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COEFFICIENT ESTIMATES FOR SUBCLASSES OF BI-UNIVALENT FUNCTIONS WITH PASCAL OPERATOR

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Abstract. In the present paper, we introduce two new subclasses of the function class Σ of bi-univalent functions defined in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. We find the bounds on the initial coefficients $|c_2|$ and $|c_3|$ and upper bounds for the Fekete-Szegö functional for the functions in this class.

1. Introduction, Definition and Preliminaries

Let $\mathcal{A}$ denote the class of normalized functions $g(z)$ of the form

$$g(z) = z + c_2 z^2 + c_3 z^3 + \cdots, \tag{1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$.

Also we let $\mathcal{S}$ to denote the subclass of functions $g \in \mathcal{A}$ which are univalent in $\mathbb{U}$.

An analytic function f is subordinate to an analytic function g, written $f(z) \prec g(z)$, provided there is a schwarz function w defined on $\mathbb{U}$ with

$$w(0) = 0 \,\text{and}\, |w(z)| < 1 \tag{2}$$

satisfying

$$f(z) = g(w(z)). \tag{3}$$

For the functions $g(z)$ of the form (1) and $h(z) = z + b_2 z^2 + b_3 z^3 + \cdots$, the Hadamard product (or convolution) of g and h is defined by

$$(g * h)(z) = z + \sum_{k=2}^{\infty} c_k\, b_k\, z^k.$$

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The Pascal distribution has been widely used in Communications and Engineering fields (see [11]). Recently, in geometric function theory, there has been a growing interest in studying the geometric properties of analytic functions associated with the Pascal distribution (see [5] , [8], [9], [11], [17]).

A variable ξ is said to be a Pascal (or Negative Binomial) distribution if it takes the values $0, 1, 2, 3, \ldots$ with probabilities

$$(1-q)^m, \; \frac{qm(1-q)^m}{1!}, \; \frac{q^2m(m+1)(1-q)^m}{2!} \cdots$$

respectively, where m and q are parameters, and hence

$$p(\xi = n) = \binom{n+m-1}{m-1} q^n (1-q)^m, \quad n = 0, 1, 2, 3, \ldots. \tag{4}$$

This distribution is based on the binomial theorem with a negative exponent and it describes the probability of m success and n failure in $(n+m-1)$ trials, and success on $(n+m)$th trials where $(1-q)$ is the probability of success.

Recently, El-Deeb et al. [19] defined and investigated the characterization of Pascal operator of the form

$$\Lambda_q^m g(z) = z + \sum_{l=2}^{\infty} \binom{l+m-2}{m-1} q^{l-1} c_l z^l \tag{5}$$

where $m \geq 1$, $0 \leq q < 1$.

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$.

Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1). The Koebe one-quarter theorem [7] ensures that the image of $\mathbb{U}$ under every univalent function $g \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$. Thus every univalent function g has an inverse g^{-1} satisfying $g^{-1}(g(z)) = z$,$(z \in \mathbb{U})$ and

$$g\left(g^{-1}(w)\right) = w, \left(|w| < r_0(g), r_0(g) \geq \frac{1}{4}\right)$$

where

$$g^{-1}(w) = w - c_2w^2 + (2c_2^2 - c_3)w^3 - (5c_2^3 - 5c_2c_3 + c_4)w^4 + \cdots. \tag{6}$$

The coefficient estimate problem for the class $\mathcal{S}$, known as the Bieberbach conjecture, is settled by de Branges [3], who proved that for a function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in the class $\mathcal{S}$, $|a_n| \leq n$, for $n = 2, 3, \cdots$, with equality only for the rotations of the Koebe function

$$K_0(z) = \frac{z}{(1-z)^2}.$$

For interesting subclasses of functions in the class Σ, see ([1],[2], [4],[6], [21]).

Lewin [15]investigated the class Σ of bi-univalent functions and showed that $|a_2| < 1.51$ for the functions belonging to Σ. Subsequently, Brannan and Clunie [4] conjectured that $|a_2| \leq \sqrt{2}$.

Motivated by the work of H. M. Srivastava et al. [20], construct a new subclass of bi-univalent functions governed by the Pascal distribution series. Then, we investigate the optimal bounds for the Taylor - Maclaurin coefficients $|c_2|$ and $|c_3|$ in our new subclass.

