Journal of Fractional Calculus and Applications
Vol. 15(1) Jan. 2024, No. 7
ISSN: 2090-5858(online).
ISSN 2090-584X (print)
http://jfca.journals.ekb.eg/

ON INTEGRAL REPRESENTATIONS OF (α, β, γ)-TYPE AND (α, β, γ)-WEAK TYPE OF MEROMORPHIC FUNCTION

SARMILA BHATTACHARYYA, TANMAY BISWAS, CHINMAY BISWAS

Abstract. In complex analysis, order and type are classical growth indicators of entire and meromorphic functions. During the past decades, several authors have made the close investigations on the properties of entire and meromorphic functions in different directions using the concepts of order, iterated p-order [8, 11], (p, q)-th order [6, 7], (p, q)-φ order [10] and achieved many valuable results. But in [3], Chyzhykov et al. showed that both definitions of iterated p-order and (p, q)-th order have the disadvantage that they do not cover arbitrary growth (see [3, Example 1.4]). They used more general scale, called the φ-order (see [3]). On the other hand, Heittokangas et al. [4] have introduced another new concept of φ-order of entire and meromorphic functions considering φ as subadditive function. Considering all these aspects, Belaïdi et al. [1, 2] have extended the above ideas and have introduced the definitions of (α, β, γ)-order and (α, β, γ)-type of entire and meromorphic functions. In this paper, we establish the integral representations of (α, β, γ)-type and (α, β, γ)-weak type of a meromorphic function. We also investigate their equivalence relation under some certain conditions.

1. Introduction

Throughout this article, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of entire and meromorphic functions which are available in [5, 9, 12, 13, 14] and therefore we do not explain those in details. For meromorphic function f, the Nevanlinna's characteristic function $T_f(r)$ is defined as

$$T_f(r) = N_f(r) + m_f(r),$$

2010 *Mathematics Subject Classification.* 30D35, 30D30.
Key words and phrases. Meromorphic function, (α, β, γ)-order, (α, β, γ)-lower order, (α, β, γ)-type, (α, β, γ)-weak type, integral representation.
Submitted October 7, 2023. Revised November 30, 2023.

where $m_f(r)$ and $N_f(r)$ are respectively called as the proximity function of f and the counting function of poles of f in $|z| \leq r$. For details about $T_f(r)$, $m_f(r)$ and $N_f(r)$ one may see [5, p.4].

First of all, let L be a class of continuous non-negative on $(-\infty, +\infty)$ function α such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x_0 \leq x \to +\infty$. We say that $\alpha \in L_1$, if $\alpha \in L$ and $\alpha(a+b) \leq \alpha(a)+\alpha(b)+c$ for all $a, b \geq R_0$ and fixed $c \in (0, +\infty)$. Further we say that $\alpha \in L_2$, if $\alpha \in L$ and $\alpha(x+O(1)) = (1+o(1))\alpha(x)$ as $x \to +\infty$. Finally, $\alpha \in L_3$, if $\alpha \in L$ and $\alpha(a+b) \leq \alpha(a)+\alpha(b)$ for all $a, b \geq R_0$, i.e., α is subadditive. Clearly $L_3 \subset L_1$.

Particularly, when $\alpha \in L_3$, then one can easily verify that $\alpha(mr) \leq m\alpha(r)$, $m \geq 2$ is an integer. Up to a normalization, subadditivity is implied by concavity. Indeed, if $\alpha(r)$ is concave on $[0, +\infty)$ and satisfies $\alpha(0) \geq 0$, then for $t \in [0, 1]$,

$$\begin{aligned} \alpha(tx) &= \alpha(tx + (1-t)\cdot 0) \\ &\geq t\alpha(x) + (1-t)\alpha(0) \geq t\alpha(x), \end{aligned}$$

so that by choosing $t = \frac{a}{a+b}$ or $t = \frac{b}{a+b}$, we obtain

$$\begin{aligned} \alpha(a+b) &= \frac{a}{a+b}\alpha(a+b) + \frac{b}{a+b}\alpha(a+b) \\ &\leq \alpha\left(\frac{a}{a+b}(a+b)\right) + \alpha\left(\frac{b}{a+b}(a+b)\right) \\ &= \alpha(a) + \alpha(b), \ a, b \geq 0. \end{aligned}$$

As a non-decreasing, subadditive and unbounded function, $\alpha(r)$ satisfies

$$\alpha(r) \leq \alpha(r + R_0) \leq \alpha(r) + \alpha(R_0)$$

for any $R_0 \geq 0$. This yields that $\alpha(r) \sim \alpha(r + R_0)$ as $r \to +\infty$. Throughout this paper we assume $\alpha \in L_1$, $\beta \in L_2$, $\gamma \in L_3$.

Heittokangas et al. [4] have introduced a new concept of φ-order of entire and meromorphic functions considering φ as subadditive function. For details one may see [4]. Later on Belaïdi et al. [1] have extended the above idea and have introduced the definitions of (α, β, γ)-order and (α, β, γ)-lower order of a meromorphic function f, which are as follows:

Definition 1.1. [1] *The (α, β, γ)-order denoted by $\rho_{(\alpha,\beta,\gamma)}[f]$ of a meromorphic function f is defined as:*

$$\rho_{(\alpha,\beta,\gamma)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log(T_f(r)))}{\beta(\log(\gamma(r)))}.$$

Definition 1.2. [1] *The (α, β, γ)-lower order denoted by $\lambda_{(\alpha,\beta,\gamma)}[f]$ of a meromorphic function f is defined as:*

$$\lambda_{(\alpha,\beta,\gamma)}[f] = \liminf_{r\to+\infty} \frac{\alpha(\log(T_f(r)))}{\beta(\log(\gamma(r)))}.$$

