Journal of Fractional Calculus and Applications
Vol. 15(1) Jan. 2024, No. 4
ISSN: 2090-5858.
ISSN 2090-584X (print)
http://jfca.journals.ekb.eg/

FRACTIONAL VARIATIONAL ITERATION METHOD FOR HIGHER-ORDER FRACTIONAL DIFFERENTIAL EQUATIONS

G. MONZÓN

Abstract. In recent decades, numerous and varied numerical methods have been proposed and studied to approximate solutions for various classes of fractional differential equations, primarily those involving single-term or multiple-order equations. However, equations incorporating fractional iterated derivatives have not received widespread attention. In this work we describe a reliable strategy to approximate the solution of higher-order fractional differential equations where both the fractional derivative and the iterated derivatives are described in the Caputo sense. Specifically, we propose a fractional variational iteration method (FVIM) where the Lagrange multiplier associated with the correction term is explicitly determined by means of the Laplace transform.

For the second-order case, we give a sufficient condition -involving the coefficients of the equation and the fractional order of the Caputo derivative- which guarantees the convergence of the sequence generated by the FVIM. Furthermore, this convergence is independent of the initial function considered for the iteration.

Finally, some examples are presented in order to illustrate the applicability of the method and the reliability of the theoretical results obtained. In particular, for most of them we observe that the FVIM leads to the exact solution which shows the power of the method in practice.

1. Introduction

Since its appearance in the late 90's, the variational iteration method (VIM) -a powerful analytical method based on the Lagrange multiplier technique- has been widely used to solve multiple and varied problems including initial value problems of fractional differential equations (see, for instance, [8, 13, 14, 19, 20, 26, 30, 33, 34, 35] and included references). Indeed, a multitude of diverse methods have

2010 *Mathematics Subject Classification.* 26A33, 65D99.

Key words and phrases. Caputo fractional derivative, Riemann-Liouville integral operator, Fractional differential equations, Variational iteration method, Laplace transform.

Submitted August 14, 2023. Revised November 1, 2023.

been proposed and thoroughly examined across various disciplines to approximate solutions for different classes of fractional differential equations. Although the list of references we give is far from being exhaustive, we recommend consulting the works [1, 2, 5, 6, 7, 9, 10, 11, 12, 15, 16, 17, 18, 27, 31] to have a comprehensive understanding of some of them. Among these methods, the variational iteration method has demonstrated remarkable reliability and efficiency in a wide range of scientific applications, both linear and non-linear. It was shown by many authors that this method is more powerful than existing techniques such as the Adomian method, perturbation method, etc. One of the main advantages of the VIM is that it provides successive approximations that converge rapidly towards the exact solution.

To the best of our knowledge, applications of the variational iteration method to higher-order fractional differential equations (HOFDEs), i.e. fractional differential equations involving sequential or iterated derivatives, have not been directly addressed. In [19] the classical VIM was implemented to give approximate solutions for linear (and non-linear) systems of differential equations of fractional order. Now, since a higher-order linear fractional differential equation can be written as a linear system of first-order fractional differential equations, we can derive an implementation of the classical VIM for HOFDEs; however, this was not explicitly mentioned or done. Given the aforementioned advantages of the VIM, exploring its applications to higher-order fractional differential equations becomes an interesting area of study.

In the present work we introduce and study a modified version of the VIM for the following general second-order fractional differential equations

$$D_0^{\overset{2}{}\alpha}u(t) = pD_0^\alpha u(t) + qu(t) + r(t) \quad \text{in } (0,a), \tag{1}$$

where $a \in \mathbb{R}^+$, $p, q \in \mathbb{R}$ and r is a bounded function on $[0,a]$, subject to the initial conditions

$$u(0) = \beta_0, \quad D_0^\alpha u(0) = \beta_1, \tag{2}$$

with $\beta_0, \beta_1 \in \mathbb{R}$. Here $\alpha \in (0,1)$, $D_0^\alpha u$ denotes the Caputo fractional derivative of u of order α and $D_0^{\overset{2}{}\alpha} = D_0^\alpha \circ D_0^\alpha$ denotes the iterated derivative.

The introduced approach can be easily generalized and extended to deal with higher-order linear fractional differential equations, however, we focus on the second-order case in order to keep our explanation as clear as possible.

In [21] a consistent approximation scheme of the finite-difference type was given for the problem (1)-(2). Such a method is based on rewriting (1) as an integro-differential equation and, in a domain discretization, using numerical rules to approximate the fractional derivative and the Riemann-Liouville integral operator. As indicated above, in this article we formulate an alternative approximation scheme according to the VIM approach which offers distinct advantages compared to the method just described. Unlike the previous approach, our method ensures pointwise convergence not only at the nodes of the domain discretization but also throughout the entire domain. Remarkably, our method produces precise approximations with only a few iterations, whereas achieving a comparable level of accuracy with the method outlined in [21] demands the consideration of a significantly larger number of nodes. This reduction in the computational workload translates to a substantially lower implementation cost, making our approach more efficient. Additionally, in certain cases, our method even yields exact solutions.

In applications of the VIM to initial value problems of differential equations there are three essential steps to follow:

(1) establishing the correction functional (which involves a Lagrange multiplier);
(2) identifying the Lagrange multiplier;
(3) determining the initial iteration.

Regarding steps 1 and 2 we consider an approach inspired by the one proposed in [34] which is based (mainly step 2) in properties of the Laplace transform and the variational theory; on the other hand, for step 3 we essentially consider the first-order generalized Taylor formula of u at a (c.f. [23]) which, as we will see later, *naturally* appears in the process.

For the fractional variational iteration method (FVIM) proposed here, we estimate the error approximation in the L^1-norm (see Theorem 4.1 below). Indeed, if $(u_n)_{n\geq 0}$ denotes the approximating sequence, then

$$\|u_{n+1}-u\|_{L^1} \leq \left(\frac{|p|a^{\alpha}}{\Gamma(\alpha+1)}+\frac{|q|a^{2\alpha}}{\Gamma(2\alpha+1)}\right)^n \|u_0-u\|_{L^1} \qquad (n\geq 0).$$

Therefore, if $\dfrac{|p|a^{\alpha}}{\Gamma(\alpha+1)}+\dfrac{|q|a^{2\alpha}}{\Gamma(2\alpha+1)} < 1$, the convergence is guaranteed and is independent of the initial data u_0.