Definition 1.1. *A function $g(z)$ given by (1) is said to be in the class $S_{\Sigma,q}^{m}(\gamma,\eta)$ if the following conditions are satisfied:*

$$g \in \Sigma \quad and \quad \left|\arg\left(\frac{z(\Lambda_q^m g(z))'}{(1-\eta)z+\eta\Lambda_q^m g(z)}\right)\right| < \frac{\gamma\pi}{2}, \quad (0<\gamma\le 1;\ 0\le\eta\le 1;\ z\in\mathbb{U}) \tag{7}$$

and

$$\left|\arg\left(\frac{w(\Lambda_q^m \psi(w))'}{(1-\eta)w+\eta\Lambda_q^m \psi(w)}\right)\right| < \frac{\gamma\pi}{2}, \quad (0<\gamma\le 1;\ 0\le\eta\le 1;\ z\in\mathbb{U}) \tag{8}$$

where the function ψ is given by

$$\psi(w) = g^{-1}(w) = w - c_2w^2 + (2c_2^2 - c_3)w^3 - (5c_2^3 - 5c_2c_3 + c_4)w^4 + \cdots. \tag{9}$$

Definition 1.2. *A function $g(z)$ given by (1) is said to be in the class $M_{\Sigma,q}^{m}(\gamma,\eta)$ if the following conditions are satisfied:*

$$g \in \Sigma \quad and \quad \mathcal{R}\left(\frac{z(\Lambda_q^m g(z))'}{(1-\eta)z+\eta\Lambda_q^m g(z)}\right) > \beta, \quad (0\le\beta\le 1;\ 0\le\eta\le 1;\ z\in\mathbb{U}) \tag{10}$$

and

$$\mathcal{R}\left(\frac{w(\Lambda_q^m \psi(w))'}{(1-\eta)w+\eta\Lambda_q^m \psi(w)}\right) > \beta, \quad (0\le\beta\le 1;\ 0\le\eta\le 1;\ z\in\mathbb{U}) \tag{11}$$

where the function ψ is given by (6).

For specifying the values of parameters γ and η, one can obtained the following examples:

Example 1.1. *A function $g(z)$ given by (1) is said to be in the class $S_{\Sigma,q}^{m}(\gamma,\eta) = S_{\Sigma,q}^{m}(\gamma,1)$ if the following conditions are satisfied:*

$$g \in \Sigma \quad and \quad \left|\arg\left(\frac{z(\Lambda_q^m g(z))'}{\Lambda_q^m g(z)}\right)\right| < \frac{\gamma\pi}{2}, \quad (0<\gamma\le 1;\ z\in\mathbb{U}) \tag{12}$$

and

$$\left|\arg\left(\frac{w(\Lambda_q^m \psi(w))'}{\Lambda_q^m \psi(w)}\right)\right| < \frac{\gamma\pi}{2}, \quad (0<\gamma\le 1;\ z\in\mathbb{U}) \tag{13}$$

where the function ψ is given by where the function ψ is given by (9).

Example 1.2. *A function $g(z)$ given by (1) is said to be in the class $M_{\Sigma,q}^{m}(\gamma,\eta) = M_{\Sigma,q}^{m}(\gamma,1)$ if the following conditions are satisfied:*

$$g \in \Sigma \quad and \quad \mathcal{R}\left(\frac{z(\Lambda_q^m g(z))'}{\Lambda_q^m g(z)}\right) > \beta, \quad (0\le\beta\le 1;\ z\in\mathbb{U}) \tag{14}$$

and

$$\mathcal{R}\left(\frac{w(\Lambda_q^m \psi(w))'}{\Lambda_q^m \psi(w)}\right) > \beta, \quad (0\le\beta\le 1;\ z\in\mathbb{U}) \tag{15}$$

where the function ψ is given by (9).

Example 1.3. *A function $g(z)$ given by (1) is said to be in the class $S^m_{\Sigma,q}(\gamma,\eta) = S^m_{\Sigma,q}(\gamma,0)$ if the following conditions are satisfied:*

$$g \in \Sigma \quad \text{and} \quad \left|\arg\left(\Lambda^m_q g(z)\right)'\right| < \frac{\gamma\pi}{2}, \quad (0 < \gamma \leq 1;\ z \in \mathbb{U}) \tag{16}$$

and

$$\left|\arg\left(\Lambda^m_q \psi(w)\right)'\right| < \frac{\gamma\pi}{2}, \quad (0 < \gamma \leq 1;\ z \in \mathbb{U}) \tag{17}$$

where the function ψ is given by where the function ψ is given by (9).