Belaïdi et al. [2] have also introduced the definition of another growth indicator, called (α, β, γ)-type of a meromorphic function f in the following way:

Definition 1.3. [2] *The (α,β,γ)-type denoted by $\sigma_{(\alpha,\beta,\gamma)}[f]$ of a meromorphic function f having finite positive (α,β,γ)-order $\left(0<\rho_{(\alpha,\beta,\gamma)}[f]<+\infty\right)$ is defined as:*

$$\sigma_{(\alpha,\beta,\gamma)}[f]=\limsup_{r\to+\infty}\frac{\exp(\alpha(\log(T_f(r))))}{(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}}.$$

Definition 1.4. *The growth indicator $\sigma_{(\alpha,\beta,\gamma)}[f]$ is alternatively defined as: If a meromorphic function f has finite positive (α,β,γ)-order $\left(0<\rho_{(\alpha,\beta,\gamma)}[f]<+\infty\right)$ then the integral $\int_{r_0}^{+\infty}\frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{[\exp(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}]^{k+1}}dr$ $(r_0>0)$ converges for $k>\sigma_{(\alpha,\beta,\gamma)}[f]$ and diverges for $k<\sigma_{(\alpha,\beta,\gamma)}[f]$.*

Analogously, to determine the relative growth of two increasing functions having same non-zero finite (α,β,γ)-lower order, one can introduce the definition of (α,β,γ)-weak type in the following way:

Definition 1.5. *The (α,β,γ)-weak type denoted by $\overline{\tau}_{(\alpha,\beta,\gamma)}[f]$ of a meromorphic function f having finite positive (α,β,γ)-lower order $\left(0<\lambda_{(\alpha,\beta,\gamma)}[f]<+\infty\right)$ is defined as:*

$$\overline{\tau}_{(\alpha,\beta,\gamma)}[f]=\liminf_{r\to+\infty}\frac{\exp(\alpha(\log(T_f(r))))}{(\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]}}.$$

Definition 1.6. *The growth indicator $\overline{\tau}_{(\alpha,\beta,\gamma)}[f]$ is alternatively defined as: If a meromorphic function f has finite positive (α,β,γ)-lower order $\left(0<\lambda_{(\alpha,\beta,\gamma)}[f]<+\infty\right)$ then the integral $\int_{r_0}^{+\infty}\frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{[\exp(\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]}]^{k+1}}dr$ $(r_0>0)$ converges for $k>\overline{\tau}_{(\alpha,\beta,\gamma)}[f]$ and diverges for $k<\overline{\tau}_{(\alpha,\beta,\gamma)}[f]$.*

In this paper, we have established the integral representations of the definitions of (α,β,γ)-type and (α,β,γ)-weak type of a meromorphic function f. We also investigate their equivalence relations under certain conditions.

2. Main results

In this section we state the main results of this paper. First of all we prove the following lemma which will be needed in the sequel.

Lemma 2.1. *Let the integral $\int_{r_0}^{+\infty}\frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{[\exp(\exp(\beta(\log(\gamma(r)))))^{A}]^{k+1}}dr$ $(r_0>0)$ converge where $0<A<+\infty$. Then*

$$\lim_{r\to+\infty}\frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{[\exp(\exp(\beta(\log(\gamma(r)))))^{A}]^{k}}=0.$$

Proof. Since the integral $\int_{r_0}^{+\infty}\frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{[\exp(\exp(\beta(\log(\gamma(r)))))^{A}]^{k+1}}dr$ converges, given ε (>0) there exists a number $m=m(\varepsilon)$ such that

$$\int_{r_0}^{+\infty}\frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{[\exp(\exp(\beta(\log(\gamma(r)))))^{A}]^{k+1}}dr<\varepsilon \text{ for } r_0>m.$$

So, for $r_0 > m$,

$$\int_{r_0}^{r_0+r} \frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{[\exp(\exp(\beta(\log(\gamma(r)))))^A]^{k+1}} dr < \varepsilon.$$

Since $\exp^{[2]}(\alpha(\log(T_f(r))))$ is an increasing function of r,

$$\begin{aligned} & \int_{r_0}^{r_0+\exp(\exp(\beta(\log(\gamma(r_0)))))^A} \frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{[\exp(\exp(\beta(\log(\gamma(r)))))^A]^{k+1}} dr \\ \geq & \frac{\exp^{[2]}(\alpha(\log(T_f(r_0))))}{[\exp(\exp(\beta(\log(\gamma(r_0)))))^A]^{k+1}} \cdot [\exp(\exp(\beta(\log(\gamma(r_0)))))^A] \\ = & \frac{\exp^{[2]}(\alpha(\log(T_f(r_0))))}{[\exp(\exp(\beta(\log(\gamma(r_0)))))^A]^{k}} \text{ for } r_0 > m, \end{aligned}$$

$$i.e., \; \frac{\exp^{[2]}(\alpha(\log(T_f(r_0))))}{[\exp(\exp(\beta(\log(\gamma(r_0)))))^A]^{k}} < \varepsilon \text{ for } r_0 > m.$$

Hence, it follows that

$$\lim_{r\to+\infty} \frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{[\exp(\exp(\beta(\log(\gamma(r)))))^A]^{k}} = 0.$$

This proves the lemma. □

Theorem 2.1. *Let a meromorphic function f has finite positive (α,β,γ)-order $\rho_{(\alpha,\beta,\gamma)}[f]$ $\left(0 < \rho_{(\alpha,\beta,\gamma)}[f] < +\infty\right)$ and (α,β,γ)-type, $\sigma_{(\alpha,\beta,\gamma)}[f]$. Then Definition 1.3 and Definition 1.4 are equivalent.*

Proof. **Case I.** $\sigma_{(\alpha,\beta,\gamma)}[f] = +\infty$.

