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ON NABLA SHEHU TRANSFORM AND ITS APPLICATIONS

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ABSTRACT. Integral transforms on time scales are persuasive and versatile mathematical operators that extend the concepts of classical integral transforms and are applied to functions defined on arbitrary time scales. Time scales can involve a combination of continuous, discrete, and some cases of mixed behaviours. Thus, integral transforms on time scales are more comprehensive for the analysis of time-varying phenomenon are therefore essential in fields where such practices are frequent. In this paper, we introduce the nabla Shehu transform, which is a generalization of the nabla Laplace and nabla Sumudu transforms on time scales, and discuss its existence with respect to fundamental properties such as linearity, transform of derivatives, transform of integrals, and convolution theorem. Further, we find the transform of the fractional integral, Riemann-Liouville fractional derivative, Liouville-Caputo fractional derivative, time scale power function, and Mittag-Leffler function and use them to solve fractional dynamic equations involving Riemann-Liouville and Liouville-Caputo type fractional derivatives in subsequent sections.

1. INTRODUCTION

Integral transforms are widely used to solve various differential and integral equations. Stefan Hilger (1988) introduced the concept of time scales in his PhD dissertation [10]. This introduction is a milestone for the development of dynamic systems in discrete and continuous cases. Calculus on time scales unifies continuous and discrete analysis therefore, it is an extension of calculus on real numbers. Time scale calculus mainly involves delta and nabla derivatives. Integral transforms on time scales are mathematical techniques used to analyze functions on time scales, that combine discrete and continuous components. These transforms generalize the classical integral transforms and are used as in traditional calculus to set of time scales. Just as integral transforms in classical calculus provide useful tools for solving differential and integral equations, integral transforms on time scales offer similar advantages for solving equations involving functions defined on time scales.

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Due to some obstacles regarding calculus on time scales under consideration for solving fractional dynamic equations on time scales, the nabla version of integral transform is more acceptable. Various integral transforms such as Laplace, Fourier, Sumudu, Shehu, etc. studied in [2, 5, 8, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 25, 26].

Motivating from this research, we studied the nabla version of the Shehu transform on time scales which is a generalization of the nabla versions of the Laplace and Sumudu transforms and hence the carrying properties of both were found to be more beneficial. Using this method, we solved fractional dynamic equations with initial conditions involving Riemann-Lioville and Liouville-Caputo type fractional derivatives. Concepts related to time scales which will serve as a prerequisite of our work are taken from [1, 3, 4, 5, 6, 9, 11, 13, 24]. For further study, we require some preliminary concepts, which are as follows.

For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is given as $\sigma(t) := inf\{\tau \in \mathbb{T} : \tau > t\}$ and backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is given as $\rho(t) := sup\{\tau \in \mathbb{T} : \tau < t\}$. If $\sigma(t) > t,\ t$ is said to be right-scattered, while if $\rho(t) < t$, then t is left-scattered. In addition, if $t < sup\mathbb{T}$ and $\sigma(t) = t$, then t is called right dense, and if $t > inf\mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. For $t \in \mathbb{T}$, the backward graininess function $\rho : \mathbb{T} \to \mathbb{R}^+ \cup \{0\}$ is $\nu(t) = t - \rho(t)$.

Definition 1.1. *A function $f : \mathbb{T} \to \mathbb{C}$ is ld-continuous if it is continuous at every left-dense point, and the right sided limit exists at every right dense point of $\mathbb{T}$. It is expressed as $f \in C_{ld}(\mathbb{T}, \mathbb{C})$.*

Note that, if $f, g \in C_{ld}(\mathbb{T}, \mathbb{C})$, then $f \oplus_\nu g = f + g - \nu fg$ and $\ominus_\nu f = \frac{-f}{1-\nu f}$.

Definition 1.2. *A function $f \in C_{ld}(\mathbb{T}, \mathbb{C})$ is called $\nu-$regressive (positively $\nu-$regressive) if $1 - \nu h \neq 0$ $(1 - \nu h \geq 0)$ for all $t \in \mathbb{T}_k$. The set of all $\nu-$regressive and positively $\nu-$regressive functions are denoted by $\mathscr{R}_\nu(\mathbb{T}, \mathbb{C})$ and $\mathscr{R}_\nu^+(\mathbb{T}, \mathbb{C})$, respectively. Here $\mathbb{T}_k = \mathbb{T} - m$, if time scale $\mathbb{T}$ has a right scattered minimum m, otherwise take $\mathbb{T}_k = \mathbb{T}$.*

For $h > 0$, we have $\mathbb{C}_h = \{z \in \mathbb{C} : z \neq \frac{1}{h}\}$ and $\mathbb{Z}_h = \{z \in \mathbb{C} : \frac{-\pi}{h} < Im(z) \leq \frac{\pi}{h}\}$, with $\mathbb{C}_0 = \mathbb{Z}_0 = \mathbb{C}$. Further, the sets $\mathcal{R}e_h(z) = \frac{1}{h}(1 - |1 - hz|)$ and $\mathcal{I}m_h(z) = \frac{1}{h}Arg(1 - hz)$ denotes Hilger real and imaginary parts of a complex number. Note that $\mathcal{R}e_0(z) = Re(z)$ and $\mathcal{I}m_0(z) = Im(z)$.