Example 1.4. *A function $g(z)$ given by (1) is said to be in the class $M^m_{\Sigma,q}(\gamma,\eta) = M^m_{\Sigma,q}(\gamma,0)$ if the following conditions are satisfied:*

$$g \in \Sigma \quad \text{and} \quad \mathcal{R}\left(\Lambda^m_q g(z)\right)' > \beta, \quad (0 \leq \beta \leq 1;\ z \in \mathbb{U}) \tag{18}$$

and

$$\mathcal{R}\left(\Lambda^m_q \psi(w)\right)' > \beta, \quad (0 \leq \beta \leq 1;\ z \in \mathbb{U}) \tag{19}$$

where the function ψ is given by (9).

Lemma 1.1. *([18]) If $h \in \mathcal{P}$, then $|d_k| \leq 2$, for each k, where $\mathcal{P}$ is the family of all functions h, analytic in $\mathbb{U}$, for which*

$$\mathcal{R}\{h(z)\} > 0,$$

where

$$h(z) = 1 + d_1 z + d_2 z^2 + \cdots . \tag{20}$$

2. Coefficient estimates

This section provides estimates for the coefficients c_2, c_3 for functions belonging to the class $S^m_{\Sigma,q}(\gamma,\eta)$ and $M^m_{\Sigma,q}(\gamma,\eta)$.

Theorem 2.1. *Let $g \in \Sigma$ given by (1) belongs to the class $S^m_{\Sigma,q}(\gamma,\eta)$. Then*

$$|c_2| \leq \frac{2\gamma}{\sqrt{[2(\eta^2-2\eta)\gamma + (1-\gamma)(2-\eta)^2]\, m^2 q^2 + \gamma(3-\eta)m(m+1)q^2}}, \tag{21}$$

$$|c_3| \leq \frac{4\gamma}{(3-\eta)m(m+1)q^2} + \frac{4\gamma^2}{(2-\eta)^2 m^2 q^2}. \tag{22}$$

Proof. Let $g \in S^m_{\Sigma,q}(\gamma,\eta)$. From (7) and (8), we have

$$\frac{z(\Lambda^m_q g(z))'}{(1-\eta)z + \eta\Lambda^m_q g(z)} = [p(z)]^\gamma \tag{23}$$

and

$$\frac{w(\Lambda^m_q \psi(w))'}{(1-\eta)w + \eta\Lambda^m_q \psi(w)} = [q(w)]^\gamma, \tag{24}$$

where $p(z)$ and $q(w)$ in $\mathcal{P}$ and have the following forms:

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \tag{25}$$

and

$$q(z) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots . \tag{26}$$

respectively. Now, equating the coefficients in (23) and (24), we get

$$(2-\eta)mqc_2 = \gamma p_1, \tag{27}$$

$$(\eta^2-2\eta)m^2q^2c_2^2+(3-\eta)\frac{m(m+1)}{2}q^2c_3=\frac{1}{2}\left[\gamma(\gamma-1)p_1^2+2\gamma p_2\right], \tag{28}$$

$$-(2-\eta)mqc_2=\gamma q_1, \tag{29}$$

and

$$(\eta^2-2\eta)m^2q^2c_2^2+(3-\eta)(2c_2^2-c_3)\frac{m(m+1)}{2}q^2=\frac{1}{2}\left[\gamma(\gamma-1)q_1^2+2\gamma q_2\right] \tag{30}$$

From (27) and (29), we find that

$$c_2=\frac{\gamma p_1}{(2-\eta)mq}=\frac{-\gamma q_1}{(2-\eta)mq}, \tag{31}$$

which implies

$$p_1=-q_1 \tag{32}$$

and

$$2\left(2-\eta\right)^2m^2q^2c_2^2=\gamma^2\left(p_1^2+q_1^2\right). \tag{33}$$

Adding (28) and (30), we obtain

$$\left[2(\eta^2-2\eta)m^2q^2+(3-\eta)m(m+1)q^2\right]c_2^2=\frac{\gamma(\gamma-1)}{2}\left(p_1^2+q_1^2\right)+\gamma\left(p_2+q_2\right). \tag{34}$$

Substituting the value of $\left(p_1^2+q_1^2\right)$ from 33 in the RHS of (36), we get

$$\left[\left((\eta-2)^2+(\eta^2-4)\gamma\right)m+(3-\eta)\gamma(m+1)\right]mq^2c_2^2=\gamma^2(p_2+q_2). \tag{35}$$

For a simple computation from (31), (32), (34), and also applying Lemma 1.1, we get

$$|c_2|\leq\frac{2\gamma}{\sqrt{\left[2(\eta^2-2\eta)\gamma+(1-\gamma)(2-\eta)^2\right]m^2q^2+\gamma(3-\eta)m(m+1)q^2}}.$$

This gives the required bound on $|c_2|$.