Definition 1.3 ⇒ Definition 1.4.

As $\sigma_{(\alpha,\beta,\gamma)}[f] = +\infty$, from Definition 1.3 we have for arbitrary positive M and for a sequence of values of r tending to infinity that

$$\exp(\alpha(\log(T_f(r)))) > M \cdot (\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]},$$

$$i.e., \; \exp^{[2]}(\alpha(\log(T_f(r)))) > [\exp(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}]^M. \tag{1}$$

By Lemma 2.1,

$$\limsup_{r\to+\infty} \frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{[\exp(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}]^M} = 0.$$

So for all sufficiently large values of r,

$$\exp^{[2]}(\alpha(\log(T_f(r)))) < [\exp(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}]^M. \tag{2}$$

Therefore from (1) and (2) we arrive at a contradiction.

Hence $\int_{r_0}^{+\infty} \frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{[\exp(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}]^{M+1}} dr$ $(r_0 > 0)$ diverges whenever M is finite, which is Definition 1.4.

Definition 1.4 ⇒ Definition 1.3.

Let M be any positive number. Since $\sigma_{(\alpha,\beta,\gamma)}[f] = +\infty$, from Definition 1.4, the divergence of the integral $\int_{r_0}^{+\infty} \frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{[\exp(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}]^{M+1}} dr$ $(r_0 > 0)$ gives for arbitrary positive ε and for a sequence of values of r tending to infinity

$$\exp^{[2]}(\alpha(\log(T_f(r)))) > [\exp(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}]^{M-\varepsilon},$$
$$i.e.,\ \exp(\alpha(\log(T_f(r)))) > (M-\varepsilon)\left(\exp(\beta(\log(\gamma(r))))\right)^{\rho_{(\alpha,\beta,\gamma)}[f]},$$

which implies that

$$\limsup_{r\to+\infty} \frac{\exp(\alpha(\log(T_f(r))))}{\left(\exp(\beta(\log(\gamma(r))))\right)^{\rho_{(\alpha,\beta,\gamma)}[f]}} \geq M-\varepsilon.$$

Since $M > 0$ is arbitrary, it follows that

$$\limsup_{r\to+\infty} \frac{\exp(\alpha(\log(T_f(r))))}{\left(\exp(\beta(\log(\gamma(r))))\right)^{\rho_{(\alpha,\beta,\gamma)}[f]}} = +\infty.$$

Thus Definition 1.3 follows.

Case II. $0 \leq \sigma_{(\alpha,\beta,\gamma)}[f] < +\infty$.
Definition 1.3 $\Rightarrow$ **Definition 1.4**.
Subcase (A). $0 < \sigma_{(\alpha,\beta,\gamma)}[f] < +\infty$.

Now according to Definition 1.3, for arbitrary positive ε and for all sufficiently large values of r, we obtain

$$\exp(\alpha(\log(T_f(r)))) < (\sigma_{(\alpha,\beta,\gamma)}[f]+\varepsilon)\left(\exp(\beta(\log(\gamma(r))))\right)^{\rho_{(\alpha,\beta,\gamma)}[f]},$$
$$i.e.,\ \exp^{[2]}(\alpha(\log(T_f(r)))) < [\exp(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}]^{\sigma_{(\alpha,\beta,\gamma)}[f]+\varepsilon},$$

$$\begin{aligned} & i.e.,\ \frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{[\exp(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}]^k} \\ < & \frac{[\exp(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}]^{\sigma_{(\alpha,\beta,\gamma)}[f]+\varepsilon}}{[\exp(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}]^k} \\ = & \frac{1}{[\exp(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}]^{k-\sigma_{(\alpha,\beta,\gamma)}[f]-\varepsilon}}. \end{aligned}$$

Therefore $\int_{r_0}^{+\infty} \frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{[\exp(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]})]^{k+1}} dr$ $(r_0 > 0)$ converges for $k > \sigma_{(\alpha,\beta,\gamma)}[f]$.

Again by Definition 1.3, we obtain for a sequence values of r tending to infinity that

$$\exp(\alpha(\log(T_f(r)))) > (\sigma_{(\alpha,\beta,\gamma)}[f]-\varepsilon)\left(\exp(\beta(\log(\gamma(r))))\right)^{\rho_{(\alpha,\beta,\gamma)}[f]},$$
$$i.e.,\ \exp^{[2]}(\alpha(\log(T_f(r)))) > [\exp(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}]^{\sigma_{(\alpha,\beta,\gamma)}[f]-\varepsilon}. \quad (3)$$

So for $k < \sigma_{(\alpha,\beta,\gamma)}[f]$, we get from (3) that

$$\begin{aligned} & \frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{[\exp(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}]^k} \\ > & \frac{1}{[\exp(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}]^{k-(\sigma_{(\alpha,\beta,\gamma)}[f]-\varepsilon)}}. \end{aligned}$$

Therefore $\int\limits_{r_0}^{+\infty} \frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{[\exp(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}]^{k+1}} dr$ $(r_0 > 0)$ diverges for $k < \sigma_{(\alpha,\beta,\gamma)}[f]$.
Hence $\int\limits_{r_0}^{+\infty} \frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{[\exp(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}]^{k+1}} dr$ $(r_0 > 0)$ converges for $k > \sigma_{(\alpha,\beta,\gamma)}[f]$ and diverges for $k < \sigma_{(\alpha,\beta,\gamma)}[f]$.

