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ZEROS AND WEIGHTED VALUE SHARING OF q-SHIFT DIFFERENCE-DIFFERENTIAL POLYNOMIALS OF ENTIRE AND MEROMORPHIC FUNCTIONS

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Abstract. In this research work, we investigate the uniqueness problems and distribution of zeroes of q-shift difference-differential polynomials of entire and meromorphic functions having zero order in the complex plane $\mathbb{C}$ of the form $(\mathbb{f}^n P(\mathbb{f})\Delta_q(\mathbb{f}))^{(k)}$ and $(\mathbb{g}^n P(\mathbb{g})\Delta_q(\mathbb{g}))^{(k)}$, where $P(\mathbb{f})$ is a polynomial with constant coefficients of degree m, which is given in Lemma 2.2 and $\Delta_q(\mathbb{f})$ is a q-difference operator defined as $\Delta_q(\mathbb{f}) = \mathbb{f}(qz+c) - \mathbb{f}(qz)$, which share a small function $\varphi(z), \infty$ CM. By considering the concept of weighted sharing introduced by I. Lahiri (Nagoya Math. J. 161 (2001), 193–206), we also investigate the uniqueness problem of q-shift difference-differential polynomials sharing a small function $\varphi(z)$ with weight $\mathcal{L}$, for the cases $\mathcal{L} \geq 2, \mathcal{L} = 1$ and $\mathcal{L} = 0$ for a zero ordered entire functions. Our results improve, generalize, and extend earlier results due to Zhao and Zhang (J. Contemp. Math. Anal. 50 (2), 63–69). We have also given suitable examples to justify our results.

1. Introduction and Main Results

We assume the reader is familiar with standard symbols and fundamental results of Nevanlinna's theory [9]. A meromorphic function $\mathbb{f}(z)$ in the complex plane $\mathbb{C}$ is a function that is analytic in $\mathbb{C}$ except for the set of isolated points, which are poles of the function. If no poles occur, then $\mathbb{f}(z)$ is an entire function. Let $\mathbb{f}$ and $\mathbb{g}$ be two meromorphic functions and a point a in $\mathbb{C} \cup \{\infty\}$. We say that $\mathbb{f}$ and $\mathbb{g}$ share a IM (Ignoring Multiplicity) when $\mathbb{f} - a$ and $\mathbb{g} - a$ have the same zeros. If $\mathbb{f} - a$ and $\mathbb{g} - a$ have the same zeros with the same multiplicities, then we say that $\mathbb{f}$ and $\mathbb{g}$ share a CM (Counting Multiplicity).

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Given a meromorphic function $\mathbb{f}(z)$, recall that $\varphi(z) \not\equiv 0, \infty$, is a small function with respect to $\mathbb{f}(z)$, if $T(r, \varphi) = S(r, \mathbb{f})$, where $S(r, \mathbb{f})$ denotes any quantity satisfying $S(r, \mathbb{f}) = o(T(r, \mathbb{f}))$ as $r \to \infty$ outside a possible exceptional set of finite logarithmic measures.

For convenience, we assume that $S(\mathbb{f})$ includes all constant functions and $\tilde{S}(\mathbb{f}) = S(\mathbb{f}) \cup \{\infty\}$. For $\varphi \in \tilde{S}(\mathbb{f})$ and $S \subset \tilde{S}$, we define

$$E(S, \mathbb{f}) = \cup_{\varphi \in S}\{z : \mathbb{f}(z) - \varphi(z) = 0, \text{counting multiplicity}\},$$
$$\overline{E}(S, \mathbb{f}) = \cup_{\varphi \in S}\{z : \mathbb{f}(z) - \varphi(z) = 0, \text{ignoring multiplicity}\}.$$

We now explain the following definitions and notations used in the paper.

Definition 1.1. [13] Let $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; \mathbb{f})$ the set of all a-points of $\mathbb{f}$ where an a-point of multiplicity m is counted m times if $m \le k$ and $k+1$ times if $m > k$. If $E_k(a; \mathbb{f}) = E_k(a; \mathbb{g})$, we say that $\mathbb{f}, \mathbb{g}$ share the value a with weight k.

