\mathfrak{x}(\mathfrak{t}_\mathfrak{z}^-)) - \mathfrak{I}_i(\mathfrak{x}(\mathfrak{t}_{\mathfrak{z}-1}^-))|.\end{aligned}$$

As $t_{\mathfrak{z}} \to \mathfrak{t}_{\mathfrak{z}-1}$, the RHS tents to 0. Hence P_1 is completely continuous.
Step 4: A priori bounds. To prove that the set

$$E = \{z \in C_0 : z = \lambda P_1(z) \text{ for some } \lambda \in (0,1)\}$$

is bounded. Let $z \in C_0$. Let $\mathfrak{x} \in C_0$, such that $z = \lambda P_1(z)$ for some $\lambda \in (0,1)$. Thus, for each $\mathfrak{t} \in \mathfrak{J}$ we have

$$\begin{aligned}\mathfrak{x}(\mathfrak{t}) &= \lambda \mathfrak{g}(\mathfrak{t}, \mathfrak{x}_{\mathfrak{t}}) + \frac{\lambda(1-\nu)}{\mathfrak{M}(\nu)}\mathfrak{f}(\mathfrak{t}) + \sum_{i=1}^{\mathfrak{z}} \frac{\lambda(1-\nu)}{\mathfrak{M}(\nu)}\mathfrak{f}(\mathfrak{t}_i) \\ &+ \frac{\lambda\nu}{\mathfrak{M}(\nu)\Gamma(\nu)} \sum_{i=1}^{\mathfrak{z}} \int_{\mathfrak{t}_{i-1}}^{\mathfrak{t}_i} (\mathfrak{t}_i - \ell)^{\nu-1}\mathfrak{f}(\ell)d\ell + \frac{\lambda\nu}{\mathfrak{M}(\nu)\Gamma(\nu)} \int_{\mathfrak{t}_{\mathfrak{z}}}^{\mathfrak{t}} (\mathfrak{t}-\ell)^{\nu-1}\mathfrak{f}(\ell)d\ell + \lambda \sum_{i=1}^{\mathfrak{z}} \mathfrak{I}_i(\mathfrak{x}(\mathfrak{t}_i^-))\end{aligned} \tag{3.3}$$

$$\begin{aligned}|\mathfrak{f}(\mathfrak{t})| &= |\mathfrak{f}(\mathfrak{t}, \bar{z}_{\mathfrak{t}} + \mathfrak{x}(\mathfrak{t}), \mathfrak{f}(\mathfrak{t}))| \\ &\leq a_1\|\bar{z}_{\mathfrak{t}} + \mathfrak{x}(\mathfrak{t})\| + a_2|\mathfrak{f}(\mathfrak{t})| \\ &\leq a_1[\|\bar{z}_{\mathfrak{t}}\| + \|\mathfrak{x}(\mathfrak{t})\|] + a_2|\mathfrak{f}(\mathfrak{t})| \\ &\leq a_1 M\|z\|_{\mathfrak{T}} + a_1 K\|\phi\| + a_2\|\mathfrak{f}(\mathfrak{t})\|_\infty\end{aligned}$$

then

$$\|\mathfrak{f}(\mathfrak{t})\|_\infty \leq \frac{a_1 M\|z\|_{\mathfrak{T}} + a_1 K\|\phi\|}{1-a_2} := \chi_1.$$