Definition 1.3. *If $f \in \mathscr{R}_\nu(\mathbb{T}, \mathbb{C})$, then the nabla exponential function is given by, $e_f(t, t_0) := exp\big[\int_{t_0}^t \xi_{\nu(\tau)}(f(\tau))\nabla\tau\big]$ for $t, t_0 \in \mathbb{T}$, where $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ is the $\nu-$cylindrical transformation given as $\xi_h(z) = \frac{-1}{h}Log(1 - zh)$.*

Definition 1.4. *Let $s, t \in \mathbb{T}$ and $\alpha, \beta > -1$ the time scale power functions $\hat{h}(t, s)$ are the nonnegative functions that satisfy the following.*

(i) $\int_s^t \hat{h}_\alpha(t, \rho(\tau))\hat{h}_\beta(\tau, s)\ \nabla\tau = \hat{h}_{\alpha+\beta+1}(t, s)$ *for* $t \geq s$.
(ii) $\hat{h}_0(s, t) = 1$ *for* $t \geq s$.
(iii) $\hat{h}_\alpha(t, t) = 0$ *for* $0 \leq \alpha \leq 1$.

Using the above definition, fractional order integrals and derivatives on time scales are defined as

Definition 1.5. *Let $a_1, a_2 \in \mathbb{T}$ and $[a_1, a_2]_\mathbb{T} = \mathbb{T} \cap [a_1, a_2]$ for an ld-continuous function $f : [a_1, a_2]_\mathbb{T} \to \mathbb{C}$ one defines.*

(i) *The fractional integral of order $\alpha > 0$ with lower limit a_1 is*

$$_{a_1}\nabla^{-\alpha} f(t) := \int_{a_1}^{t} \hat{h}_{\alpha-1}(t,\rho(s)) f(s) \nabla s$$

when $\alpha = 0$, we get $_{a_1}\nabla^{0} f(t) = f(t)$.

(ii) *The Riemann-Liouville fractional derivative of order $\beta > 0$ with lower limit a_1 is given by,*

$$_{a_1}\nabla^{\beta} f(t) = [_{a_1}\nabla^{-(n-\beta)} f]^{\nabla^n}(t) \ for \ t \in [\sigma(a_1), a_2]_{\mathbb{T}}$$

where $n = [\beta] + 1$.

(iii) *The Liouville-Caputo-fractional derivative of order $\gamma > 0$ with lower limit a_1 is given by,*

$$^{c}_{a_1}\nabla^{\gamma} f(t) = (_{a_1}\nabla^{-(n-\gamma)} f^{\nabla^n})(t) \ for \ t \in [\sigma(a_1), a_2]_{\mathbb{T}}$$

where $n = [\gamma] + 1$.

Definition 1.6. *Suppose $\tau \in \mathbb{T}$, then an ld-continuous function $f : \mathbb{T} \to \mathbb{C}$, is said to be of exponential order c on $[\tau, \infty)_{\mathbb{T}}$ with $c \in \mathscr{R}^{+}_{\gamma}([\tau,\infty)_{\mathbb{T}}, \mathbb{C})$ if there exists a constant $M > 0$, such that $|f(t)| \leq M e_c(t,\tau)$ for $t \in [\tau,\infty)_{\mathbb{T}}$.*

Definition 1.7. *The minimal graininess function $\nu_\star : \mathbb{T} \to [0, \infty)$*

$$\nu_\star(\tau) := \inf_{t \in [\tau,\infty)} \nu(t) \ for \ \tau \in \mathbb{T}.$$

Also if $h \geq 0$ and $\delta \in \mathbb{R}$ then

$$\mathbb{C}_h(\delta) := \Big\{ z \in \mathbb{C}_h : \mathcal{R}_h(z) > \delta \Big\}.$$

Lemma 1.1. *If $f \in \mathscr{R}_\nu(\mathbb{T}, \mathbb{C})$, then $e^{\rho}_{\ominus f}(t,t_0) = \frac{e_{\ominus f}(t,t_0)}{1-\nu(t) f}$.*

Theorem 1.1 (Decay of nabla-exponential function)**.** *For an unbounded time scale $\mathbb{T}$, let $s \in \mathbb{T}$ and $\omega \in \mathcal{R}^{+}_{\nu}([s,\infty)_{\mathbb{T}}, \mathbb{R})$. Then for any $z \in \mathbb{C}_{\nu_\star(s)}(\omega)$ we have the following properties.*

(1) $|e_{\omega \ominus_\nu z}(t,s)| \leq e_{\omega \ominus_\nu \mathcal{R} e_{\nu_\star(s)}(z)}(t,s)$ *for all $t \in [s,\infty)_{\mathbb{T}}$.*
(2) $\lim\limits_{t\to\infty} e_{\omega \ominus_\nu \mathcal{R} e_{\nu_\star(s)}(z)}(t,s) = 0$.
(3) $\lim\limits_{t\to\infty} e_{\omega \ominus_\nu z}(t,s) = 0$.

2. Nabla Shehu Transform

In this section, we define the nabla Shehu transform and give its existence. Hence,fourth, we are going to assume that $t_0 \in \mathbb{T}$ and *sup* $\mathbb{T} = \infty$.

Definition 2.8 (Nabla Shehu Transform)**.** *For $f : \mathbb{T} \to \mathbb{C}$ with $f \in C_{ld}(\mathbb{T}, \mathbb{C})$, we define nabla-Shehu transform of f as*

$$Sh_{\nabla}\{f\}(s,u) := \int_{t_0}^{\infty} e^{\rho}_{\ominus_\nu \frac{s}{u}}(t,t_0) f(t) \ \nabla t \tag{1}$$

for all $\frac{s}{u} \in \mathscr{D}_\nu\{f\}$, where $\mathscr{D}_\nu\{f\}$ consists of all $\frac{s}{u} \in \mathscr{R}_\nu(\mathbb{T}, \mathbb{C})$ for which the improper integral exists.

The below lemma can be proved similarly as lemma 2.1 in [13].