In order to illustrate the applicability and convergence of the introduced FVIM we exhibit numerical examples where the theoretical results are confirmed (c.f. Section 5); moreover, for most of them we obtain the exact analytical solution that shows the power of the method in practice.

The organization of the paper is as follows: In Section 2 we introduce definitions, notations and basic properties related to the Riemann-Liouville integral operator, the Caputo fractional derivative and the Laplace transform which are essential for the development of our approach. In Section 3, the fractional variational iteration method associated to (1)-(2) is derived and, in Section 4, we prove some convergence results and error estimates. Finally, in Section 5, numerical examples are presented with the aim to illustrate the applicability and convergence of the method and, in Section 6, some conclusions are given.

2. Definitions, notations and preliminary results

Let $\alpha \in (0,1)$. The Riemann-Liouville integral operator I_0^{α} of $u \in L^1[a,b]$ of order α is defined as

$$I_0^{\alpha}u(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}u(t)\,dt \quad (0\leq x\leq a).$$

From [4, Theorem 2.1] we know that $I_0^{\alpha}u \in L^1[0,a]$, moreover

$$\|I_0^{\alpha}u\|_{L^1} \leq \frac{1}{\Gamma(\alpha)}\frac{a^{\alpha}}{\alpha}\|u\|_{L^1}=\frac{a^{\alpha}}{\Gamma(\alpha+1)}\|u\|_{L^1}.$$

Remark 1. *Last equality in the previous equation is due to*

$$z\Gamma(z)=\Gamma(z+1) \quad \forall z\in\mathbb{C}. \tag{3}$$

We will use this well-known property on the Γ *function again in* Section 5*.*

The Caputo fractional derivative D_0^α of u of order α is given by

$$D_0^\alpha u = I_0^{1-\alpha} u'$$

where u' denotes the ordinary derivative of u, i.e.

$$D_0^\alpha u(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{u'(t)}{(x-t)^\alpha}\, dt \quad (x \geq 0). \tag{4}$$

In the sequel, we recall the definition of the Laplace transform and some basic properties which will be useful for our purposes.

Let v be a function defined on $[0, +\infty)$, then the Laplace transform $L[v]$ of v is given by

$$L[v](s) = \int_0^\infty e^{-st} v(t)\, dt. \tag{5}$$

We assume that $L[v]$ exists for $s > 0$ and, as we just did, we skip writing the variable s unless absolutely necessary.

The operator L is linear, i.e.

$$L[k_1 v_1 + k_2 v_2] = k_1 L[v_1] + k_2 L[v_2] \quad k_1, k_2 \in \mathbb{R} \tag{6}$$

and also satisfies

$$L[1] = \frac{1}{s}, \tag{7}$$

$$L\left[t^\alpha\right] = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}. \tag{8}$$

On the other hand, the Laplace transform of the Caputo derivative is given by

$$L[D_0^\alpha v] = s^\alpha L[v] - s^{\alpha-1} v(0) \tag{9}$$

while the Laplace transform of the Riemann-Liouville integral operator verifies

$$L[I_0^\alpha v] = \frac{1}{s^\alpha} L[v] \tag{10}$$

(see [4, Theorem 7.1] or [24] for details).

Finally, as usual, we use L^{-1} to denote the inverse Laplace transform. In particular, L^{-1} is a linear operator which satisfies

$$L^{-1}\left[\frac{1}{s}\right] = 1 \tag{11}$$

and

$$L^{-1}\left[\frac{1}{s^{\alpha+1}}\right] = \frac{t^\alpha}{\Gamma(\alpha+1)}. \tag{12}$$

3. The fractional variational iteration method

In short, the variational iteration method consists of defining an approximating sequence $(u_n)_{n\geq 0}$ of the solution u by means of a recurrence relation in which a correction term is incorporated. As usual, we assume that the correction term is determined by the fractional differential equation in question -in our case

$$D_0^{2\alpha} u - p D_0^\alpha u - qu - r = 0 \tag{13}$$

subject to the initial conditions $u(0) = \beta_0,\ D_0^\alpha u(0) = \beta_1$- and we incorporate it by means of a Lagrange multiplier λ. In concrete, the iteration formula is given by

$$u_{n+1} = u_n + \lambda \left[D_0^{2\alpha} u_n - p D_0^\alpha u_n - q u_n - r\right] \quad (n \geq 0) \tag{14}$$

where the initial data u_0 must be determined as well as the value of λ.

In order to do this we explore an approach based on the Laplace transform L. This approach has similarities with the one discussed in [34], however, obvious modifications must be made since the type of equations under consideration here is different from the one studied in said work (for example, the equations under study in [34] have integer values as orders for the highest order derivative whereas, in our case, the highest order derivative is fractional).

We are going to begin by applying L to both sides of (14). Indeed, after doing this we have

$$\begin{aligned} L[u_{n+1}] &= L[u_n] + \lambda \left[L[D_0^{2\alpha} u_n - pD_0^\alpha u_n - qu_n - r] \right] \\ &= L[u_n] + \lambda \left[L[D_0^{2\alpha} u_n] - L[pD_0^\alpha u_n + qu_n + r] \right] \end{aligned} \tag{15}$$

Now, assuming that

$$u_n(0) = \beta_0 \qquad \text{and} \qquad D_0^\alpha u_n(0) = \beta_1, \tag{16}$$

and, by repeatedly using the identity (9), it follows that

$$\begin{aligned} L\left[D_0^{2\alpha} u_n \right] = L[D_0^\alpha (D_0^\alpha u_n)] &= s^\alpha L[D_0^\alpha u_n] - s^{\alpha-1} D_0^\alpha u_n(0) \\ &= s^\alpha \left(s^\alpha L[u_n] - s^{\alpha-1} u_n(0) \right) - s^{\alpha-1} D_0^\alpha u_n(0) \\ &= s^{2\alpha} L[u_n] - s^{2\alpha-1} u_n(0) - s^{\alpha-1} D_0^\alpha u_n(0) \\ &= s^{2\alpha} L[u_n] - s^{2\alpha-1}\beta_0 - s^{\alpha-1}\beta_1. \end{aligned}$$