Thus,

$$\begin{aligned}|P_1 z(\mathfrak{t})| &= |\mathfrak{g}(\mathfrak{t}, z_{\mathfrak{t}})| + \frac{(1-\nu)}{\mathfrak{M}(\nu)}|\mathfrak{f}(\mathfrak{t})| + \sum_{i=1}^{\mathfrak{z}} \frac{(1-\nu)}{\mathfrak{M}(\nu)}|\mathfrak{f}(\mathfrak{t}_i)| \\ &+ \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)} \sum_{i=1}^{\mathfrak{z}} \int_{\mathfrak{t}_{i-1}}^{\mathfrak{t}_i} (\mathfrak{t}_i - \ell)^{\nu-1}|\mathfrak{f}(\ell)|d\ell \\ &+ \frac{\nu}{\mathfrak{M}(\nu)\Gamma(\nu)} \int_{\mathfrak{t}_{\mathfrak{z}}}^{\mathfrak{t}} (\mathfrak{t}-\ell)^{\nu-1}|\mathfrak{f}(\ell)|d\ell + \sum_{i=1}^{\mathfrak{z}} |\mathfrak{I}_i(z(\mathfrak{t}_i^-))|, \\ &\leq d_1\|\mathfrak{z}\| + d_2 + \frac{(1-\nu)}{\mathfrak{M}(\nu)}\chi_1 + m\frac{(1-\nu)}{\mathfrak{M}(\nu)}\chi_1 + \frac{m\mathfrak{T}^\nu}{\mathfrak{M}(\nu)\Gamma(\nu)}\chi_1 + \frac{\mathfrak{T}^\nu}{\mathfrak{M}(\nu)\Gamma(\nu)}\chi_1 + m(n_1\|z(\mathfrak{t}_i^-)\| + n_2) \\ &\leq d_1\|z\|_{\mathfrak{T}} + d_2 + \left[\frac{(1-\nu)}{\mathfrak{M}(\nu)} + \frac{\mathfrak{T}^\nu}{\mathfrak{M}(\nu)\Gamma(\nu)}\right](m+1)\chi_1 + m(n_1\|z\|_{\mathfrak{T}} + n_2) := l_2.\end{aligned}$$

Hence,

$$\|P_1(z)\|_{\mathfrak{T}} \leq l_2.$$

Hence the set E is bounded. By theorem 2.4, fixed point of the operator P is a solution of the problem (1.1).

3. Example

Consider the following problem

$$\begin{cases} {}_0^{\mathcal{ABC}}D_{\mathfrak{t}}^{\frac{1}{2}}\left[\mathfrak{x}(\mathfrak{t}) - \dfrac{\tan^{-1}|\mathfrak{x}(\mathfrak{t})|}{35}\right] = \dfrac{\mathfrak{t}^3 + \sin|\mathfrak{x}(\mathfrak{t})|}{45} - \dfrac{e^{-\mathfrak{t}}}{11+e^{\mathfrak{t}}}\dfrac{|{}_0^{\mathcal{ABC}}D_{\mathfrak{t}}^{\frac{1}{2}}\mathfrak{x}(\mathfrak{t})|}{1+|{}_0^{\mathcal{ABC}}D_t^{\frac{1}{2}}\mathfrak{x}(\mathfrak{t})|}, \ \mathfrak{t}\in[0,1], \\ \Delta\mathfrak{x}(\mathfrak{t}) = \dfrac{\mathfrak{x}(\frac{1}{2}^-)}{10+\mathfrak{x}(\frac{1}{2}^-)}, \\ \mathfrak{x}(\mathfrak{t}) = \varphi(\mathfrak{t}), \ \mathfrak{t}\in(-\infty,0], \\ \mathfrak{x}(0) = \displaystyle\int_0^1 \dfrac{(1-\ell)^{\nu-1}}{\Gamma(\nu)}\dfrac{1}{25}\exp(-\mathfrak{x}(\ell))d\ell, \end{cases} \tag{4.1}$$

Let $\delta > 0$ be a real constant and

$$B_\delta = \{\mathfrak{x}\in C(-\infty,0],R,) : \lim_{\eta\to\infty} e^{\delta\eta}\mathfrak{x}(\eta) \text{ exists in } R\}.$$

The norm B_δ is provided by

$$\|\mathfrak{x}\|_\delta = \sup_{\eta\in(-\infty,0]} e^{\delta\eta}\mathfrak{x}(\eta).$$

where

$$\mathfrak{g}(\mathfrak{t},\mathfrak{x}(\mathfrak{t})) = \frac{\tan^{-1}|\mathfrak{x}(\mathfrak{t})|}{35}, \ f(\mathfrak{t},\mathfrak{x},\mathfrak{y}) = \frac{\mathfrak{t}^3+\sin|\mathfrak{x}(\mathfrak{t})|}{45} - \frac{e^{-\mathfrak{t}}}{11+e^{\mathfrak{t}}}\frac{|\mathfrak{y}|}{1+|\mathfrak{y}|}, \ \mathfrak{h}(\mathfrak{t},\mathfrak{x}(\mathfrak{t})) = \frac{1}{25}\exp(-\mathfrak{x}(\mathfrak{t})).$$