Subcase (B). $\sigma_{(\alpha,\beta,\gamma)}[f] = 0$.

When $\sigma_{(\alpha,\beta,\gamma)}[f] = 0$, Definition 1.3 gives for all sufficiently large values of r that

$$\frac{\exp(\alpha(\log(T_f(r))))}{(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}} < \varepsilon.$$

Then as before we obtain that $\int\limits_{r_0}^{+\infty} \frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{[\exp(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}]^{k+1}} dr$ $(r_0 > 0)$ converges for $k > 0$ and diverges for $k < 0$.

Thus combining Subcase (A) and Subcase (B), Definition 1.4 follows.

Definition 1.4 $\Rightarrow$ Definition 1.3.

From Definition 1.4, the integral

$$\int\limits_{r_0}^{+\infty} \frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{[\exp(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}]^{\sigma_{(\alpha,\beta,\gamma)}[f]+\varepsilon+1}} dr \ (r_0 > 0)$$

converges for arbitrary positive ε. Then by Lemma 2.1, we get

$$\limsup_{r\to+\infty} \frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{[\exp(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}]^{\sigma_{(\alpha,\beta,\gamma)}[f]+\varepsilon}} = 0.$$

So we obtain all sufficiently large values of r that

$$\frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{[\exp(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}]^{\sigma_{(\alpha,\beta,\gamma)}[f]+\varepsilon}} < \varepsilon,$$

$$i.e., \ \exp^{[2]}(\alpha(\log(T_f(r)))) < \varepsilon \cdot [\exp(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}]^{\sigma_{(\alpha,\beta,\gamma)}[f]+\varepsilon},$$

$$i.e., \ \exp(\alpha(\log(T_f(r)))) < \log\varepsilon + (\sigma_{(\alpha,\beta,\gamma)}[f] + \varepsilon)(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]},$$

$$i.e., \ \limsup_{r\to+\infty} \frac{\exp(\alpha(\log(T_f(r))))}{(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}} \le \sigma_{(\alpha,\beta,\gamma)}[f] + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows from above that

$$\limsup_{r\to+\infty} \frac{\exp(\alpha(\log(T_f(r))))}{(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}} \le \sigma_{(\alpha,\beta,\gamma)}[f]. \tag{4}$$

On the other hand the divergence of the integral $\int\limits_{r_0}^{+\infty} \frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{[\exp(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}]^{\sigma_{(\alpha,\beta,\gamma)}[f]-\varepsilon+1}} dr$ $(r_0 > 0)$ implies that there exists a sequence of values of r tending to infinity such that

$$\begin{aligned} & \frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{[\exp(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}]^{\sigma_{(\alpha,\beta,\gamma)}[f]-\varepsilon+1}} \\ > \ & \frac{1}{[\exp((\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]})]^{1+\varepsilon}}, \end{aligned}$$

$$i.e.,\ \exp^{[2]}(\alpha(\log(T_f\,(r)))) > [\exp((\exp\,(\beta\,(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]})]^{\sigma_{(\alpha,\beta,\gamma)}[f]-2\varepsilon},$$

$$i.e.,\ \exp(\alpha(\log(T_f\,(r)))) > (\sigma_{(\alpha,\beta,\gamma)}[f]-2\varepsilon)\cdot(\exp\,(\beta\,(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]},$$

$$i.e.,\ \frac{\exp(\alpha(\log(T_f\,(r))))}{(\exp(\beta\,(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}} > (\sigma_{(\alpha,\beta,\gamma)}[f]-2\varepsilon).$$

As $\varepsilon>0$ is arbitrary, it follows from above that

$$\limsup_{r\to+\infty}\frac{\exp(\alpha(\log(T_f\,(r))))}{(\exp(\beta\,(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}} \geq \sigma_{(\alpha,\beta,\gamma)}[f]. \tag{5}$$

So from (4) and (5) , we obtain that

$$\limsup_{r\to+\infty}\frac{\exp(\alpha(\log(T_f\,(r))))}{(\exp(\beta\,(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}} = \sigma_{(\alpha,\beta,\gamma)}[f].$$

This proves the theorem. □

Theorem 2.2. *Let a meromorphic function f has finite positive (α,β,γ)-lower order $\lambda_{(\alpha,\beta,\gamma)}[f]$ $\left(0<\lambda_{(\alpha,\beta,\gamma)}[f]<+\infty\right)$ and (α,β,γ)-weak type, $\overline{\tau}_{(\alpha,\beta,\gamma)}[f]$. Then Definition 1.5 and Definition 1.6 are equivalent.*

Proof. **Case I.** $\overline{\tau}_{(\alpha,\beta,\gamma)}[f]=+\infty$.
Definition 1.5 ⇒ Definition 1.6.
As $\overline{\tau}_{(\alpha,\beta,\gamma)}[f]=+\infty$, from Definition 1.5 we obtain for arbitrary positive M and for all sufficiently large values of r that

$$\exp(\alpha(\log(T_f\,(r)))) > M\cdot(\exp(\beta\,(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]},$$

$$i.e.,\ \exp^{[2]}(\alpha(\log(T_f\,(r)))) > [\exp((\exp(\beta\,(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]})]^{M}. \tag{6}$$

By Lemma 2.1,

$$\liminf_{r\to+\infty}\frac{\exp^{[2]}(\alpha(\log(T_f\,(r))))}{[\exp((\exp(\beta\,(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]})]^{M}}=0.$$

So for a sequence of values of r tending to infinity we get

$$\exp^{[2]}(\alpha(\log(T_f\,(r)))) < [\exp((\exp(\beta\,(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]})]^{M}. \tag{7}$$

Therefore from (6) and (7), we arrive at a contradiction.