Moreover, if we subtract (30) from (28), we have

$$m(m+1)(3-\eta)q^2\left(c_3-c_2^2\right)=\gamma\left(p_2-q_2\right)+\frac{\gamma(\gamma-1)}{2}\left(p_1^2-q_1^2\right). \tag{36}$$

It follows from (31), (32), (36) and Lemma 1.1 that

$$|c_3|\leq\frac{4\gamma}{(3-\eta)m(m+1)q^2}+\frac{4\gamma^2}{(2-\eta)^2m^2q^2}.$$

This completes the proof of Theorem 2.1. □

Putting $\eta=1$ in Theorem 2.1, we have the following corollary.

Corollary 2.0. *A function $g(z)$ given by (1) is said to be in the class $S^m_{\Sigma,q}(\gamma,\eta)$ $=S^m_{\Sigma,q}(\gamma,1)$. Then*

$$|c_2|\leq\frac{2\gamma}{\sqrt{(1-\gamma)m^2q^2+2\gamma mq^2}}\quad\text{and}\quad|c_3|\leq\frac{4\gamma^2}{m^2q^2}+\frac{2\gamma}{m(m+1)q^2}.$$

Putting $\eta=0$ in Theorem 2.1, we have the following corollary.

Corollary 2.0. *A function $g(z)$ given by (1) is said to be in the class $S^m_{\Sigma,q}(\gamma,\eta)$ $=S^m_{\Sigma,q}(\gamma,0)$. Then*

$$|c_2|\leq\frac{2\gamma}{\sqrt{(4-\gamma)m^2q^2+3\gamma mq^2}}\quad\text{and}\quad|c_3|\leq\frac{\gamma^2}{m^2q^2}+\frac{4\gamma}{m(m+1)q^2}.$$

Theorem 2.2. *Let $g \in \Sigma$ given by (1) belongs to the class $S^{m}_{\Sigma,q}(\beta,\eta)$. Then*

$$|c_2| \leq \sqrt{\frac{4(1-\beta)}{2(\eta^2-2\eta)m^2q^2+(3-\eta)m(m+1)q^2}}, \tag{37}$$

$$|c_3| \leq \frac{4(1-\beta)}{(3-\eta)m(m+1)q^2} + \frac{4(1-\beta)^2}{(2-\eta)^2m^2q^2}. \tag{38}$$

Proof. It follows from (10) and (11) that there exist $p,\, q \in \mathcal{P}$ such that

$$\frac{z(\Lambda_q^m g(z))'}{(1-\eta)z+\eta\Lambda_q^m g(z)} = \beta + (1-\beta)p(z) \tag{39}$$

and

$$\frac{w(\Lambda_q^m \psi(w))'}{(1-\eta)w+\eta\Lambda_q^m \psi(w)} = \beta + (1-\beta)q(w). \tag{40}$$

where $p(z)$ and $q(w)$ have the forms (25) and (26) respectively. Equating coefficients in (39) and (40), we get

$$(2-\eta)mqc_2 = (1-\beta)p_1, \tag{41}$$

$$(\eta^2-2\eta)m^2q^2c_2^2 + (3-\eta)\frac{m(m+1)}{2}q^2c_3 = (1-\beta)p_2, \tag{42}$$

$$-(2-\eta)mqc_2 = (1-\beta)q_1, \tag{43}$$

and

$$(\eta^2-2\eta)m^2q^2c_2^2 + (3-\eta)(2c_2^2-c_3)\frac{m(m+1)}{2}q^2 = (1-\beta)q_2. \tag{44}$$

From (41) and (43), we have

$$c_2 = \frac{(1-\beta)p_1}{(2-\eta)mq} = \frac{-(1-\beta)q_1}{(2-\eta)mq}, \tag{45}$$

which implies

$$p_1 = -q_1. \tag{46}$$

Also,

$$2(2-\eta)^2m^2q^2c_2^2 = (1-\beta)^2\left(p_1^2+q_1^2\right). \tag{47}$$

From (42) and (44), we get

$$\left[2(\eta^2-2\eta)m^2q^2+(3-\eta)m(m+1)q^2\right]c_2^2 = (1-\beta)\left[p_2+q_2\right]. \tag{48}$$