Lemma 2.2. *Suppose $\tau \in \mathbb{T}$ and $f : [\tau,\infty)_{\mathbb{T}} \to \mathbb{C}$ is an ld-continuous function of exponential order c, then $\lim\limits_{t\to\infty} f(t) e_{\ominus_\nu \frac{s}{u}}(t,\tau) = 0$, where $\frac{s}{u} \in \mathbb{C}_{\nu_\star(\tau)}(c)$.*

Theorem 2.2 (Existence theorem). *Let $f \in C_{ld}([t_0,\infty)_{\mathbb{T}},\mathbb{C})$ be a function of the exponential order c, then the nabla Shehu transform of f exists and the integral*

$$\int_{t_0}^{\infty} e^{\rho}_{\ominus_\nu \frac{s}{u}}(t,t_0) f(t)\nabla t$$

converges absolutely for all $\frac{s}{u} \in \mathbb{C}_{\nu_\star(t_0)}(c)$.

Proof. For some fixed $z \in \mathbb{C}$, the Hilger real part [3] is a non decreasing function of $h \geq 0$,

$$\text{then } Re_{\nu(t)}\left(\frac{s}{u}\right) \geq Re_{\nu_\star(t_0)}\left(\frac{s}{u}\right) \text{ for all } t \in [t_0,\infty)_{\mathbb{T}}.$$

We get the inequality

$$\left|1-\nu\frac{s}{u}\right| \geq 1-\nu Re_{\nu_\star(t_0)}\left(\frac{s}{u}\right) \text{ for all } t \in [t_0,\infty)_{\mathbb{T}}.$$

Also for $t \in [t_0,\infty)_{\mathbb{T}}$ and $\frac{s}{u} \in \mathbb{C}_{\nu_\star(t_0)}(c)$ using Definition 1.6 and Lemma 1.1 we get

$$\begin{aligned}
\left|\int_{t_0}^{\tau} e^{\rho}_{\ominus_\nu(\frac{s}{u})}(t,t_0) f(t)\nabla t\right| &\leq \int_{t_0}^{\tau} \left|e^{\rho}_{\ominus_\nu(\frac{s}{u})}(t,t_0) f(t)\right| \nabla t \\
&\leq M\int_{t_0}^{\tau} \left|e_c(t,t_0) e^{\rho}_{\ominus_\nu \frac{s}{u}}(t,t_0)\right| \nabla t \\
&= M\int_{t_0}^{\tau} \frac{e_{c\ominus_\nu \frac{s}{u}}(t,t_0)}{\left|1-\nu\frac{s}{u}\right|} \nabla t \\
&\leq M\int_{t_0}^{\tau} \frac{e_{c\ominus_\nu Re_{\nu_\star(t_0)}(\frac{s}{u})}(t,t_0)}{1-\nu Re_{\nu_\star(t_0)}\left(\frac{s}{u}\right)} \nabla t \\
&= \frac{M}{c-Re_{\nu_\star(t_0)}\left(\frac{s}{u}\right)} \int_{t_0}^{\tau} e^{\nabla}_{c\ominus_\nu Re_{\nu_\star(t_0)}(\frac{s}{u})}(t,t_0)\ \nabla t \\
&= \frac{M}{c-Re_{\nu_\star(t_0)}\left(\frac{s}{u}\right)} \left[e_{c\ominus_\nu Re_{\nu_\star(t_0)}(\frac{s}{u})}(\tau,t_0)-1\right] \\
&= \frac{M}{Re_{\nu_\star(t_0)}\left(\frac{s}{u}\right)-c} \left[1-e_{c\ominus_\nu Re_{\nu_\star(t_0)}(\frac{s}{u})}(\tau,t_0)\right]
\end{aligned}$$

using Theorem 1.1 as $\tau \to \infty$ we get

$$\left|\int_{t_0}^{\infty} e^{\rho}_{\ominus_\nu \frac{s}{u}}(t,t_0)\ f(t)\ \nabla t\right| \leq \frac{M}{Re_{\nu_\star(t_0)}-c} \text{ for all } \frac{s}{u} \in \mathbb{C}_{\nu_\star(t_0)}(c)$$

thus the integral $\int_{t_0}^{\infty} e^{\rho}_{\ominus_\nu \frac{s}{u}}(t,t_0)\ \nabla t$ converges absolutely for all $\frac{s}{u} \in \mathbb{C}_{\nu_\star(t_0)}(c)$. □

3. Fundamental Properties

Now, we give some fundamental properties of the nabla Shehu transform through the following theorems.

Theorem 3.3 (Linearity). *Let $\alpha f : \mathbb{T} \to \mathbb{C},\ \beta g : \mathbb{T} \to \mathbb{C}$ with $f,g \in C_{ld}(\mathbb{T},\mathbb{C})$ are of exponential order c_1, c_2 respectively then for any $\alpha,\beta \in \mathbb{R}$ then we have,*

$$Sh_{\nabla}\{\alpha f+\beta g\}(s,u) = \alpha Sh_{\nabla}\{f\}(s,u)+\beta Sh_{\nabla}\{g\}(s,u)$$

for all $\frac{s}{u} \in \mathbb{C}_{\nu_\star(t_0)}(max\{c_1,c_2\})$.

The proof follows directly from Definition 2.8.