Hence $\int_{r_0}^{+\infty}\frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{[\exp((\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]})]^{M+1}}dr$ $(r_0>0)$ diverges whenever M is finite, which is Definition 1.6.
Definition 1.6 ⇒ Definition 1.5.
Let M be any positive number. Since $\overline{\tau}_{(\alpha,\beta,\gamma)}[f]=+\infty$, from Definition 1.6, the divergence of the integral $\int_{r_0}^{+\infty}\frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{[\exp((\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]})]^{M+1}}dr$ $(r_0>0)$ gives for arbitrary positive ε and for all sufficiently large values of r that

$$\exp^{[2]}(\alpha(\log(T_f\,(r)))) > [\exp((\exp(\beta\,(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]})]^{M-\varepsilon},$$

$$i.e.,\ \exp(\alpha(\log(T_f\,(r)))) > (M-\varepsilon)\cdot(\exp(\beta\,(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]},$$

which implies that

$$\liminf_{r\to+\infty}\frac{\exp(\alpha(\log(T_f\,(r))))}{(\exp(\beta\,(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]}} \geq M-\varepsilon.$$

Since $M > 0$ is arbitrary, it follows that

$$\liminf_{r\to+\infty} \frac{\exp(\alpha(\log(T_f(r))))}{(\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]}} = +\infty.$$

Thus Definition 1.5 follows.
Case II. $0 \leq \overline{\tau}_{(\alpha,\beta,\gamma)}[f] < +\infty$.
Definition 1.5 $\Rightarrow$ **Definition 1.6**.
Subcase (C). Let $0 < \overline{\tau}_{(\alpha,\beta,\gamma)}[f] < +\infty$.

Then according to Definition 1.5, for a sequence of values of r tending to infinity we get

$$\exp(\alpha(\log(T_f(r)))) < (\overline{\tau}_{(\alpha,\beta,\gamma)}[f] + \varepsilon)(\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]},$$
$$i.e.,\ \exp^{[2]}(\alpha(\log(T_f(r))))) < [\exp((\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]})]^{\overline{\tau}_{(\alpha,\beta,\gamma)}[f]+\varepsilon},$$

$$\begin{aligned} & i.e.,\ \frac{\exp^{[2]}(\alpha(\log(T_f(r)))))}{[\exp((\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]})]^k} \\ < \ & \frac{[\exp((\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]})]^{\overline{\tau}_{(\alpha,\beta,\gamma)}[f]+\varepsilon}}{[\exp((\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]})]^k} \\ = \ & \frac{1}{[\exp((\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]})]^{k-(\overline{\tau}_{(\alpha,\beta,\gamma)}[f]+\varepsilon)}}. \end{aligned}$$

Therefore $\int_{r_0}^{+\infty} \frac{\exp^{[2]}(\alpha(\log(T_f(r)))))}{[\exp((\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]})]^{k+1}} dr\ (r_0 > 0)$ converges for $k > \overline{\tau}_{(\alpha,\beta,\gamma)}[f]$.

Again by Definition 1.5, we obtain for all sufficiently large values of r that

$$\exp(\alpha(\log(T_f(r)))) > (\overline{\tau}_{(\alpha,\beta,\gamma)}[f] - \varepsilon)(\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]},$$
$$i.e.,\ \exp^{[2]}(\alpha(\log(T_f(r))))) > [\exp((\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]})]^{\overline{\tau}_{(\alpha,\beta,\gamma)}[f]-\varepsilon}. \quad (8)$$

So for $k < \overline{\tau}_{(\alpha,\beta,\gamma)}[f]$, we get from (8) that

$$\begin{aligned} & \frac{\exp^{[2]}(\alpha(\log(T_f(r)))))}{[\exp((\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]})]^k} \\ > \ & \frac{1}{[\exp((\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]})]^{k-(\overline{\tau}_{(\alpha,\beta,\gamma)}[f]-\varepsilon)}}. \end{aligned}$$

Therefore $\int_{r_0}^{+\infty} \frac{\exp^{[2]}(\alpha(\log(T_f(r)))))}{[\exp((\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]})]^{k+1}} dr\ (r_0 > 0)$ diverges for $k < \overline{\tau}_{(\alpha,\beta,\gamma)}[f]$.
Hence $\int_{r_0}^{+\infty} \frac{\exp^{[2]}(\alpha(\log(T_f(r)))))}{[\exp((\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]})]^{k+1}} dr\ (r_0 > 0)$ converges for $k > \overline{\tau}_{(\alpha,\beta,\gamma)}[f]$ and diverges for $k < \overline{\tau}_{(\alpha,\beta,\gamma)}[f]$.
Subcase (D). $\overline{\tau}_{(\alpha,\beta,\gamma)}[f] = 0$.

When $\overline{\tau}_{(\alpha,\beta,\gamma)}[f] = 0$, Definition 1.5 gives for a sequence of values of r tending to infinity that

$$\frac{\exp(\alpha(\log(T_f(r))))}{(\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]}} < \varepsilon.$$

Then as before we obtain that $\int\limits_{r_0}^{+\infty} \frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{[\exp((\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]})]^{k+1}} dr\ (r_0>0)$ converges for $k>0$ and diverges for $k<0$.

Thus combining Subcase (C) and Subcase (D), Definition 1.6 follows.

Definition 1.6 ⇒ Definition 1.5.