By applying (45), (46) and also using the Lemma 1.1, we obtain

$$|c_2| \leq \sqrt{\frac{4(1-\beta)}{2(\eta^2-2\eta)m^2q^2+(3-\eta)m(m+1)q^2}}.$$

This gives the bound on $|c_2|$. Next, in order to obtain the estimate on $|c_3|$, by subtracting (44) from (42), we get

$$(3-\eta)m(m+1)q^2\left(c_3-c_2^2\right) = (1-\beta)\left[p_2-q_2\right]. \tag{49}$$

It follows from (45), (49) and Lemma 1.1 that

$$|c_3| \leq \frac{4(1-\beta)}{(3-\eta)m(m+1)q^2} + \frac{4(1-\beta)^2}{(2-\eta)^2m^2q^2}.$$

This completes the proof of Theorem 2.2. □

Putting $\eta = 1$ in Theorem 2.1, we have the following corollary.

Corollary 2.0. *A function $g(z)$ given by (1) is said to be in the class $S^m_{\Sigma,q}(\beta,\eta) = S^m_{\Sigma,q}(\beta,1)$. Then*

$$|c_2| \leq \sqrt{\frac{2(1-\beta)}{mq}} \quad \textit{and} \quad |c_3| \leq \frac{4(1-\beta)}{2m(m+1)q^2} + \frac{4(1-\beta)^2}{m^2q^2}.$$

Putting $\eta = 0$ in Theorem 2.1, we have the following corollary.

Corollary 2.0. *A function $g(z)$ given by (1) is said to be in the class $S^m_{\Sigma,q}(\beta,\eta) = S^m_{\Sigma,q}(\beta,0)$. Then*

$$|c_2| \leq \sqrt{\frac{4(1-\beta)}{3m(m+1)q^2}} \quad \textit{and} \quad |c_3| \leq \frac{4(1-\beta)}{3m(m+1)q^2} + \frac{(1-\beta)^2}{m^2q^2}.$$

3. Fekete-Szegö Inequality for the Function Class $S^m_{\Sigma,q}(\gamma,\eta)$ and $S^m_{\Sigma,q}(\beta,\eta)$

In this section, our aim to provide the Fekete-Szego Inequality for the function classes defined by the previous section 2.

Theorem 3.3. *Let $g \in \Sigma$ given by (1) belongs to the class $S^m_{\Sigma,q}(\gamma,\eta)$. Then*

$$\left|c_3 - \vartheta c_2^2\right| \leq \begin{cases} \frac{2|\gamma|}{m(m+1)(3-\eta)q^2}, & \textit{if} \quad |\vartheta - 1| \leq \left|\frac{\left[(\eta-2)^2+(\eta^2-4)\gamma\right]m+\gamma(3-\eta)(m+1)}{\gamma(m+1)(3-\eta)}\right| \\ \frac{2|\gamma|^2|1-\vartheta|}{|[(\eta-2)^2+(\eta^2-4)\gamma]m+\gamma(3-\eta)(m+1)|}, & \textit{if} \quad |\vartheta - 1| \geq \left|\frac{\left[(\eta-2)^2+(\eta^2-4)\gamma\right]m+\gamma(3-\eta)(m+1)}{\gamma(m+1)(3-\eta)}\right|. \end{cases} \tag{50}$$

Proof. From (35) and (36), we have

$$[alignment] c_3 - \vartheta c_2^2 = (1-\vartheta)\frac{\gamma^2(p_2+q_2)}{[(\eta-2)^2+(\eta^2-4)\gamma]\, m^2q^2 + \gamma(3-\eta)m(m+1)q^2} + \frac{\gamma(p_2-q_2)}{(3-\eta)m(m+1)q^2}$$
$$= \gamma\left[\left(\Phi(\vartheta) + \frac{1}{(3-\eta)m(m+1)q^2}\right)p_2 + \left(\Phi(\vartheta) - \frac{1}{(3-\eta)m(m+1)q^2}\right)q_2\right], \tag{51}$$

where,

$$\Phi(\vartheta) = \frac{\gamma(1-\vartheta)}{[(\eta-2)^2+(\eta^2-4)\gamma]\, m^2q^2 + \gamma(3-\eta)m(m+1)q^2}. \tag{52}$$