Then, the equation (15) can be written as

$$\begin{aligned} L[u_{n+1}] &= L[u_n] + \lambda \left[s^{2\alpha} L[u_n] - s^{2\alpha-1}\beta_0 - s^{\alpha-1}\beta_1 - L[pD_0^\alpha u_n + qu_n + r] \right] \\ &= (1 + \lambda s^{2\alpha}) L[u_n] - \lambda \left[s^{2\alpha-1}\beta_0 + s^{\alpha-1}\beta_1 + L[pD_0^\alpha u_n + qu_n + r] \right]. \end{aligned} \tag{17}$$

Setting $L[pD_0^\alpha u_n + qu_n + r]$ as a restricted variation term, from the stationary condition it follows that

$$1 + \lambda s^{2\alpha} = 0.$$

In this way, λ can be identified as $\lambda = -\dfrac{1}{s^{2\alpha}}$. Under this consideration, the equation (17) reduces to

$$\begin{aligned} L[u_{n+1}] &= \frac{1}{s^{2\alpha}} \left[s^{2\alpha-1}\beta_0 + s^{\alpha-1}\beta_1 + L[pD_0^\alpha u_n + qu_n + r] \right] \\ &= \frac{\beta_0}{s} + \frac{\beta_1}{s^{\alpha+1}} + \frac{1}{s^{2\alpha}} L[pD_0^\alpha u_n + qu_n + r]. \end{aligned}$$

Now, taking into account (6)-(8) we get

$$L[u_{n+1}] = L\left[\beta_0 + \frac{\beta_1}{\Gamma(\alpha+1)} t^\alpha \right] + \frac{1}{s^{2\alpha}} L[pD_0^\alpha u_n + qu_n + r].$$

Therefore, the iteration equation reads as

$$u_{n+1}(t) = \beta_0 + \frac{\beta_1}{\Gamma(\alpha+1)} t^\alpha + L^{-1}\left[\frac{1}{s^{2\alpha}} L[pD_0^\alpha u_n + qu_n + r] \right].$$

At this point it should be noted that the function that appears at the beginning of the right-hand side of the previous equation is independent of n and can be written as follows

$$\beta_0 + \frac{\beta_1}{\Gamma(\alpha+1)} t^\alpha = u(0) + \frac{D_0^\alpha u(0)}{\Gamma(\alpha+1)} t^\alpha$$

so it turns out to be the first-order generalized Taylor formula for u at $t = 0$ (c.f. [23]). On the other hand, it is immediate to check that such a function satisfies the conditions imposed in (16). For this reasons we consider it as the initial data u_0, this is

$$u_0(t) = \beta_0 + \frac{\beta_1}{\Gamma(\alpha+1)} t^\alpha$$

and therefore

$$u_{n+1}(t) = u_0(t) + L^{-1}\left[\frac{1}{s^{2\alpha}} L[pD_0^\alpha u_n + qu_n + r]\right].$$

Finally, by using (6) and (9) again, it follows that

$$\begin{aligned} u_{n+1}(t) &= u_0(t) + L^{-1}\left[\frac{1}{s^{2\alpha}}\left(pL[D_0^\alpha u_n] + qL[u_n] + L[r]\right)\right] \\ &= u_0(t) + L^{-1}\left[\frac{1}{s^{2\alpha}}\left(p(s^\alpha L[u_n] - s^{\alpha-1}\beta_0) + qL[u_n] + L[r]\right)\right] \\ &= u_0(t) + L^{-1}\left[\frac{ps^\alpha + q}{s^{2\alpha}} L[u_n] - \frac{p\beta_0}{s^{\alpha+1}} + \frac{L[r]}{s^{2\alpha}}\right] \\ &= \beta_0 + \frac{\beta_1 - p\beta_0}{\Gamma(\alpha+1)} t^\alpha + L^{-1}\left[\frac{ps^\alpha + q}{s^{2\alpha}} L[u_n] + \frac{L[r]}{s^{2\alpha}}\right]. \end{aligned}$$

In sum, the proposed fractional variational iteration method reduces to

$$u_{n+1}(t) = \beta_0 + \frac{\beta_1 - p\beta_0}{\Gamma(\alpha+1)} t^\alpha + L^{-1}\left[\frac{(ps^\alpha + q)L[u_n] + L[r]}{s^{2\alpha}}\right] \quad (n \geq 0) \tag{18}$$

with the inital data

$$u_0(t) = \beta_0 + \frac{\beta_1}{\Gamma(\alpha+1)} t^\alpha. \tag{19}$$

4. Convergence and the error treatment

In some cases, where the classical version of the VIM was used, the convergence of the recurrence sequence was studied (see for instance [22, 29, 32]). In this section we present our main results on the convergence of the recurrence sequence obtained by the fractional variational iteration method that we propose (see Theorem 4.1 below). In fact, we prove that the sequence given by (18)-(19) converges to the real solution of the fractional differential equation in the L^1-norm under certain assumption involving the coefficients p and q, the fractional order α and the length a of the interval domain (c.f. (20)).

Theorem 4.1. *Let $\alpha \in (0,1)$, $a \in \mathbb{R}^+$ and $p, q \in \mathbb{R}$ be the values involved in* (13). *Assume that*

$$\gamma = \frac{|p|a^\alpha}{\Gamma(\alpha+1)} + \frac{|q|a^{2\alpha}}{\Gamma(2\alpha+1)} < 1, \tag{20}$$

then the sequence $(u_n)_{n\geq 0}$ given by (18)-(19) *is L^1-convergent to the solution u of* (13). *Moreover, the L^1-error estimate verifies*

$$\|u_{n+1} - u\|_{L^1} \leq \gamma^n \|u_0 - u\|_{L^1} \qquad n \geq 0. \tag{21}$$

Proof. From (18) we have

$$u_{n+1} = \bar{u}_0 + L^{-1}\left[\frac{(ps^\alpha + q)L[u_n] + L[r]}{s^{2\alpha}}\right]$$

where

$$\bar{u}_0(t) = \beta_0 + \frac{\beta_1 - p\beta_0}{\Gamma(\alpha+1)} t^\alpha.$$