From Definition 1.6, the integral

$$\int\limits_{r_0}^{+\infty} \frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{[\exp((\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]})]^{\overline{\tau}_{(\alpha,\beta,\gamma)}[f]+\varepsilon+1}} dr\ (r_0>0)$$

converges for arbitrary positive ε. Then by Lemma 2.1, we get

$$\liminf_{r\to+\infty} \frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{[\exp((\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]})]^{\overline{\tau}_{(\alpha,\beta,\gamma)}[f]+\varepsilon}} = 0.$$

So we get for a sequence of values of r tending to infinity that

$$\frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{[\exp((\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]})]^{\overline{\tau}_{(\alpha,\beta,\gamma)}[f]+\varepsilon}} < \varepsilon,$$

$$i.e.,\ \exp^{[2]}(\alpha(\log(T_f(r)))) < \varepsilon\cdot[\exp((\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]})]^{\overline{\tau}_{(\alpha,\beta,\gamma)}[f]+\varepsilon},$$

$$i.e.,\ \exp(\alpha(\log(T_f(r)))) < \log\varepsilon + (\overline{\tau}_{(\alpha,\beta,\gamma)}[f]+\varepsilon)(\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]},$$

$$i.e.,\ \liminf_{r\to+\infty} \frac{\exp(\alpha(\log(T_f(r))))}{(\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]}} \le \overline{\tau}_{(\alpha,\beta,\gamma)}[f]+\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, it follows from above that

$$\liminf_{r\to+\infty} \frac{\exp(\alpha(\log(T_f(r))))}{(\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]}} \le \overline{\tau}_{(\alpha,\beta,\gamma)}[f]. \tag{9}$$

On the other hand the divergence of the integral $\int\limits_{r_0}^{+\infty} \frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{[\exp((\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]})]^{\overline{\tau}_{(\alpha,\beta,\gamma)}[f]-\varepsilon+1}} dr\ (r_0>0)$ implies for all sufficiently large values of r that

$$\begin{aligned} & \frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{[\exp((\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]})]^{\overline{\tau}_{(\alpha,\beta,\gamma)}[f]-\varepsilon+1}} \\ > & \frac{1}{[\exp((\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]})]^{1+\varepsilon}}, \end{aligned}$$

$$i.e.,\ \exp^{[2]}(\alpha(\log(T_f(r)))) > [\exp((\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]})]^{\overline{\tau}_{(\alpha,\beta,\gamma)}[f]-2\varepsilon},$$

$$i.e.,\ \exp(\alpha(\log(T_f(r)))) > (\overline{\tau}_{(\alpha,\beta,\gamma)}[f]-2\varepsilon)((\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]}),$$

$$i.e.,\ \frac{\exp(\alpha(\log(T_f(r))))}{(\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]}} > (\overline{\tau}_{(\alpha,\beta,\gamma)}[f]-2\varepsilon).$$

As $\varepsilon>0$ is arbitrary, it follows from above that

$$\liminf_{r\to+\infty} \frac{\exp(\alpha(\log(T_f(r))))}{(\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]}} \ge \overline{\tau}_{(\alpha,\beta,\gamma)}[f]. \tag{10}$$

So from (9) and (10) we obtain

$$\liminf_{r\to+\infty} \frac{\exp(\alpha(\log(T_f(r))))}{(\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]}} = \overline{\tau}_{(\alpha,\beta,\gamma)}[f].$$

This proves the theorem. □

Next we introduce the following two growth indicators which will also help our subsequent study.

Definition 2.7. *The (α,β,γ)-lower type denoted by $\overline{\sigma}_{(\alpha,\beta,\gamma)}[f]$ of a meromorphic function f having finite positive (α,β,γ)-order $\left(0<\rho_{(\alpha,\beta,\gamma)}[f]<+\infty\right)$ is defined as:*

$$\overline{\sigma}_{(\alpha,\beta,\gamma)}[f]=\liminf_{r\to+\infty}\frac{\exp(\alpha(\log\left(T_f\left(r\right)\right)))}{\left(\exp\left(\beta\left(\log(\gamma(r))\right)\right)\right)^{\rho_{(\alpha,\beta,\gamma)}[f]}}.$$

Definition 2.8. *The growth indicator $\overline{\sigma}_{(\alpha,\beta,\gamma)}[f]$ is alternatively defined as: If a meromorphic function f has finite positive (α,β,γ)-order $\left(0<\rho_{(\alpha,\beta,\gamma)}[f]<+\infty\right)$, then the integral $\int\limits_{r_0}^{+\infty}\frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{[\exp(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}]^{k+1}}dr$ $(r_0>0)$ converges for $k>\overline{\sigma}_{(\alpha,\beta,\gamma)}[f]$ and diverges for $k<\overline{\sigma}_{(\alpha,\beta,\gamma)}[f]$.*

Definition 2.9. *The (α,β,γ)-upper weak type denoted by $\tau_{(\alpha,\beta,\gamma)}[f]$ of a meromorphic function f having finite positive (α,β,γ)-lower order $\left(0<\lambda_{(\alpha,\beta,\gamma)}[f]<+\infty\right)$ is defined as:*

$$\tau_{(\alpha,\beta,\gamma)}[f]=\limsup_{r\to+\infty}\frac{\exp(\alpha(\log\left(T_f\left(r\right)\right)))}{\left(\exp\left(\beta\left(\log(\gamma(r))\right)\right)\right)^{\lambda_{(\alpha,\beta,\gamma)}[f]}}.$$

Definition 2.10. *The growth indicator $\tau_{(\alpha,\beta,\gamma)}[f]$ is alternatively defined as: If a meromorphic function f has finite positive (α,β,γ)-lower order $\left(0<\lambda_{(\alpha,\beta,\gamma)}[f]<+\infty\right)$, then the integral $\int\limits_{r_0}^{+\infty}\frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{[\exp(\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]}]^{k+1}}dr$ $(r_0>0)$ converges for $k>\tau_{(\alpha,\beta,\gamma)}[f]$ and diverges for $k<\tau_{(\alpha,\beta,\gamma)}[f]$.*

Now we state the following two theorems without their proofs as those can easily be carried out with help of Lemma 2.1 and in the line of Theorem 2.1 and Theorem 2.2 respectively.