The definition implies that if $\mathbb{f}, \mathbb{g}$ share a value a with weight k, then z_0 is an a-point of $\mathbb{f}$ with multiplicity $m(\le k)$ if and only if it is an a-point of $\mathbb{g}$ with multiplicity $m(\le k)$ and z_0 is an a-point of $\mathbb{f}$ with multiplicity $m(> k)$ if and only if it is an a-point of $\mathbb{g}$ with multiplicity $n(> k)$, where m is not necessarily equal to n. We write $\mathbb{f}, \mathbb{g}$ share (a, k) to mean that $\mathbb{f}, \mathbb{g}$ share the value a with weight k. If $\mathbb{f}, \mathbb{g}$ share (a, k), then $\mathbb{f}, \mathbb{g}$ share (a, p) for any integer $p, 0 \le p < k$. Also, we note that $\mathbb{f}, \mathbb{g}$ share a value a IM or CM if and only if $\mathbb{f}, \mathbb{g}$ share $(a, 0)$ or (a, ∞) respectively.

Definition 1.2. [14] Let $\mathbb{f}, \mathbb{g}$ share the value a IM. We denote by $N_*(r, a; \mathbb{f}, \mathbb{g})$ the reduced counting function of those a-points of $\mathbb{f}$ whose multiplicities differ from the corresponding a-points of $\mathbb{g}$.

Clearly, $\overline{N}_*(r, a; \mathbb{f}, \mathbb{g}) \equiv \overline{N}_*(r, a; \mathbb{g}, \mathbb{f})$ and $\overline{N}_*(r, a; \mathbb{f}, \mathbb{g}) \equiv \overline{N}_L(r, a; \mathbb{f}) + \overline{N}_L(r, a; \mathbb{g})$.

Definition 1.3. [12] Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$.

(i) $N(r, a; \mathbb{f}| \ge p)(\overline{N}(r, a; \mathbb{f}| \ge p))$ denotes the counting function (reduced counting function) of those a-points of $\mathbb{f}$ whose multiplicities are not less than p.

(ii) $N(r, a; \mathbb{f}| \le p)(\overline{N}(r, a; \mathbb{f}| \le p)$ denotes the counting function (reduced counting function) of those a-points of $\mathbb{f}$ whose multiplicities are not greater than p.

In 1959, it was shown by Hayman [9] that if meromorphic function $\mathbb{f}$ and its derivative $\mathbb{f}'$ take every non-zero complex value infinitely often if $n \ge 3$. Yang and Hua [30] obtained some results about the uniqueness problems for entire functions. Since then, the difference has become a subject of significant interest (see [17, 21, 22, 23, 31, 32]).

Recently, many mathematicians have been working on difference equations, the difference product, and the q-difference analogues of the value distribution of entire and meromorphic functions in the complex plane ([4, 5, 6, 10, 16, 19, 24, 25, 26, 33]). In 2006, Halburd and Korhonen [7] established a difference analogue of the Logarithmic Derivative Lemma, and then applying it, many results on meromorphic solutions of complex difference equations have been proved. After that, Barnett, Halburd, Korhonen, and Morgan [3] also established a q-difference analogue of the Logarithmic Derivative Lemma.

In 2011, Liu and Cao [21] obtained results on the uniqueness and value distributions of q-shift difference polynomials. Here, we only state some results.

Theorem A. [[21], Theorem 1.1] Let $\mathbb{f}(z)$ be a transcendental meromorphic (resp. entire) function with zero order and let m, n be positive integers and a, q be non-zero complex constants. If $n \geq 6$ (resp. $n \geq 2$), then $(\mathbb{f}(z)^n(\mathbb{f}(z)^m - a)\mathbb{f}(qz + c)) - \varphi(z)$ has infinitely many zeros, where $\varphi(z)$ is a non-zero small function with respect to $\mathbb{f}(z)$. In particular, if $\mathbb{f}(z)$ is a transcendental entire function and $\varphi(z)$ is a non-zero rational function, then m and n can be any positive integers.

Theorem B. [[21], Theorem 1.5] Let $\mathbb{f}(z)$ and $\mathbb{g}(z)$ be transcendental entire functions with zero order. If $n \geq m+5$, and $\mathbb{f}(z)^n(\mathbb{f}(z)^m - a)\mathbb{f}(qz+c)$ and $\mathbb{g}(z)^n(\mathbb{g}(z)^m - a)\mathbb{g}(qz + c)$ share a non-zero polynomial $p(z)$ CM, then $\mathbb{f}(z) \equiv \mathbb{g}(z)$.

Recently, Zhao and Zhang [34], based on Theorems A and B, studied the k^{th} derivative of q-shift difference polynomials and obtained the following results.

Theorem C. Let $\mathbb{f}(z)$ be a transcendental meromorphic function with zero order, and let n, k be positive integers. If $n > k + 5$, then $(\mathbb{f}(z)^n\mathbb{f}(qz + c))^{(k)} - 1$ has infinitely many zeros.

Theorem D. Let $\mathbb{f}(z)$ be a transcendental entire function with zero order and let n, k be positive integers, then $(\mathbb{f}(z)^n\mathbb{f}(qz + c))^{(k)} - 1$ has infinitely many zeros.