Then

$$\left|c_3 - \vartheta c_2^2\right| \leq \begin{cases} \frac{2|\gamma|}{m(m+1)(3-\eta)q^2}, & \text{if} \quad 0 \leq |\Phi(\vartheta)| \leq \frac{1}{m(m+1)(3-\eta)q^2} \\ 2|\gamma|\,|\Phi(\vartheta)|, & \text{if} \quad |\Phi(\vartheta)| \geq \frac{1}{m(m+1)(3-\eta)q^2}. \end{cases} \tag{53}$$

Hence (50) can be easily obtained from (53). □

Corollary 3.0. *A function $g(z)$ given by (1) is said to be in the class $S^m_{\Sigma,q}(\gamma,\eta) = S^m_{\Sigma,q}(\gamma,1)$. Then*

$$\left|c_3 - \vartheta c_2^2\right| \leq \begin{cases} \frac{|\gamma|}{m(m+1)q^2}, & \textit{if} \quad |\vartheta - 1| \leq \left|\frac{(1-\gamma)m+2\gamma}{2\gamma(m+1)}\right| \\ \frac{2|\gamma|^2|1-\vartheta|}{|(1-\gamma)m+2\gamma|}, & \textit{if} \quad |\vartheta - 1| \geq \left|\frac{(1-\gamma)m+2\gamma}{2\gamma(m+1)}\right|. \end{cases} \tag{54}$$

Corollary 3.0. *A function $g(z)$ given by (1) is said to be in the class $S_{\Sigma,q}^{m}(\gamma,\eta) = S_{\Sigma,q}^{m}(\gamma,0)$. Then*

$$\left|c_3-\vartheta c_2^2\right| \leq \begin{cases} \frac{2|\gamma|}{3m(m+1)q^2}, & \text{if} \quad |\vartheta-1| \leq \left|\frac{(4-\gamma)m+3\gamma}{3\gamma(m+1)}\right| \\ \frac{2|\gamma|^2|1-\vartheta|}{|(4-\gamma)m+3\gamma|}, & \text{if} \quad |\vartheta-1| \leq \left|\frac{(4-\gamma)m+3\gamma}{3\gamma(m+1)}\right|. \end{cases} \tag{55}$$

Theorem 3.4. *Let $g \in \Sigma$ given by (1) belongs to the class $S_{\Sigma,q}^{m}(\beta,\eta)$. Then*

$$\left|c_3-\vartheta c_2^2\right| \leq \begin{cases} \frac{2(1-\beta)}{m(m+1)(3-\eta)q^2}, & \text{if} \quad |\vartheta-1| \leq \left|\frac{2\eta(\eta-2)m+(3-\eta)(m+1)}{(1-\beta)(m+1)(3-\eta)}\right| \\ \frac{4(1-\beta)^2|1-\vartheta|}{|[(\eta-2)^2+(\eta^2-4)\gamma]m+\gamma(3-\eta)(m+1)|}, & \text{if} \quad |\vartheta-1| \geq \left|\frac{2\eta(\eta-2)m+(3-\eta)(m+1)}{(1-\beta)(m+1)(3-\eta)}\right|. \end{cases} \tag{56}$$

Proof. From (47) and (49), we have

$$[alignment]c_3-\vartheta c_2^2 = (1-\vartheta)\frac{(1-\beta)^2(p_2+q_2)}{2\eta(\eta-2)m^2q^2+(3-\eta)m(m+1)q^2} + \frac{(1-\beta)(p_2-q_2)}{(3-\eta)m(m+1)q^2}$$
$$= (1-\beta)\left[\left(\Psi(\vartheta)+\frac{1}{(3-\eta)m(m+1)q^2}\right)p_2 + \left(\Phi(\vartheta)-\frac{1}{(3-\eta)m(m+1)q^2}\right)q_2\right], \tag{57}$$

where,

$$\Psi(\vartheta) = \frac{(1-\beta)(1-\vartheta)}{2\eta(\eta-2)m^2q^2+(3-\eta)m(m+1)q^2}. \tag{58}$$

Then

$$\left|c_3-\vartheta c_2^2\right| \leq \begin{cases} \frac{2(1-\beta)}{m(m+1)(3-\eta)q^2}, & \text{if} \quad 0 \leq |\Psi(\vartheta)| \leq \frac{1}{m(m+1)(3-\eta)q^2} \\ 4(1-\beta)^2\left|\Phi(\vartheta)\right|, & \text{if} \quad |\Psi(\vartheta)| \geq \frac{1}{m(m+1)(3-\eta)q^2}. \end{cases} \tag{59}$$

Hence (56) can be easily obtained from (59). □

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