Theorem 2.3. *Let a meromorphic function f has finite positive (α,β,γ)-order $\rho_{(\alpha,\beta,\gamma)}[f]$ $\left(0<\rho_{(\alpha,\beta,\gamma)}[f]<+\infty\right)$ and (α,β,γ)-lower type, $\overline{\sigma}_{(\alpha,\beta,\gamma)}[f]$. Then Definition 2.7 and Definition 2.8 are equivalent.*

Theorem 2.4. *Let a meromorphic function f has finite positive (α,β,γ)-lower order $\lambda_{(\alpha,\beta,\gamma)}[f]$ $\left(0<\lambda_{(\alpha,\beta,\gamma)}[f]<+\infty\right)$ and (α,β,γ)-upper weak type, $\tau_{(\alpha,\beta,\gamma)}[f]$. Then Definition 2.9 and Definition 2.10 are equivalent.*

Theorem 2.5. *If f is a meromorphic function such that $0<\lambda_{(\alpha,\beta,\gamma)}[f]\le\rho_{(\alpha,\beta,\gamma)}[f]<+\infty$, then*

$$(i)\ \sigma_{(\alpha,\beta,\gamma)}[f]=\limsup_{r\to+\infty}\frac{\exp(\alpha(\log\left(T_f\left(\gamma^{-1}(\exp r)\right)\right)))}{\left(\exp\left(\beta\left(r\right)\right)\right)^{\rho_{(\alpha,\beta,\gamma)}[f]}},$$

$$(ii)\ \overline{\sigma}_{(\alpha,\beta,\gamma)}[f]=\liminf_{r\to+\infty}\frac{\exp(\alpha(\log\left(T_f\left(\gamma^{-1}(\exp r)\right)\right)))}{\left(\exp\left(\beta\left(r\right)\right)\right)^{\rho_{(\alpha,\beta,\gamma)}[f]}},$$

$$(iii)\ \overline{\tau}_{(\alpha,\beta,\gamma)}[f]=\liminf_{r\to+\infty}\frac{\exp(\alpha(\log\left(T_f\left(\gamma^{-1}(\exp r)\right)\right)))}{\left(\exp\left(\beta\left(r\right)\right)\right)^{\lambda_{(\alpha,\beta,\gamma)}[f]}}$$

and

$$(iv)\ \tau_{(\alpha,\beta,\gamma)}[f]=\limsup_{r\to+\infty}\frac{\exp(\alpha(\log\left(T_f\left(\gamma^{-1}(\exp r)\right)\right)))}{(\exp\left(\beta\left(r\right)\right))^{\lambda_{(\alpha,\beta,\gamma)}[f]}}.$$

Proof. Theorem 2.5 follows from the definitions of $\sigma_{(\alpha,\beta,\gamma)}[f]$, $\overline{\sigma}_{(\alpha,\beta,\gamma)}[f]$, $\overline{\tau}_{(\alpha,\beta,\gamma)}[f]$ and $\tau_{(\alpha,\beta,\gamma)}[f]$ respectively by taking $\log(\gamma(r))=R$. □

In the following theorem we obtain a relationship between $\sigma_{(\alpha,\beta,\gamma)}[f]$, $\overline{\sigma}_{(\alpha,\beta,\gamma)}[f]$, $\overline{\tau}_{(\alpha,\beta,\gamma)}[f]$ and $\tau_{(\alpha,\beta,\gamma)}[f]$.

Theorem 2.6. *If f is a meromorphic function such that $0<\lambda_{(\alpha,\beta,\gamma)}[f]=\rho_{(\alpha,\beta,\gamma)}[f]<+\infty$, then the following quantities*

$$(i)\ \sigma_{(\alpha,\beta,\gamma)}[f],\ (ii)\ \overline{\tau}_{(\alpha,\beta,\gamma)}[f],\ (iii)\ \overline{\sigma}_{(\alpha,\beta,\gamma)}[f]\ \text{and}\ (iv)\ \tau_{(\alpha,\beta,\gamma)}[f]$$

are all equivalent.

Proof. From Definition 1.6, it follows that the integral $\int\limits_{r_0}^{+\infty}\frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{[\exp((\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]})]^{k+1}}dr$ $(r_0>0)$ converges for $k>\overline{\tau}_{(\alpha,\beta,\gamma)}[f]$ and diverges for $k<\overline{\tau}_{(\alpha,\beta,\gamma)}[f]$. On the other hand, Definition 1.4 implies that the integral $\int\limits_{r_0}^{+\infty}\frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{[\exp((\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]})]^{k+1}}dr$ $(r_0>0)$ converges for $k>\sigma_{(\alpha,\beta,\gamma)}[f]$ and diverges for $k<\sigma_{(\alpha,\beta,\gamma)}[f]$.
$(i)\Rightarrow(ii)$.