Theorem 3.4 (Shehu transform of derivative)**.** *Let* $f:\mathbb{T}\to\mathbb{C}$ *with* $f, f^{\nabla}\in C_{ld}(\mathbb{T},\mathbb{C})$ *be a function of exponential order c then,*

$$Sh_{\nabla}\{f^{\nabla}\}(s,u)=\tfrac{s}{u}Sh_{\nabla}\{f\}(s,u)-f(t_0)\ \textit{for all}\ \tfrac{s}{u}\in\mathbb{C}_{\nu_{\star}(t_0)}(c).$$

Proof. Applying Definition 2.8,

$$\begin{aligned}
Sh_{\nabla}\{f^{\nabla}\}(s,u) &= \int_{t_0}^{\infty} e^{\rho}_{\ominus\frac{s}{u}}(t,t_0)f^{\nabla}(t)\ \nabla t\\
&= \int_{t_0}^{\infty}\left[e_{\ominus\frac{s}{u}}(t,t_0)f(t)\right]^{\nabla}-e^{\nabla}_{\ominus\frac{s}{u}}(t,t_0)f(t)\ \nabla t\\
&= -f(t_0)-\int_{t_0}^{\infty} e^{\nabla}_{\ominus\frac{s}{u}}(t,t_0)f(t)\ \nabla t\\
&= -f(t_0)+\frac{s}{u}\int_{t_0}^{\infty} f(t)\frac{e_{\ominus\frac{s}{u}(t,t_0)}}{1-\nu(t)\frac{s}{u}}\ \nabla t\\
&= -f(t_0)+\frac{s}{u}\int_{t_0}^{\infty} f(t)e^{\rho}_{\ominus\frac{s}{u}}(t,t_0)\ \nabla t\\
&= -f(t_0)+\frac{s}{u}Sh_{\nabla}\{f\}(s,u).
\end{aligned}$$

□

From the above theorem, we have if $f:\mathbb{T}\to\mathbb{C}$ is ld-continuous such that $f^{\nabla\nabla}:\mathbb{T}\to\mathbb{C}$ is also ld-continuous and is of exponential order c, then

$$Sh_{\nabla}\{f^{\nabla\nabla}\}(s,u)=\tfrac{s^2}{u^2}Sh_{\nabla}\{f\}(s,u)-\tfrac{s}{u}f(t_0)-f^{\nabla}(t_0).$$

Hence in general we can show that

$$Sh_{\nabla}\{f^{\nabla^n}\}(s,u)=\frac{s^n}{u^n}Sh_{\nabla}\{f\}(s,u)-\sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-(k+1)}f^{\nabla^k}(t_0)$$

for all $\frac{s}{u}\in\mathbb{C}_{\nu_{\star}(t_0)}(c)$.

The theorem, which relates transform of integral of a function to the transform of a function can be proved as theorem 3.3 in [14].

Theorem 3.5. *Let* $f:\mathbb{T}\to\mathbb{C}$ *is ld-continuous function such that* $F(t)=\int_{t_0}^{t}f(\tau)\nabla\tau$ *for all* $t\in\mathbb{T}$ *is ld-continuous and is of exponential order c, then*

$$Sh_{\nabla}\{F\}(s,u)=\frac{u}{s}Sh_{\nabla}\{f\}(s,u).$$

Using our Definition 2.8 we can show that.

(1) If $f(t)\equiv 1$, then $Sh_{\nabla}\{1\}=\frac{u}{s}$ provided $\lim_{t\to\infty}e_{\ominus\frac{s}{u}}(t,t_0)=0$ for $\frac{s}{u}\in\mathscr{R}_{\nu}(\mathbb{T},\mathbb{C})$.

(2) If $f(t)=h_n(t,t_0)$, Taylor's monomials introduced in [3] for $n\in\mathbb{N}\cup\{0\}$ then

$$Sh_{\nabla}\{h_n(t,t_0)\}(s,u)=\frac{1}{\left(\frac{s}{u}\right)^{n+1}}$$

with $\frac{s}{u}\in\mathscr{R}_{\nu}(\mathbb{T},\mathbb{C}), s\neq 0, u\neq 0$ with $\lim_{t\to\infty}h_n(t,t_0)e_{\ominus\frac{s}{u}}(t,t_0)=0$.

(3) If $f(t) = \hat{h}_\alpha(t,s)$, is the time scale power functions defined in [13] then

$$Sh_\nabla\{\hat{h}_\alpha(t,t_0)\}(s,u) = \frac{1}{\left(\frac{s}{u}\right)^{\alpha+1}}.$$

Convolutions of two functions on time scales is given in the following definition.

Definition 3.9. [26] *For functions $f, g : \mathbb{T} \to \mathbb{C}$, the convolution $f * g$ of f and g is defined as follows*

$$(f * g)(t) = \int_{t_0}^{\infty} \hat{f}(t, \rho(\tau))g(\tau)\ \nabla\tau \ \ for\ t \in \mathbb{T},$$

where for a given $f : [t_0, \infty) \to \mathbb{C}$, $\hat{f}$ called shift (or decay) is the solution of the Shifting problem.

$$\begin{aligned} u^{\nabla_t}(t, \rho(s)) \ &= -u^{\nabla_s}(t,s)\ t, s \in \mathbb{T}\ t \geq s \geq t_0, \\ u(t,t_0) &= f(t)\ t \in \mathbb{T},\ t \geq t_0. \end{aligned}$$

Using above definition of convolution we are ready to give convolution theorem for nabla Shehu transform as follows.

Theorem 3.6 (The Convolution Theorem)**.** *Let $f, g : \mathbb{T} \to \mathbb{C}$ be ld-continuous functions having the nabla Shehu transforms $Sh_\nabla\{f\}$ and $Sh_\nabla\{g\}$ respectively, then*

$$Sh_\nabla\{f * g\}(s,u) = Sh_\nabla\{f\}(s,u) \cdot Sh_\nabla\{g\}(s,u)$$

for all $\frac{s}{u} \in \mathscr{D}_\nu\{f\} \cap \mathscr{D}_\nu\{g\}$.