Theorem E. Let $\mathbb{f}(z)$ and $\mathbb{g}(z)$ be transcendental entire functions with zero order and let n, k be positive integers. If $n > 2k + 5$, and $(\mathbb{f}(z)^n\mathbb{f}(qz + c))^{(k)}$ and $(\mathbb{g}(z)^n\mathbb{g}(qz + c))^{(k)}$ share z CM, then $\mathbb{f} \equiv t\mathbb{g}$ for a constant t with $t^{n+1} = 1$.

Theorem F. Let $\mathbb{f}(z)$ and $\mathbb{g}(z)$ be transcendental entire functions with zero order, and let n, k be positive integers. If $n > 2k + 5$, and $(\mathbb{f}(z)^n\mathbb{f}(qz + c))^{(k)}$ and $(\mathbb{g}(z)^n\mathbb{g}(qz + c))^{(k)}$ share 1 CM, then $\mathbb{f} \equiv t\mathbb{g}$ for a constant t with $t^{n+1} = 1$.

When sharing a single value IM, they proved the following two results.

Theorem G. Let $\mathbb{f}(z)$ and $\mathbb{g}(z)$ be transcendental entire functions with zero order and let n, k be positive integers. If $n > 5k + 11$, and $(\mathbb{f}(z)^n\mathbb{f}(qz + c))^{(k)}$ and $(\mathbb{g}(z)^n\mathbb{g}(qz + c))^{(k)}$ share z IM, then $\mathbb{f} \equiv t\mathbb{g}$ for a constant t with $t^{n+1} = 1$.

Theorem H. Let $\mathbb{f}(z)$ and $\mathbb{g}(z)$ be transcendental entire functions with zero order and let n, k be positive integers. If $n > 5k + 11$, and $(\mathbb{f}(z)^n\mathbb{f}(qz + c))^{(k)}$ and $(\mathbb{g}(z)^n\mathbb{g}(qz + c))^{(k)}$ share 1 IM, then $\mathbb{f} \equiv t\mathbb{g}$ for a constant t with $t^{n+1} = 1$.

For a meromorphic (entire) function $\mathbb{f}(z)$ and a non-zero complex constant c, for $n \in \mathbb{N}$, we define its q-shift by $\mathbb{f}(qz + c)$ and q-difference operator by $\Delta_q(\mathbb{f}) = \mathbb{f}(qz + c) - \mathbb{f}(qz)$.

It is quite interesting to check the uniqueness of $\mathbb{f}(z)$ and $\mathbb{g}(z)$ when one can replace $\mathbb{f}(qz + c)$ with $\Delta_q(\mathbb{f})$. Therefore, asking the following question in the above-stated Theorems C-H is inevitable.

Question 1.1. What can be said about the uniqueness of $\mathbb{f}(z)$ and $\mathbb{g}(z)$ if one replace $(\mathbb{f}(z)^n\mathbb{f}(qz+c))^{(k)}$ by the difference polynomial $(\mathbb{f}(z)^nP(\mathbb{f})\Delta_q(\mathbb{f}))^{(k)}$ in Theorems C-H?

In this article, we generalize Theorems C-H to the case of difference-differential polynomials defined above. We now present the following theorems, which are the main results of this paper.

Theorem 1.1. Let $\mathbb{f}(z)$ be a transcendental meromorphic function of zero order such that $\Delta_q(\mathbb{f}) \not\equiv 0$ and $\varphi(z)$ be a small function with respect to $\mathbb{f}(z)$. If $n > k+7$, then the difference-differential polynomial $(\mathbb{f}(z)^n P(\mathbb{f})\Delta_q(\mathbb{f}))^{(k)} - \varphi(z)$ has infinitely many zeros.

Theorem 1.2. Let $\mathbb{f}(z)$ be a transcendental entire function with zero order, and let n, m, k be positive integers such that $\Delta_q(\mathbb{f}) \not\equiv 0$ and $\varphi(z)$ be a small function with respect to $\mathbb{f}(z)$, then the difference-differential polynomial $(\mathbb{f}(z)^n P(\mathbb{f})\Delta_q(\mathbb{f}))^{(k)} - \varphi(z)$ has infinitely many zeros.