Now it is obvious that all the quantities in the expression

$$\left[\frac{\exp^{[2]}(\alpha(\log(T_f\left(r\right))))}{[\exp((\exp(\beta\left(\log(\gamma(r))\right)))^{\lambda_{(\alpha,\beta,\gamma)}[f]})]^{k+1}}-\frac{\exp^{[2]}(\alpha(\log(T_f\left(r\right))))}{[\exp((\exp(\beta\left(\log(\gamma(r))\right)))^{\rho_{(\alpha,\beta,\gamma)}[f]})]^{k+1}}\right]$$

are of non-negative type. So

$$\int\limits_{r_0}^{+\infty}\left[\frac{\exp^{[2]}(\alpha(\log(T_f\left(r\right))))}{[\exp((\exp(\beta\left(\log(\gamma(r))\right)))^{\lambda_{(\alpha,\beta,\gamma)}[f]})]^{k+1}}-\frac{\exp^{[2]}(\alpha(\log(T_f\left(r\right))))}{[\exp((\exp(\beta\left(\log(\gamma(r))\right)))^{\rho_{(\alpha,\beta,\gamma)}[f]})]^{k+1}}\right]dr\geq 0 \text{ for } r_0>0,$$

$$i.e.,\ \int\limits_{r_0}^{+\infty}\frac{\exp^{[2]}(\alpha(\log(T_f\left(r\right))))}{[\exp((\exp(\beta\left(\log(\gamma(r))\right)))^{\lambda_{(\alpha,\beta,\gamma)}[f]})]^{k+1}}dr\geq\int\limits_{r_0}^{+\infty}\frac{\exp^{[2]}(\alpha(\log(T_f\left(r\right))))}{[\exp((\exp(\beta\left(\log(\gamma(r))\right)))^{\rho_{(\alpha,\beta,\gamma)}[f]})]^{k+1}}dr \text{ for } r_0>0.$$

Hence,

$$\overline{\tau}_{(\alpha,\beta,\gamma)}[f]\geq\sigma_{(\alpha,\beta,\gamma)}[f]. \tag{11}$$

Further as $\rho_{(\alpha,\beta,\gamma)}[f] = \lambda_{(\alpha,\beta,\gamma)}[f]$, we get that

$$\begin{aligned}
\sigma_{(\alpha,\beta,\gamma)}[f] &= \limsup_{r\to+\infty} \frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}} \\
&\geq \liminf_{r\to+\infty} \frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}} \\
&= \liminf_{r\to+\infty} \frac{\exp(\alpha(\log(T_f(r))))}{(\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]}} = \overline{\tau}_{(\alpha,\beta,\gamma)}[f]. \qquad (12)
\end{aligned}$$

Hence from (11) and (12) we obtain

$$\sigma_{(\alpha,\beta,\gamma)}[f] = \overline{\tau}_{(\alpha,\beta,\gamma)}[f]. \qquad (13)$$

$(ii) \Rightarrow (iii)$.

Since $\rho_{(\alpha,\beta,\gamma)}[f] = \lambda_{(\alpha,\beta,\gamma)}[f]$, we get

$$\begin{aligned}
\overline{\tau}_{(\alpha,\beta,\gamma)}[f] &= \liminf_{r\to+\infty} \frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{(\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]}} \\
&= \liminf_{r\to+\infty} \frac{\exp^{[2]}(\alpha(\log(T_f(r))))}{(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}} = \overline{\sigma}_{(\alpha,\beta,\gamma)}[f].
\end{aligned}$$

$(iii) \Rightarrow (iv)$.

In view of (13) and the condition $\rho_{(\alpha,\beta,\gamma)}[f] = \lambda_{(\alpha,\beta,\gamma)}[f]$, it follows that

$$\begin{aligned}
\overline{\sigma}_{(\alpha,\beta,\gamma)}[f] &= \liminf_{r\to+\infty} \frac{\exp(\alpha(\log(T_f(r))))}{(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}} \\
&= \liminf_{r\to+\infty} \frac{\exp(\alpha(\log(T_f(r))))}{(\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]}},
\end{aligned}$$

$$\begin{aligned}
i.e.,\ \overline{\sigma}_{(\alpha,\beta,\gamma)}[f] &= \overline{\tau}_{(\alpha,\beta,\gamma)}[f] \\
&= \sigma_{(\alpha,\beta,\gamma)}[f] \\
&= \limsup_{r\to+\infty} \frac{\exp(\alpha(\log(T_f(r))))}{(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}} \\
&= \limsup_{r\to+\infty} \frac{\exp(\alpha(\log(T_f(r))))}{(\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]}} \\
&= \tau_{(\alpha,\beta,\gamma)}[f].
\end{aligned}$$

$(iv) \Rightarrow (i)$.

As $\rho_{(\alpha,\beta,\gamma)}[f] = \lambda_{(\alpha,\beta,\gamma)}[f]$, we obtain that

$$\begin{aligned}
\tau_{(\alpha,\beta,\gamma)}[f] &= \limsup_{r\to+\infty} \frac{\exp(\alpha(\log(T_f(r))))}{(\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]}} \\
&= \limsup_{r\to+\infty} \frac{\exp(\alpha(\log(T_f(r))))}{(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}} = \sigma_{(\alpha,\beta,\gamma)}[f].
\end{aligned}$$

Thus the theorem follows. □

Acknowledgement. The authors are very much thankful to the reviewer for his/her valuable suggestions to bring the paper in its present form.

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S. Bhattacharyya
Department of Mathematics, Netaji Mahavidyalaya, P.O.- Arambagh
District-Hooghly, PIN-712601, West Bengal, India.
Email address: sarmilabhattacharyya1982@gmail.com

T. Biswas
Rajbari, Rabindrapally, R. N. Tagore Road
P.O.-Krishnagar, P.S.-Katwali, Dist-Nadia, PIN- 741101, West Bengal, India.
Email address: tanmaybiswas_math@rediffmail.com

C. Biswas
Department of Mathematics, Nabadwip Vidyasagar College
Nabadwip, Dist.- Nadia, PIN-741302, West Bengal, India.
Email address: chinmay.shib@gmail.com