On the other hand, if u is the real solution of (13), from the same arguments used in Section 3 in order to deduce (18), it follows that

$$u = \bar{u}_0 + L^{-1}\left[\frac{(ps^\alpha + q)L[u] + L[r]}{s^{2\alpha}}\right].$$

Then

$$u_{n+1} - u = L^{-1}\left[\frac{ps^\alpha + q}{s^{2\alpha}} L[u_n - u]\right]$$

or, equivalently,

$$L[u_{n+1} - u] = \frac{ps^\alpha + q}{s^{2\alpha}} L[u_n - u]\,.$$

From this fact and making use of properties (6) and (10) we get

$$\begin{aligned} L[u_{n+1} - u] &= \frac{1}{s^\alpha} L[p(u_n - u)] + \frac{1}{s^{2\alpha}} L[q(u_n - u)] \\ &= L[I_0^\alpha p(u_n - u)] + L[I_0^{2\alpha} q(u_n - u)] \\ &= L[pI_0^\alpha (u_n - u) + qI_0^{2\alpha}(u_n - u)]. \end{aligned}$$

Now, thanks to Lerch's theorem (see, for instance, [3, Theorem 2.1]),

$$u_{n+1} - u = pI_0^\alpha (u_n - u) + qI_0^{2\alpha}(u_n - u) \tag{22}$$

and then, from the triangular inequality, it follows that

$$\|u_{n+1} - u\|_{L^1} \le |p|\|I_0^\alpha(u_n - u)\|_{L^1} + |q|\|I_0^{2\alpha}(u_n - u)\|_{L^1}.$$

Taking into account (2) we have

$$\|u_{n+1} - u\|_{L^1} \le |p|\frac{a^\alpha}{\Gamma(\alpha+1)}\|u_n - u\|_{L^1} + |q|\frac{a^{2\alpha}}{\Gamma(2\alpha+1)}\|u_n - u\|_{L^1} = \gamma\,\|u_n - u\|_{L^1}$$

where γ is given by (20). Finally, (21) is a direct consequence of this inequality and the convergence of $(u_n)_n$ follows immediately by combining (21) and (20). □

Remark 2. *Thanks to the linearity of the operator* I_0^α *and the following fact* (c.f. [4, Theorem 2.2])

$$I_0^{2\alpha} = I_0^\alpha \circ I_0^\alpha,$$

the equation (22) *can be written as*

$$u_{n+1} - u = I_0^\alpha\left[p(u_n - u) + qI_0^\alpha(u_n - u)\right].$$

Then, making use of (2) *again together with the triangular inequality, we get*

$$\begin{aligned} \|u_{n+1} - u\|_{L^1} &\le \frac{a^\alpha}{\Gamma(\alpha+1)}\|p(u_n - u) + qI_0^\alpha(u_n - u)\|_{L^1} \\ &\le \frac{a^\alpha}{\Gamma(\alpha+1)}\left[|p|\|u_n - u\|_{L^1} + |q|\|I_0^\alpha(u_n - u)\|_{L^1}\right] \\ &\le \frac{a^\alpha}{\Gamma(\alpha+1)}\left[|p|\|u_n - u\|_{L^1} + |q|\frac{a^\alpha}{\Gamma(\alpha+1)}\|u_n - u\|_{L^1}\right] \\ &= \left(\frac{|p|a^\alpha}{\Gamma(\alpha+1)} + \frac{|q|a^{2\alpha}}{\Gamma^2(\alpha+1)}\right)\|u_n - u\|_{L^1}. \end{aligned}$$

In consequence, $\|u_{n+1}-u\|_{L^1} \leq \left(\frac{|p|a^{\alpha}}{\Gamma(\alpha+1)}+\frac{|q|a^{2\alpha}}{\Gamma^2(\alpha+1)}\right)^n \|u_0-u\|_{L^1}$*. Therefore, under the assumption*

$$\frac{|p|a^{\alpha}}{\Gamma(\alpha+1)}+\frac{|q|a^{2\alpha}}{\Gamma^2(\alpha+1)}<1 \tag{23}$$

we also obtain the L^1*-convergence of* $(u_n)_n$*. However, since*

$$\frac{1}{\Gamma(2\alpha+1)} \leq \frac{1}{\Gamma^2(\alpha+1)} \quad \forall \alpha \in (0,1)$$

the assumption (20) *made in* Theorem 4.1 *is slightly weaker than* (23)*.*

Remark 3. *The* L^1*-convergence of the sequence* $(u_n)_n$ *given by* (18) *is independent of the choice of* u_0*.*

As we claim before, our election for u_0 *in* (19) *is due that this function naturally appears in the deduction of* u_{n+1} *and agrees with the first-order generalized Taylor formula for* u *at the point* $t=0$*.*

Although the convergence of u_n to u in the L^1-norm does not imply the pointwise convergence $u_n(t) \to u(t)$, $t \in (0,a)$, from Theorem 4.1 and a well-known fact (see for instance [28, Theorem 3.12]) we have the following result.

Corollary 4.3. *Let* $\alpha \in (0,1)$, $a \in \mathbb{R}^+$ *and* $p, q \in \mathbb{R}$ *be the values involved in* (13)*. Under the assumption* (20) *there is a subsequence* $(u_{n_k})_k$ *of the sequence given by* (18)-(19) *which converges pointwise almost everywhere to the solution* u *of* (13) *on* $[0,a]$ *i.e.*

$$\lim_{k\to\infty} u_{n_k}(t) = u(t) \quad a.e.$$

Remark 4. *Although we have not shown pointwise convergence for the entire sequence* $(u_n)_n$*, the numerical experiments we consider in* Section 5 *suggest that pointwise convergence holds for the entire sequence and is performed at every point of the interval* $[0,a]$*. Moreover, in most cases, the analytical expression of the exact solution* u *can be obtained from the proposed approximation scheme.*

In the next section we illustrate the applicability of the method and also confirm the theoretical results obtained here by displaying numerical examples.