As $\mathfrak{T} = 1$ and $\nu = \frac{1}{2}$, let $\mathfrak{x},\mathfrak{y}\in B_b$

$$\begin{aligned} |\mathfrak{g}(\mathfrak{t},\mathfrak{x}(\mathfrak{t})) - \mathfrak{g}(\mathfrak{t},\mathfrak{y}(\mathfrak{t}))| &= \left|\frac{\tan^{-1}|\mathfrak{x}(\mathfrak{t})|}{35} - \frac{\tan^{-1}|\mathfrak{y}(\mathfrak{t})|}{35}\right| \\ &\le \frac{1}{35}|\mathfrak{x}(\mathfrak{t}) - \mathfrak{y}(\mathfrak{t})|, \\ |\mathfrak{f}(\mathfrak{t},\mathfrak{x},\mathfrak{y}) - \mathfrak{f}(\mathfrak{t},\bar{\mathfrak{x}},\bar{\mathfrak{y}})| &= \left|\frac{\mathfrak{t}^3+\sin|\mathfrak{x}(\mathfrak{t})|}{45} - \frac{\mathfrak{t}^3+\sin|\mathfrak{y}(\mathfrak{t})|}{45}\right| + \frac{e^{-\mathfrak{t}}}{11+e^{\mathfrak{t}}}\left|\frac{|\bar{x}|}{1+|\bar{\mathfrak{x}}|}\frac{|\bar{\mathfrak{y}}|}{1+|\bar{\mathfrak{y}}|}\right| \\ &\le \frac{19}{180}|\mathfrak{x}(\mathfrak{t}) - \mathfrak{y}(\mathfrak{t})| + \frac{19}{180}|\bar{\mathfrak{x}}(\mathfrak{t}) - \bar{\mathfrak{y}}(\mathfrak{t})| \\ |\mathfrak{I}_{\mathfrak{z}}\mathfrak{x}(\mathfrak{t}) - \mathfrak{I}_k\mathfrak{y}(\mathfrak{t})| &= \left|\frac{\mathfrak{x}}{10+\mathfrak{x}} - \frac{\mathfrak{y}}{10+\mathfrak{y}} = \frac{10|\mathfrak{x}-\mathfrak{y}|}{(10+\mathfrak{x})(10+\mathfrak{y})}\right| \le \frac{1}{10}|\mathfrak{x}-\mathfrak{y}| \\ &\text{and} \end{aligned}$$

Thus we have $a_g = \frac{1}{35}$, $a_f = b_f = \frac{19}{180}$, $a_i = \frac{1}{10}$ and choose $m = 1$, $\mathfrak{T} = 1$.
Now examine the condition of the theorem 3.1 and attain

$$\Theta = \left\{a_g + \left[\frac{1-\nu}{\mathfrak{M}(\nu)} + \nu\frac{\mathfrak{T}^\nu}{\mathfrak{M}(\nu)\Gamma(\nu+1)}\right](m+1)\frac{a_f}{1-b_f} + ma_i\right\} = 0.63026 < 1.$$

Therefore, the problem (4.1) has a unique solution.

4. Conclusion

This study has successfully examined the existence and uniqueness findings for the integral boundary conditions and fractional implicit differential equation. Numerous mathematical models of human diseases and dynamical issues are applicable to this kind of issue. We have established adequate results for at least one solution based on the fixed point theorems of Schaefer and Banach. The results that were deduced have been supported by a good problem. We will eventually add numerical results to our work.

References

[1] M.A. Almalahi, A.B. Ibrahim, A. Almutairi, O. Bazighifan, T.A. Aljaaidi and J. Awrejcewicz.: A Qualitative study on second-order nonlinear fractional differential evolution equations with generalized ABC operator, symmetry, 14, 207 (2021).

[2] M.A. Almalahi, S.K. Panchal, M.S. Abdo and F. Jarad.: On Atangana-Baleanu-Type nonlocal boundary fractional differential equations, Hindawi, Vol.2022, 1812445, (2022), 17 pages.