Proof. We have,

$$\begin{aligned} Sh_\nabla\{f * g\}(s,u) &= \int_{t_0}^{\infty} e^{\rho}_{\ominus_\nu \frac{s}{u}}(t,t_0)(f(t) * g(t))\ \nabla t \\ &= \int_{t_0}^{\infty} e_{\ominus_\nu \frac{s}{u}}(\rho(t),t_0) \int_{t_0}^{t} \hat{f}(t,\rho(\tau))g(\tau)\ \nabla\tau \\ &= \int_{t_0}^{\infty} g(\tau)e_{\ominus_\nu \frac{s}{u}}(\rho(\tau),t_0)\Big[\int_{\rho(\tau)}^{\infty} e(\rho(t),\rho(\tau))\hat{f}(t,\rho(\tau))\nabla t\Big]\nabla\tau \\ &= \int_{t_0}^{\infty} g(\tau)e_{\ominus_\nu \frac{s}{u}}(\rho(\tau),t_0)\ \psi(\rho(\tau))\ \nabla\tau \end{aligned}$$

where, $\psi(\tau) = \int_\tau^\infty e_{\ominus_\nu \frac{s}{u}}(\rho(t),\tau)\hat{f}(t,\tau)\nabla t$.
Now, applying the nabla-versions of lemma 2.4 and lemma 3.3 from [6], it is clear that $\psi(\tau)$ is a constant function, and hence is independent of τ. So we have,

$$\psi(t_0) = \int_{t_0}^{\infty} e_{\ominus_\nu \frac{s}{u}}(\rho(t),t_0)\hat{f}(t,t_0)\ \nabla t = \int_{t_0}^{\infty} e_{\ominus_\nu \frac{s}{u}}(\rho(t),t_0)f(t)\nabla t = Sh_\nabla\{f\}.$$

And we get $Sh_\nabla\{f * g\}(s,u) = Sh_\nabla\{f\} \cdot Sh_\nabla\{g\}$. □

Theorem 3.7 (Shehu Transform Of Fractional Integral)**.** *For $a = t_0$, the Shehu transform of fractional integral ${}_{t_0}\nabla^{-\alpha} f$ is $Sh_\nabla\{{}_{t_0}\nabla^{-\alpha} f\}(s,u)$, which is given as*

$$Sh_\nabla\{{}_{t_0}\nabla^{-\alpha} f\}(s,u) = \left(\frac{s}{u}\right)^{-\alpha} Sh_\nabla\{f(t)\}(s,u).$$

Proof. First of all note that for $a = t_0$, Definition 1.5 (i) can be rewritten as

$$\begin{aligned} {}_{t_0}\nabla^{-\alpha} f(t) &= \int_{t_0}^{t} \hat{h}_{\alpha-1}(t, \rho(\tau)) f(\tau)\ \nabla\tau. \\ {}_{t_0}\nabla^{-\alpha} f(t) &= \hat{h}_{\alpha-1}(t, t_0) * f(t) \\ Sh_{\nabla}\{{}_{t_0}\nabla^{-\alpha} f\}(s,u) &= Sh_{\nabla}\{\hat{h}_{\alpha-1}(t,t_0) * f(t)\}(s,u) \\ &= Sh_{\nabla}\{\hat{h}_{\alpha-1}(t,t_0)\}(s,u) \cdot Sh_{\nabla}\{f(t)\}(s,u) \\ &= \frac{1}{\left(\frac{s}{u}\right)^{\alpha}} Sh_{\nabla}\{f(t)\}(s,u) \\ &= \left(\frac{s}{u}\right)^{-\alpha} Sh_{\nabla}\{f(t)\}(s,u). \end{aligned}$$

□

Theorem 3.8 (Shehu Transform of Riemann-Liouville Fractional Derivative)**.** *For $a = t_0$, the Shehu transform of Riemann-Liouville Fractional derivative ${}_{t_0}\nabla^{\beta} f$ is $Sh_{\nabla}\{{}_{t_0}\nabla^{\beta} f\}(s,u)$ which is given as*

$$Sh_{\nabla}\{{}_{t_0}\nabla^{\beta} f\}(s,u) = \left(\frac{s}{u}\right)^{\beta} Sh_{\nabla}\{f\}(s,u) - \sum_{j=1}^{n} \left(\frac{s}{u}\right)^{j-1} (\ {}_{t_0}\nabla^{\beta-j} f)(t_0).$$

Proof. We have,

$$\begin{aligned} Sh_{\nabla}\{{}_{t_0}\nabla^{\beta} f\}(s,u) &= Sh_{\nabla}\{[\ {}_{t_0}\nabla^{-(n-\beta)} f]^{\nabla^n}\}(s,u) \\ &= \frac{s^n}{u^n} Sh_{\nabla}\{\ {}_{t_0}\nabla^{-(n-\beta)} f\}(s,u) - \sum_{j=0}^{n-1} \left(\frac{s}{u}\right)^{n-(j+1)} \left(\ {}_{t_0}\nabla^{-(n-\beta)} f\right)^{\nabla^j}(t_0) \\ &= \frac{s^n}{u^n} \left[\frac{1}{\left(\frac{s}{u}\right)^{n-\beta}} Sh_{\nabla}\{f\}(s,u) \right] - \sum_{j=0}^{n-1} \left(\frac{s}{u}\right)^{n-j-1} \left(\ {}_{t_0}\nabla^{j-n+\beta} f\right)(t_0) \\ &= \left(\frac{s}{u}\right)^{\beta} Sh_{\nabla}\{f\}(s,u) - \sum_{j=1}^{n} \left(\frac{s}{u}\right)^{(j-1)} \left(\ {}_{t_0}\nabla^{\beta-j} f\right)(t_0) \end{aligned}$$

where $\left(\ {}_{t_0}\nabla^{-(n-\beta)} f\right)^{\nabla^j}(t_0) = \left(\ {}_{t_0}\nabla^{j-n+\beta} f\right)(t_0)$ follows from the definition of Riemann-Liouville fractional $\nabla-$derivative, also note that $\left(\ {}_{t_0}\nabla^{\beta-n} f\right)(t_0) = \lim_{t\to t_0} \left(\ {}_{t_0}\nabla^{-(n-\beta)} f\right)(t_0)$ (t_0 is right dense) and $\left({}_{t0}\nabla^{\beta-1} f\right)(t_0) = 0$ (t_0 is right scattered). □