5. Numerical examples

In [21] several second-order fractional differential equations were presented and analyzed to demonstrate the application and convergence of the method proposed in that study. Interestingly, some of these examples were previously examined in reference [23], where the generalized Taylor formula was employed to derive the analytical solution. In this section, we will explore these examples, or slightly generalized variations of them, to illustrate the application of the FVIM proposed and studied in the preceding sections. However, before introducing such examples, we point out an elementary fact about the gamma function related to those terms involved in (20), namely

$$1 \leq \frac{1}{\Gamma(\alpha+1)}, \frac{1}{\Gamma(2\alpha+1)} \leq 1.13 \quad \forall\, \alpha \in (0,1). \tag{24}$$

Example 5.1. *Let $0 < \alpha < 1$. Consider the following second-order fractional differential equation*

$$\begin{cases} D_0^{2\,\alpha} u(t) = 3^{-1} D_0^{\alpha} u(t) + \alpha\Gamma(2\alpha)(u(t)-1) \quad \textit{in } (0,1), \\ u(0) = 1, \quad D_0^{\alpha} u(0) = 0. \end{cases} \tag{25}$$

We start by noting that, in this case, the requirement (20) is satisfied. Indeed, for any $\alpha \in (0,1)$ and taking into account that $a = 1$, $p = 3^{-1}$ and (thanks to (3)) $q = \alpha\Gamma(2\alpha) = \Gamma(2\alpha+1)/2$, from (24) it follows that

$$\frac{|p|a^{\alpha}}{\Gamma(\alpha+1)} + \frac{|q|a^{2\alpha}}{\Gamma(2\alpha+1)} = \frac{1}{3\Gamma(\alpha+1)} + \frac{1}{2} < 0.88\,.$$

On the other hand, the initial data u_0 (c.f. (19)) is given by

$$u_0(t) = 1.$$

In regards to the iteration formula (18), for $n \geq 0$ and thanks to (6) we have

$$\begin{aligned} u_{n+1}(t) &= 1 - \frac{t^{\alpha}}{3\Gamma(\alpha+1)} + L^{-1}\left[\frac{(s^{\alpha}/3 + \alpha\Gamma(2\alpha))L[u_n] + L[-\alpha\Gamma(2\alpha)]}{s^{2\alpha}}\right] \\ &= 1 - \frac{t^{\alpha}}{3\Gamma(\alpha+1)} + L^{-1}\left[\frac{L[u_n]}{3s^{\alpha}} + \frac{\alpha\Gamma(2\alpha)}{s^{2\alpha}} L[u_n - 1]\right]. \end{aligned}$$

In particular, for $n = 0$ (making use of (7) and (12)) we obtain

$$u_1(t) = 1 - \frac{t^{\alpha}}{3\Gamma(\alpha+1)} + \frac{1}{3}L^{-1}\left[\frac{L[1]}{s^{\alpha}}\right] = 1 - \frac{t^{\alpha}}{3\Gamma(\alpha+1)} + \frac{1}{3}L^{-1}\left[\frac{1}{s^{\alpha+1}}\right] = 1.$$

Additionally, from what has been done it follows that, for any $n > 1$, $u_n(t) = 1$. Then, for any $t \in [0,1]$

$$\lim_{n\to\infty} u_n(t) = 1$$

and we obtain the exact solution $u \equiv 1$ of (25).

Example 5.2. *Consider the following second-order fractional differential equation*

$$\begin{cases} 2D_0^{2\,\frac{1}{2}} u(t) = u(t) + 4t - t^2 \quad \textit{in } (0,1), \\ u(0) = 0, \quad D_0^{\frac{1}{2}} u(0) = 0. \end{cases} \tag{26}$$

In this case, the parameters a, p, q and α involved in (20) are $1, 0, 1/2$ and $1/2$ respectively. Moreover, such requirement is fulfilled since

$$\frac{|p|a^{\alpha}}{\Gamma(\alpha+1)} + \frac{|q|a^{2\alpha}}{\Gamma(2\alpha+1)} = \frac{1}{2}.$$

Therefore, according to Corollary 4.3, pointwise convergence is expected for a subsequence of $(u_n)_n$ given by (18)-(19). In fact, as in the previous example, pointwise convergence is observed for the entire sequence and the explicit formula is derived for the real solution $u(t) = t^2$. Indeed, the iteration formula (18)-(19) reads as

$$\begin{cases} u_0(t) = 0, \\ u_{n+1}(t) = L^{-1}\left[\frac{1/2L[u_n] + L[2t - t^2/2]}{s}\right] = \frac{1}{2}L^{-1}\left[\frac{L[u_n]}{s} + \frac{4}{s^3} - \frac{2}{s^4}\right]. \end{cases}$$

After an straightforward calculation, we get

$$u_n(t) = t^2 - \frac{1}{2^{n-1}}\frac{t^{n+2}}{(n+2)!} \qquad n \geq 1.$$

Then, for any $t \in [0,1]$ (due to $\lim_{n\to\infty} \frac{1}{2^{n-1}} \frac{t^{n+2}}{(n+2)!} = 0$), we have

$$\lim_{n\to\infty} u_n(t) = t^2$$

which give us the explicit expression of the exact solution $u(t) = t^2$ as we claim before. Furthermore, the pointwise approximation error is given by

$$|u_n(t) - u(t)| = \frac{1}{2^{n-1}} \frac{t^{n+2}}{(n+2)!} \leq \frac{1}{2^{n-1}(n+2)!} \qquad \forall t \in [0,1]$$

allowing to conclude that with very few iterations a good approximation is achieved.

Example 5.3. *Let $0 < \alpha < 1$. Consider the following second-order fractional differential equation*

$$\begin{cases} D_0^{2\alpha} u(t) = \dfrac{\alpha+1}{2} D^\alpha u(t) - \dfrac{\Gamma(\alpha+2)}{2} & \text{in } (0,1), \\ u(0) = 0, \quad D_0^\alpha u(0) = \Gamma(\alpha+1). \end{cases} \tag{27}$$

We start by noting that the condition (20) is also satisfied in this case since the parameters a, p and q involved are 1, $\frac{\alpha+1}{2}$ and 0 respectively, and holds

$$\frac{|p|a^\alpha}{\Gamma(\alpha+1)} + \frac{|q|a^{2\alpha}}{\Gamma(2\alpha+1)} = \frac{\alpha+1}{2\Gamma(\alpha+1)} < 1 \quad \forall \alpha \in (0,1).$$