[3] A.S. Alnahdi, M.B. Jeelani, M.S. Abdo, S.M. Ali and S. Saleh.: On a nonlocal implicit problem under Atangana-Baleanu-Caputo fractional derivative, Boundary Value Problems, No. 104, (2021), 18 pages.

[4] E. Bazhlekova; Fractional Evolution Equations in Banach Spaces, PhD. Thesis, Eindhoven University of Technology, Eindhoven, (2001).

[5] A. Din, Y. Li, F.M. Khan, Z.U. Khan and P. Liu.: On analysis of fractional order mathematical model of Hepatitis B using Atangana-Baleanu Caputo (ABC) derivative, Fractals, Vol. 30, No. 1 (2022), 18 pages.

[6] G. S. Ahmad, F.A. Rihan, A. Ullah, Q.M. Al-Mdallal.: Nonlinear analysis of a nonlinear modified KdV equation under Atangana Baleanu Caputo derivative, AIMS Mathematics, Vol. 7, No. 5, (2021), 7847-7865.

[7] R. Gul, K. Shah, Z.A. Khan, F. Jarad.: On a class of boundary value problems under ABC fractional derivatives, Advances in Differce Equations, No. 437, (2021), https://doi.org/10.1186/s13662-021-03595-3.

[8] M. Kosti´c, Abstract Volterra Integro-Differential Equations, CRC Press, Boca Raton, Fl., (2015).

[9] K. Karthikeyan, J. Reunsumrit, P. Karthikeyan and S. Poornima, D. Tamizharasan and T. Sitthiwirattham.: Existence results for impulsive fractional integrodifferential equations involving integral boundary conditions, Mathematical Problems in Engineering, Vol. 2022, Article ID 6599849, (2022), 12 pages.

[10] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo.: Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, (2006).

[11] S. Krim, S. Abbas and M. Benchohra.: Caputo-Hadamard implicit fractional differential equations with delay, São Paulo Journal of Mathematical Sciences, Vol. 15, (2021), 463-484.

[12] S.K. Panda, T. Abdeljawad and C. Ravichandran.: Novel fixed point approach to Atangana-Baleanu fractional and Lp-Fredholm integral equations, Alex. Eng. J. Vol.59, No. 4, (2020), 1959–1970.

[13] I. Podlubny.: Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications, Academic Press, New York, (1998).

[14] D. Prathumwan, I. Chaiya and K. Trachoo.: Study of Transmission Dynamics of Streptococcus suis Infection Mathematical Model between Pig and Human under ABC Fractional Order Derivative, symmetry, Vol. 14, No. 10, (2022), 1-21.

[15] C. Ravichandran, K. Logeswari and F. Jarad.: New results on existence in the framework of Atangana-Baleanu derivative for fractional integro differential equations, Chaos Solitons Fractals, Vol. 125, (2019), 194–200.

[16] J. Reunsumrit, P. Karthikeyan, S. Poornima, K. Karthikeyan and T. Sitthiwirattham.: Analysis of Existence and Stability Results for Impulsive Fractional Integro-Differential Equations Involving the Atangana–Baleanu–Caputo Derivative under Integral Boundary Conditions, Mathematical Problems in Engineering, Vol. 2022, Article ID 5449680, (2022), 18 pages.

[17] Q. Tul Ain, T. Sathiyaraj, S. Karim, M. Nadeem and P.K. Mwanakatwe.: ABC Fractional Derivative for the Alcohol Drinking Model using Two-Scale Fractal Dimension, Complexity, Vol. 2022, Article ID 8531858, (2022), 11 pages.

[18] S.T. Sutar and K.D. Kucche.: On nonlinear hybrid fractional differential equations with Atangana-Baleanu-Caputo derivative, Chaos, Solitons and Fractals, Vol. 143, (2021), 110557, 11 pages.

[19] V. Wattanakejorn, P. Karthikeyan, S. Poornima, K. Karthikeyan and T. Sitthiwirattham.: Existence solutions for implicit fractional relaxation differential equations with impulsive delay boundary conditions, axioms 2022, Vol. 11, No. 611, (2022). https://dx.doi.org/10.3390/axioms11110611.

S. Poornima, P. Karthikeyan

Department of Mathematics, Sri Vasavi College, Erode, India

Email address: `spoorni26@gmail.com, pkarthisvc@gmail.com`