Theorem 3.9 (Shehu Transform of Liouville-Caputo Fractional Derivative)**.** *For $a = t_0$, the Shehu transform of the Liouville-Caputo fractional derivative ${}^{c}_{t_0}\nabla^{\nu} f$ is $Sh_{\nabla}\{\ {}^{c}_{t_0}\nabla^{\nu} f\}(s,u)$, which is given as*

$$Sh_{\nabla}\{\ {}^{c}_{t_0}\nabla^{\nu} f\}(s,u) = \left(\frac{s}{u}\right)^{\nu} Sh_{\nabla}\{f\}(s,u) - \sum_{j=0}^{n-1} \left(\frac{s}{u}\right)^{\nu-j-1} (f^{\nabla^j})(t_0).$$

Proof.

$$
\begin{aligned}
Sh_{\nabla}\{\,{}^{c}_{t_0}\nabla^{\nu} f\}(s,u) &= Sh_{\nabla}\{\,{}_{t_0}\nabla^{-(n-\nu)}[f^{\nabla^n}]\}(s,u) \\
&= \frac{1}{\left(\frac{s}{u}\right)^{n-\nu}} Sh_{\nabla}\{f^{\nabla^n}\}(s,u) \\
&= \frac{1}{\left(\frac{s}{u}\right)^{n-\nu}} \left[\frac{s^n}{u^n} Sh_{\nabla}\{f\}(s,u) - \sum_{j=0}^{n-1}\left(\frac{s}{u}\right)^{n-(j+1)} (f^{\nabla^j})(t_0)\right] \\
&= \left(\frac{s}{u}\right)^{\nu} Sh_{\nabla}\{f\}(s,u) - \sum_{j=0}^{n-1}\left(\frac{s}{u}\right)^{\nu-j-1} (f^{\nabla^j})(t_0).
\end{aligned}
$$

□

Solutions of fractional dynamic equations on time scales involve the fractional order case of the exponential function e^z called the Mittag-Leffler function $E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha k+\beta)}$ introduced in [7, 13, 25, 26] .

Definition 3.10. [25] *The time scale $\nabla-$Mittag-Leffler function is defined as,*

$$
{}_{\nabla}E_{\alpha,\beta}(\lambda,t,t_0) = \sum_{j=0}^{\infty} \lambda^j \ \hat{h}_{\alpha j+\beta-1}(t,t_0)
$$

for $\alpha > 0, \beta, \lambda \in \mathbb{R}$.

Now we give the nabla Shehu transform of $\nabla-$Mittag-Leffler function through the following theorem.

Theorem 3.10. *Shehu transform of time scale $\nabla-$ Mittag-Leffler function is*

$$
Sh_{\nabla}\{{}_{\nabla}E_{\alpha,\beta}(\lambda,t,t_0)\} = \frac{\left(\frac{s}{u}\right)}{\left(\frac{s}{u}\right)^{\alpha} - \lambda}
$$

provided $\left|\lambda\left(\frac{s}{u}\right)^{-\alpha}\right| < 1$.

Proof.

$$\begin{aligned}
Sh_{\nabla}\{{}_{\nabla}E_{\alpha,\beta}(\lambda,t,t_0)\} &= Sh_{\nabla}\ \Big\{\sum_{j=0}^{\infty}\lambda^j \hat{h}_{\alpha j+\beta-1}(t,t_0)\Big\}(s,u)\\
&= \sum_{j=0}^{\infty}\lambda^j Sh_{\nabla}\{\hat{h}_{\alpha j+\beta-1}(t,t_0)\}(s,u)\\
&= \sum_{j=0}^{\infty}\lambda^j\ \frac{1}{\left(\frac{s}{u}\right)^{\alpha j+\beta}}\\
&= \frac{1}{\left(\frac{s}{u}\right)^{\beta}}\ \Big\{1+\frac{\lambda}{\left(\frac{s}{u}\right)^{\alpha}}\frac{\lambda^2}{\left(\frac{s}{u}\right)^{2\alpha}}+\cdots\Big\}\\
&= \frac{1}{\left(\frac{s}{u}\right)^{\beta}}\ \sum_{j=0}^{\infty}\frac{\lambda^j}{\left(\frac{s}{u}\right)^{\alpha j}}\\
&= \frac{1}{\left(\frac{s}{u}\right)^{\beta}}\left[\frac{1}{1-\frac{\lambda}{\left(\frac{s}{u}\right)^{\alpha}}}\right] \qquad \text{provided } \Big|\lambda\Big(\frac{s}{u}\Big)^{-\alpha}\Big|<1 \text{ for convergence of geometric series,}\\
&= \frac{1}{\left(\frac{s}{u}\right)^{\beta}}\left[\frac{\left(\frac{s}{u}\right)^{\alpha}}{\left(\frac{s}{u}\right)^{\alpha}-\lambda}\right]\\
&= \frac{\left(\frac{s}{u}\right)^{\alpha-\beta}}{\left(\frac{s}{u}\right)^{\alpha}-\lambda}.
\end{aligned}$$

□

4. Applications

Now in this section, we present some applications of the nabla Shehu transforms for solving fractional dynamic equations involving Riemann-Liouville and Liouville-Caputo type fractional derivative.