Moreover, the behavior of the approximating sequence $(u_n)_n$ given by (18)-(19) is similar to that observed in Example 5.1 since its terms are equal to each other and equal to the real solution at the same time. In this way the pointwise convergence of the entire sequence to the real solution is obviously guaranteed and it is reached at every point of the interval $[0,1]$. Indeed, from (19) we have

$$u_0(t) = t^\alpha$$

and, according to (18), for any $n \geq 0$,

$$\begin{aligned} u_{n+1}(t) &= t^\alpha + L^{-1}\left[\frac{\frac{\alpha+1}{2} s^\alpha L[u_n] + L\left[-\frac{\Gamma(\alpha+2)}{2}\right]}{s^{2\alpha}}\right] \\ &= t^\alpha + \frac{1}{2} L^{-1}\left[\frac{\alpha+1}{s^\alpha} L[u_n] - \frac{\Gamma(\alpha+2)}{s^{2\alpha+1}}\right]. \end{aligned}$$

In particular, for $n = 0$ and thanks to (8) we have

$$\begin{aligned} u_1(t) &= t^\alpha + \frac{1}{2} L^{-1}\left[\frac{\alpha+1}{s^\alpha} L[t^\alpha] - \frac{\Gamma(\alpha+2)}{s^{2\alpha+1}}\right] \\ &= t^\alpha + \frac{1}{2} L^{-1}\left[\frac{\alpha+1}{s^\alpha} \frac{\Gamma(\alpha+1)}{s^{\alpha+1}} - \frac{\Gamma(\alpha+2)}{s^{2\alpha+1}}\right] = t^\alpha. \end{aligned}$$

In short, we have seen that $u_0(t) = t^\alpha$ implies $u_1(t) = t^\alpha$. From this, and given the recurrence that defines u_n, we conclude that $u_n(t) = t^\alpha$ for all $n \geq 1$.

Finally, for any $t \in [0,1]$

$$\lim_{n\to\infty} u_n(t) = t^\alpha$$

which agrees with the exact solution of (27).

Example 5.4. *Let* $0 < \alpha < 1$*. Consider the following second-order fractional differential equation*

$$\begin{cases} D_0^{2\alpha} u(t) = 0.5 D_0^{\alpha} u(t) + 0.5 u(t) \quad \textit{in } (0,1) \\ u(0) = 1, \quad D_0^{\alpha} u(0) = 1. \end{cases} \tag{28}$$

As usual, we will denote by E_α the Mittag-Leffler function of order α, that is

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}.$$

Taking into account that the Mittag-Leffler function of order α verifies (c.f. [4, Theorem 4.3])

$$D_0^{\alpha} E_\alpha(z^\alpha) = E_\alpha(z^\alpha),$$

it follows that $u(t) = E_\alpha(t^\alpha) = \sum_{k=0}^{\infty} \frac{t^{\alpha k}}{\Gamma(\alpha k + 1)}$ is the exact solution of (28).

Unlike the previous examples, the application of our method in this specific case does not yield the exact solution, at least not in a straightforward manner. Nevertheless, even after a limited number of iterations, our approach enables us to obtain highly accurate approximations of the solution. This holds true, particularly within the range of α where the condition (20) is satisfied.

In this regard, it should be noted that the condition (20) can be expressed as $\frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(2\alpha+1)} < 2$ and is valid for any $\alpha \in (0.625, 1)$.

On the other hand, the iteration formula (18)-(19) reduces to

$$\begin{cases} u_0(t) = 1 + \dfrac{1}{\Gamma(\alpha+1)} t^\alpha, \\ u_{n+1}(t) = 1 + \dfrac{0.5}{\Gamma(\alpha+1)} t^\alpha + 0.5 L^{-1}\left[\dfrac{s^\alpha + 1}{s^{2\alpha}} L[u_n]\right], \quad n \geq 0, \end{cases}$$

and, after an straightforward calculation, for any $n \geq 1$ we get

$$u_n(t) = \sum_{k=0}^{n+1} \frac{t^{k\alpha}}{\Gamma(k\alpha + 1)} + \sum_{k=n+2}^{2n+1} a_{n,k} \frac{t^{k\alpha}}{\Gamma(k\alpha+1)} \tag{29}$$

where

$$a_{1,3} = \frac{1}{2} \quad \text{and } a_{n,k} = \frac{a_{n-1,k-1} + a_{n-1,k-2}}{2} \quad \text{with } a_{n-1,n} = 1 \quad \text{and } a_{n-1,2n} = 0$$

(notice that the first term in the right-hand side of (29) is a truncation of the real solution).

In particular,

$$u_1(t) = \sum_{k=0}^{2} \frac{t^{k\alpha}}{\Gamma(k\alpha+1)} + \frac{1}{2}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

$$u_2(t) = \sum_{k=0}^{3} \frac{t^{k\alpha}}{\Gamma(k\alpha+1)} + \frac{3}{4}\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \frac{1}{4}\frac{t^{5\alpha}}{\Gamma(5\alpha+1)}$$

$$u_3(t) = \sum_{k=0}^{4} \frac{t^{k\alpha}}{\Gamma(k\alpha+1)} + \frac{7}{8}\frac{t^{5\alpha}}{\Gamma(5\alpha+1)} + \frac{1}{2}\frac{t^{6\alpha}}{\Gamma(6\alpha+1)} + \frac{1}{8}\frac{t^{7\alpha}}{\Gamma(7\alpha+1)}$$

$$u_4(t) = \sum_{k=0}^{5} \frac{t^{k\alpha}}{\Gamma(k\alpha+1)} + \frac{15}{16}\frac{t^{6\alpha}}{\Gamma(6\alpha+1)} + \frac{11}{16}\frac{t^{7\alpha}}{\Gamma(7\alpha+1)} + \frac{5}{16}\frac{t^{8\alpha}}{\Gamma(8\alpha+1)} + \frac{1}{16}\frac{t^{9\alpha}}{\Gamma(9\alpha+1)}$$

In Figure 1, the graph on the left displays the real solution u and the approximating function u_4 for $\alpha = 0.63$, represented by solid and dashed lines respectively. On the right side, the graph shows the real solution u and the approximating function u_4 for $\alpha = 0.9$, also distinguished by solid and dashed lines respectively.

In accordance with our findings (see Theorem 4.1 and Corollary 4.3) and observing that $\alpha = 0.63$ and $\alpha = 0.9$ fall within the interval $(0.621, 1)$ (as previously observed, the condition (20) is satisfied over this interval) we anticipate a good approximation of u through the terms of the sequence u_n across the domain $[0, 1]$. In fact, as we can see from the overlap of the curves, the approximation is remarkably satisfactory with just a few iterations. Furthermore, consistent with expectations derived from (21), better performance is observed as α approaches 1 because the quantity $\gamma = \dfrac{1}{2\Gamma(\alpha+1)} + \dfrac{1}{2\Gamma(2\alpha+1)}$ decreases as α increases.

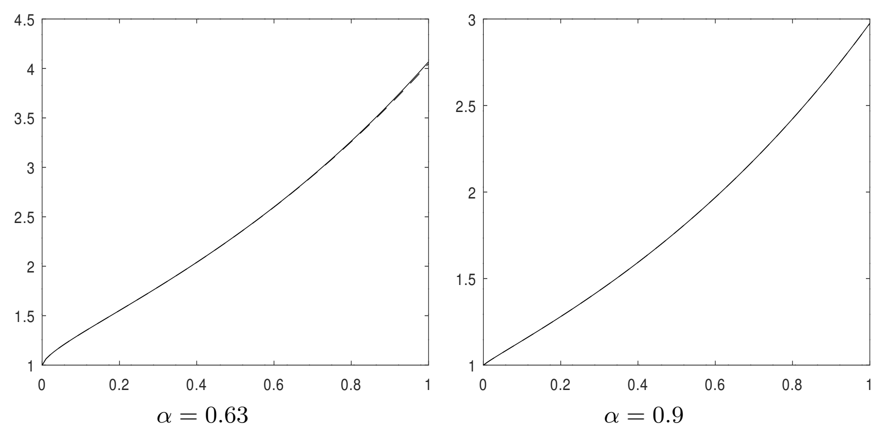

FIGURE 1. The function $u(t) = E_\alpha(t^\alpha)$ is represented by a solid line, while the function u_4 is represented by a dashed line (the Mittag-Leffler function values were obtained from [25]).

On the other hand, in Figure 2, we depict the real solution u and its corresponding approximating function u_4 for $\alpha = 0.4$ on the left, using solid and dashed lines respectively. Similarly, on the right side, we present the real solution u and its approximating function u_4 for $\alpha = 0.1$, also represented by solid and dashed lines respectively.

In this case, for both values of α, condition (20) is not satisfied. Consequently, the approximation of u using the sequence terms is not guaranteed within the domain $[0, 1]$. In fact, for these specific values of α, it is evident from the graphical representation that u_4 does not provide a reliable approximation of u, unlike what we observe when $\alpha \in (0.625, 1)$. Moreover, it is noteworthy that the behavior deteriorates as α decreases, since $\gamma = \dfrac{1}{2\Gamma(\alpha+1)} + \dfrac{1}{2\Gamma(2\alpha+1)} > 1$ increases as α approaches 0.

Additionally, note that the condition (20) is satisfied for these α values if the interval $[0, a]$, where $a << 1$, is under consideration instead of $[0, 1]$. In other words, the approximation property holds on $[0, a]$ with $a = a(\alpha) << 1$ (but not necessarily on $[0, 1]$) when α does not belong to $(0.625, 1)$.

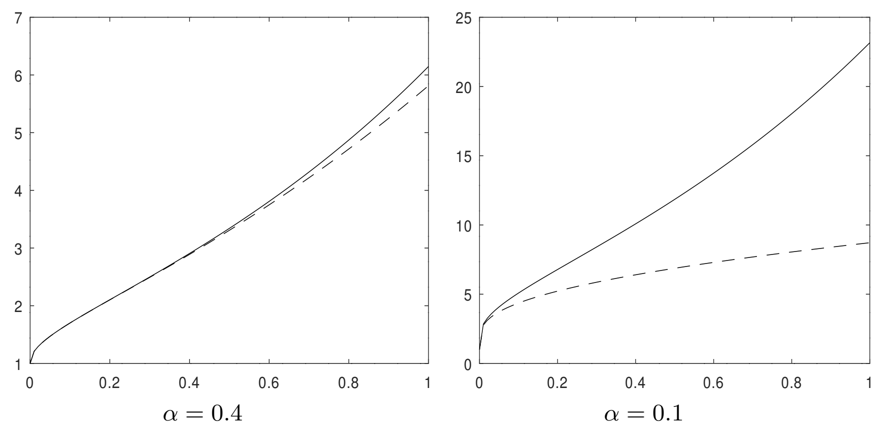

FIGURE 2. The function $u(t) = E_\alpha(t^\alpha)$ is represented by a solid line, while the function u_4 is represented by a dashed line (the Mittag-Leffler function values were obtained from [25]).

6. Conclusions

A fractional variational iteration method, which can be easily generalized and extended to higher-order fractional differential equations, is proposed and studied in detail for the second-order case. In this approach, the Lagrange multiplier corresponding to the correction term is explicitly determined by means of the Laplace transform.

The convergence of the approximating sequence is proved under the assumption (20) involving the coefficients of the equation, the fractional order and the length of the interval domain. On a first reading, condition (20) may seem too restrictive, however, Example 5.4 shows that it is not really so. Indeed, this example illustrates that if the requirement (20) is not verified, the approximation to the real solution is not guaranteed, or at the very least, it deteriorates considerably. Furthermore, associated with this example, the following two significant observations can be made: First, for the domain interval $[0, 1]$, a good approximation is achieved when

α falls within $(0.625, 1)$ which is in accordance with the fact that the condition (20) is satisfied. Second, if α does not belong to $(0.625, 1)$, then convergence is observed over a considerably smaller subinterval $[0, a] \subset [0, 1]$ which is in accordance with the fact that the condition (20) is satisfied for $a << 1$.

Finally, the applicability of the method, as well as the confirmation of the theoretical results obtained, is observed in the study of the proposed numerical examples. In particular, exact analytical solutions were obtained for most of them, demonstrating the practical effectiveness of the method.

References

[1] Arafa A. A. M., Hagaga A. M. SH., *A new analytic solution of fractional coupled Ramani equation.* Chin. J. Phys., 60, 388–406 (2019).