Example 4.1. *Consider following Riemann-Liouville type fractional dynamic equation of order* $\alpha,\ n-1<\alpha\leq n$

$$\big(\ {}_{t_0}\nabla^{\alpha}y\big)(t)-\lambda y(t)=h(t),\ \ t\in\mathbb{T},$$

with initial conditions

$$\big(\ {}_{t_0}\nabla^{\alpha-k}\ y\big)(t_0)=a_k\ \ (a_k\in\mathbb{R},\ \ k=1,2,3,\cdots,n=[\alpha]).$$

Applying the nabla Shehu transform on both sides

$$\begin{aligned}
\Big(\frac{s}{u}\Big)^{\alpha}\ Sh_{\nabla}\{y\}(s,u) - \sum_{k=1}^{n}\Big(\frac{s}{u}\Big)^{k-1}\big(\ {}_{t_0}\nabla^{\alpha-k}y\big)(t_0) - \lambda\ Sh_{\nabla}\{y\}(s,u) &= Sh_{\nabla}\{h\}(s,u)\\
\Big(\frac{s}{u}\Big)^{\alpha}\ Sh_{\nabla}\{y\}(s,u) - \sum_{k=1}^{n}\Big(\frac{s}{u}\Big)^{k-1}\ a_k - \lambda\ Sh_{\nabla}\{y\}(s,u) &= Sh_{\nabla}\{h\}(s,u)\\
\Big[\Big(\frac{s}{u}\Big)^{\alpha} - \lambda\Big]\ Sh_{\nabla}\{y\}(s,u) &= \sum_{k=1}^{n}\Big(\frac{s}{u}\Big)^{k-1}\ a_k + Sh_{\nabla}\{h\}(s,u)\\
Sh_{\nabla}\{y\}(s,u) &= \frac{1}{\Big[\big(\frac{s}{u}\big)^{\alpha}-\lambda\Big]}\sum_{k=1}^{n}\Big(\frac{s}{u}\Big)^{k-1}\ a_k + \frac{1}{\Big[\big(\frac{s}{u}\big)^{\alpha}-\lambda\Big]}Sh_{\nabla}\{h\}(s,u)\\
Sh_{\nabla}\{y\}(s,u) &= \sum_{k=1}^{n}\frac{\big(\frac{s}{u}\big)^{\alpha-(\alpha-k+1)}}{\Big[\big(\frac{s}{u}\big)^{\alpha}-\lambda\Big]}a_k + \frac{\big(\frac{s}{u}\big)^{\alpha-1}}{\Big[\big(\frac{s}{u}\big)^{\alpha}-\lambda\Big]}\ Sh_{\nabla}\{h\}(s,u)\\
Sh_{\nabla}\{y\}(s,u) &= \sum_{k=1}^{n} Sh_{\nabla}\{{}_{\nabla}E_{\alpha,\alpha-1+k}(\lambda,t,t_0)\}(s,u)\ a_k\\
&\quad + Sh_{\nabla}\{{}_{\nabla}E_{\alpha,\alpha}(\lambda,t,t_0)\}(s,u)\cdot Sh_{\nabla}\{h\}(s,u)\\
Sh_{\nabla}\{y\}(s,u) &= \sum_{k=1}^{n} Sh_{\nabla}\{{}_{\nabla}E_{\alpha,\alpha-k+1}(\lambda,t,t_0)\}(s,u)\ \cdot\ a_k\\
&\quad + Sh_{\nabla}\{{}_{\nabla}E_{\alpha,\alpha}(\lambda,t,t_0)\ *\ h(t)\}(s,u)\\
y(t) &= \sum_{k=1}^{n} {}_{\nabla}E_{\alpha-k+1}(\lambda,t,t_0)\cdot a_k\\
&\quad + {}_{\nabla}E_{\alpha,\alpha}(\lambda,t,t_0)\ *\ h(t)\\
y(t) &= \sum_{k=1}^{n} {}_{\nabla}E_{\alpha,\alpha-k+1}(\lambda,t,t_0)\cdot a_k\\
&\quad + {}_{\nabla}E_{\alpha,\alpha}(\lambda,t,t_0) * h(t)\\
y(t) &= \sum_{k=1}^{n} {}_{\nabla}E_{\alpha,\alpha-k+1}(\lambda,t,t_0)\cdot a_k\\
&\quad + \int_{t_0}^{t} {}_{\nabla}\hat{E}_{\alpha,\alpha}(\lambda,t,\rho(\tau))\cdot h(\tau)\ \nabla\tau.
\end{aligned}$$

Example 4.2. *Consider the following fractional Cauchy problem of order* $\alpha,\ 0 < \alpha \leq 1$,

$$\big(\ {}_{t_0}\nabla^{\alpha}y(t)\big) - \lambda y(t) = h(t),\ \ t \in \mathbb{T},$$

with initial conditions

$$\big(\ {}_{t_0}\nabla^{\alpha-1}y\big)(t_0) = a_0,\ \ (a_0 \in \mathbb{R}).$$

Applying the nabla Shehu transform on both sides

$$\begin{aligned}
\Big(\frac{s}{u}\Big)^{\alpha} Sh_{\nabla}\{y\}(s,u) - \{\ {}_{t_0}\nabla^{\alpha-1}y\}(t_0) - \lambda Sh_{\nabla}\{y\}(s,u) \\
= Sh_{\nabla}\{h\}(s,u) \\
\Big[\Big(\frac{s}{u}\Big)^{\alpha} - \lambda\Big] Sh_{\nabla}\{y\}(s,u) &= \{\ {}_{t_0}\nabla^{\alpha-1}y\}(t_0) + Sh_{\nabla}\{h\}(s,u) \\
Sh_{\nabla}\{y\}(s,u) &= \frac{1}{\Big[\big(\frac{s}{u}\big)^{\alpha} - \lambda\Big]}\ (\ {}_{t_0}\nabla^{\alpha}y)(t_0) + \frac{1}{\Big[\big(\frac{s}{u}\big)^{\alpha} - \lambda\Big]} Sh_{\nabla}\{h\}(s,u) \\
Sh_{\nabla}\{y\}(s,u) &= \frac{1}{\Big[\big(\frac{s}{u}\big)^{\alpha} - \lambda\Big]}\ a_0 + \frac{1}{\Big[\big(\frac{s}{u}\big)^{\alpha} - \lambda\Big]} Sh_{\nabla}\{h\}(s,u) \\
Sh_{\nabla}\{y\}(s,u) &= \frac{\big(\frac{s}{u}\big)^{\alpha-\alpha}}{\Big[\big(\frac{s}{u}\big)^{\alpha} - \lambda\Big]}\ a_0 + \frac{\big(\frac{s}{u}\big)^{\alpha-\alpha}}{\Big[\big(\frac{s}{u}\big)^{\alpha} - \lambda\Big]} Sh_{\nabla}\{h\}(s,u) \\
Sh_{\nabla}\{y\}(s,u) &= a_0\ Sh_{\nabla}\{{}_{\nabla}E_{\alpha,\alpha}(\lambda,t,t_0)\}(s,u) \\
&\quad + Sh_{\nabla}\{\ {}_{\nabla}E_{\alpha,\alpha}(\lambda,t,t_0)\}(s,u)\cdot Sh_{\nabla}\{h\}(s,u) \\
Sh_{\nabla}\{y\}(s,u) &= a_0\ Sh_{\nabla}\{\ {}_{\nabla}E_{\alpha,\alpha}(\lambda,t,t_0)\}(s,u) \\
&\quad + Sh_{\nabla}\{\ {}_{\nabla}E_{\alpha,\alpha}(\lambda,t,t_0) * h(t)\}(s,u) \\
y(t) &= a_0\ {}_{\nabla}E_{\alpha,\alpha}(\lambda,t,t_0)\ +\ {}_{\nabla}E_{\alpha,\alpha}(\lambda,t,t_0) * h(t) \\
y(t) &= a_0\ {}_{\nabla}E_{\alpha,\alpha}(\lambda,t,t_0) + \int_{t_0}^{t} {}_{\nabla}\hat{E}_{\alpha,\alpha}(\lambda,t,\rho(\tau)) * h(\tau)\ \nabla\tau.
\end{aligned}$$

Example 4.3. *Consider the following fractional Cauchy problem of order* $\alpha,\ 0 < \alpha \leq 1$

$$\big(\ {}^{c}_{t_0}\nabla^{\nu}y\big)(t)\ =\ g(t),\ t\in\mathbb{T},$$

with initial condition,

$$y(t_0) = b_0,\ b_0\in\mathbb{R}.$$

Applying nabla Shehu transform on both sides

$$\begin{aligned}
\Big(\frac{s}{u}\Big)^{\nu} Sh_{\nabla}\{y\}(s,u) - \Big(\frac{s}{u}\Big)^{\nu-1} y(t_0) = Sh_{\nabla}\{g\}(s,u) \\
\Big(\frac{s}{u}\Big)^{\nu} Sh_{\nabla}\{y\}(s,u) &= \Big(\frac{s}{u}\Big)^{\nu-1} b_0 + Sh_{\nabla}\{g\}(s,u) \\
Sh_{\nabla}\{y\}(s,u) &= \frac{\big(\frac{s}{u}\big)^{\nu-1}}{\big(\frac{s}{u}\big)^{\nu}} b_0 + \frac{1}{\big(\frac{s}{u}\big)^{\nu}} Sh_{\nabla}\{g\}(s,u) \\
Sh_{\nabla}\{y\}(s,u) &= \frac{1}{\big(\frac{s}{u}\big)^{\nu}} b_0 + \frac{1}{\big(\frac{s}{u}\big)^{\nu}} Sh_{\nabla}\{g\}(s,u) \\
Sh_{\nabla}\{y\}(s,u) &= Sh_{\nabla}\{1\}(s,u)\ b_0 + Sh_{\nabla}\{\hat{h}_{\nu-1}(t,t_0)\}(s,u)\cdot Sh_{\nabla}\{g\}(s,u)
\end{aligned}$$

$$Sh_{\nabla}\{y\}(s,u) = b_0 Sh_{\nabla}\{1\}(s,u) + Sh_{\nabla}\{\hat{h}_{\nu-1}(t,t_0) * g(t)\}(s,u)$$
$$y(t) = b_0 + \hat{h}_{\nu-1}(t,t_0) * g(t)$$
$$y(t) = b_0 + \int_{t_0}^{t} \hat{h}_{\nu-1}(t,\rho(t))g(t)\nabla\tau$$
$$y(t) = b_0 + {}_{t_0}\nabla^{-\nu} g(t).$$

5. Conclusions

We introduced the nabla Shehu transform, which generalizes the nabla Laplace and nabla Sumudu transforms on time scales. We discuss its existence conditions and provide some fundamental properties, including convolution theorem. Transforms of the fractional integral, Riemann-Liouville fractional derivative, Liouville-Caputo fractional derivative, time scale power function, and Mittag-Leffler function are found. We applied this transform to solve some fractional dynamic equations involving Riemann-Liouville and Liouville-Caputo type fractional derivatives.

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