[2] Arafa A. A. M., Khalil M., Sayed A., *A Non-Integer Variable Order Mathematical Model of Human Immunodeficiency Virus and Malaria Coinfection with Time Delay.* Hindawi, Complexity, Vol. 2019, 13 pages (2019).

[3] Cohen M. A., *Numerical methods for Laplace transform inversion.* Numerical Methods and Algorithms, Springer, Boston (2007).

[4] Diethelm K., *The analysis of fractional differential equations. An application-oriented exposition using differential operators of Caputo type.* Lecture Notes in Mathematics, Springer, Berlin (2010).

[5] Diethelm K., Ford J., Ford N., Weilbeer M., *Pitfalls in fast numerical solvers for fractional differential equations.* J. Comput. Appl. Math., 186, 482–503 (2006).

[6] El-Saka H. A. A., Arafa A. A. M., Gouda M. I., *Dynamical analysis of a fractional SIRS model on homogenous networks.* Adv. Differ. Equ., 2019:144 (2019).

[7] El-Sayed A. M. A., Arafa A. A. M., Khalil M., Hassan A., *A Mathematical Model with Memory for Propagation of Computer Virus Under Human Intervention.* Progr. Fract. Differ. Appl., 2, 105-113 (2016).

[8] Elbeleze A. A., Kiliçman A., Taib B. M., *Fractional variational iteration method and its application to fractional partial differential equation.* Math. Probl. Eng., 10 pages (2013).

[9] EsmaeiliS., Shamsi M., Luchko Y., *Numerical solution of fractional differential equations with a collocation method based on Müntz polynomials.* Comput. Math. Appl., 62, 918–929 (2011).

[10] Ford N., Connolly J., *Comparison of numerical methods for fractional differential equations.* Commun. Pure Appl. Anal., 5, 289–307 (2006).

[11] Ghoreishi F., Yazdani S., *An extension of the spectral Tau method for numerical solution of multi-order fractional differential equations with convergence analysis.* Comput. Math. Appl., 61, 30–43 (2011).

[12] Jafari H., Das S., Tajadodi H., *Solving a multi-order fractional differential equation using homotopy analysis method.* J. King Saud Univ. Sci, 23, 151-155 (2011).

[13] Jafari H., Tajadodi H., Baleanu D., *A modified variational iteration method for solving fractional Ricatti Differential equation by Adomian polynomials.* Fract. Calc. Appl. Anal., 16, 109–122 (2013).

[14] Khan Y., Faraz N., Yildirim A., Wu Q., *Fractional variational iteration method for fractional initial-boundary value problems arising in the application of nonlinear science.* Comput. Math. with Appl., 62, 2273–2278 (2011).

[15] Krishnasamy V., Mashayekhi S., Razzaghi M., *Numerical Solutions of Fractional Differential Equations by Using Fractional Taylor Basis.* IEEE/CAA J. Autom. Sin., 4, 98-106 (2017).

[16] Luchko Y., Gorenflo R., *An operational method for solving fractional differential equations with the Caputo derivatives.* Acta Math. Vietnam., 24, 207–233 (1999).

[17] Li C., Wang Y., *Numerical algorithm based on Adomian decomposition for fractional differential equations.* Comput. Math. Appl., 57, 1672–1681 (2009).

[18] Li C., Zeng F., *Finite Difference Methods for Fractional Diffrential Equations.* Int. J. Bifurcat. Chaos, 22, 1230014 (2012).

[19] Momani S., Odibat Z., *Numerical approach to differential equations of fractional order.* J. Comput. Appl. Math., 207, 96-110 (2007).

[20] Momani S., Odibat Z., *The variational iteration method: An efficient scheme for handling fractional differential equations in fluid mechanics*. Comput. Math. with Appl., 58, 2199-2208 (2009).

[21] Monzón G., *On numerical approximation of second-order fractional differential equations in the frame of the Caputo fractional derivative*. J. Fractional Calc. & Appl., 14, 200-213 (2023).

[22] Odibat Z., *A study on the convergence of variational iteration method*. Math. Comput. Model., 51, 1181-1192 (2010).

[23] Odibat Z. M., Shawagfeh N., *Generalized Taylor's formula*. Appl. Math. Comput., 186, 286–293 (2007).

[24] Podlubny I., *Fractional Differential Equations*. Academic Press, San Diego, 1999.

[25] Podlubny I., Mittag-Leffler function, `https://www.mathworks.com/matlabcentral/fileexchange/8738-mittag-leffler-fbiunction`

[26] Prakash A., Goyal M., Gupta S., *Fractional variational iteration method for solving time-fractional Newell-Whitehead-Segel equation*. Nonlinear Eng., 8, 164–171 (2019).

[27] Rida S., Arafa A., Abedl-Rady A., Abdl-Rahaim H., *Fractional physical differential equations via natural transform*. Chin. J. Phys., 55, 1569–1575 (2017).

[28] Rudin W., *Real and Complex Analysis*. Mc Graw Hill, 3rd ed., 1986.

[29] Salkuyeh D. K., *Convergence of the variational iteration method for solving linear systems of ODEs with constant coefficients*. Comput. Math. with Appl., 56, 2027-2033 (2008).

[30] Si-yuan B., *Fractional variational iteration method for fractional Cauchy problems*. Math. Probl. Eng., 2014, 5 pages (2014).

[31] Saadatmandi A., Dehghan M., *A new operational matrix for solving fractional-order differential equations*. Comput. Math. Appl., 59, 1326–1336 (2010).

[32] Tatari M., Dehghan M., *On the convergence of He's variational iteration method*. J. Comput. Appl. Math., 207, 121-128 (2007).

[33] Wu GC., *A fractional variational iteration method for solving fractional nonlinear differential equations*. Comput. Math. with Appl., 61, 2186–2190 (2011).

[34] Wu GC., Baleanu D., *Variational iteration method for fractional calculus - a universal approach by Laplace transform*. Adv. Differ. Equ., 2013:18 (2013).

[35] Wu GC., Baleanu D., *Variational iteration method for the Burger's flow with fractional derivatives - New Lagrange multipliers*. Appl. Math. Model., 37, 6183-6190 (2013).

G. Monzón

Instituto de Ciencias, Universidad Nacional de General Sarmiento, Buenos Aires, Argentina

Email address: `gmonzon@campus.ungs.edu.ar`