Theorem 1.3. Let $\mathbb{f}(z)$ and $\mathbb{g}(z)$ be transcendental meromorphic (resp. entire) functions of zero order, and n, k, m be three positive integers. Suppose that c is a non-zero complex constant such that $\Delta_q(\mathbb{f}) \not\equiv 0$ and $\Delta_q(\mathbb{g}) \not\equiv 0$. If $n > 2k+m+10$ (resp. $n > 2k+m+4$) and $(\mathbb{f}(z)^n P(\mathbb{f})\Delta_q(\mathbb{f}))^{(k)}$ and $(\mathbb{g}(z)^n P(\mathbb{g})\Delta_q(\mathbb{g}))^{(k)}$ share $\varphi(z), \infty$ CM, then one of the following two results holds:

(a) $\mathbb{f}(z) \equiv t\mathbb{g}(z)$, where t is a constant such that $t^d = 1, d = gcd(\Upsilon_0, \Upsilon_1, \ldots, \Upsilon_m)$, where $\Upsilon_i' s$ are defined by

$$\Upsilon_i = \begin{cases} n+i+1, & \text{if} \quad a_i \neq 0 \\ n+m+1, & \text{if} \quad a_i = 0 \end{cases} \qquad i = 0, 1, \ldots, m.$$

(b) $\mathbb{f}$ and $\mathbb{g}$ satisfy the algebraic equation $R(\omega_1, \omega_2) = 0$, where $R(\omega_1, \omega_2)$ is given by

$$R(\omega_1, \omega_2) = \omega_1^n P(\omega_1)\Delta_q(\omega_1) - \omega_2^n P(\omega_2)\Delta_q(\omega_2).$$

We also proved the following result by using weighted sharing for transcendental entire functions of zero order.

Theorem 1.4. Let $\mathbb{f}(z)$ and $\mathbb{g}(z)$ be transcendental entire functions of zero order, and let n, k, m be three positive integers. Suppose that c is a non-zero complex constant such that $\Delta_q(\mathbb{f}) \not\equiv 0$ and $\Delta_q(\mathbb{g}) \not\equiv 0$. Let $(\mathbb{f}(z)^n P(\mathbb{f})\Delta_q(\mathbb{f}))^{(k)}$ and $(\mathbb{g}(z)^n P(\mathbb{g})\Delta_q(\mathbb{g}))^{(k)}$ share $(\varphi(z), \mathcal{L})$ and one of the following conditions hold:

(i) $\mathcal{L} \geq 2$ and $n > 2k+m+5$.
(ii) $\mathcal{L} = 1$ and $n > \frac{5k+4m}{2} + 6$.
(iii) $\mathcal{L} = 0$ and $n > 5k+4m+11$.

Then the conclusion of Theorem 1.3 holds:

The following examples show that the conclusions of Theorems 1.3 and 1.4 occur.

Example 1.1. Let $\mathbb{f}(z) = sin(z), \mathbb{g}(z) = cos(z)$ and $P(z) = (z-1)^6(z+1)^6$. Take $c = \pi, q = 1, k = 0$, then it is easy to verify that $(\mathbb{f}(z)^n P(\mathbb{f})\Delta_q(\mathbb{f}))^{(k)}$ and $(\mathbb{g}(z)^n P(\mathbb{g})\Delta_q(\mathbb{g}))^{(k)}$ share $\varphi(z), \infty$ CM. Here $\mathbb{f}(z)$ and $\mathbb{g}(z)$ satisfy the algebraic equation $R(\mathbb{f}, \mathbb{g}) = 0$,
i.e.,

$$(\mathbb{f}(z)^n P(\mathbb{f})\Delta_q(\mathbb{f}))^{(k)} - (\mathbb{g}(z)^n P(\mathbb{g})\Delta_q(\mathbb{g}))^{(k)} = 0.$$

Clearly, $\mathbb{f}(z)$ and $\mathbb{g}(z)$ satisfies the conclusion (b) of Theorems 1.3 and 1.4.

Example 1.2. Let $P(z) = a_m z^m, \varphi(z) = 1, q = 1, k = 0, \mathbb{f}(z) = e^z, \mathbb{g}(z) = te^z$, where $t^{n+m+1} = 1, n, m \in \mathbb{N}$. Then it is easy to verify that $(\mathbb{f}(z)^n P(\mathbb{f})\Delta_q(\mathbb{f}))^{(k)}$ and $(\mathbb{g}(z)^n P(\mathbb{g})\Delta_q(\mathbb{g}))^{(k)}$ share $\varphi(z)$ CM. Clearly, $\mathbb{f}(z)$ and $\mathbb{g}(z)$ satisfy the conclusions of Theorem 1.3 and 1.4.

To further generalize $\Delta_q(\mathbb{f})$, we now define the Linear q-difference operator of a meromorphic (entire) function $\mathbb{f}$ as $L_q(\mathbb{f}) = \mathbb{f}(qz+c) + c_0\mathbb{f}(qz)$, where c_0 is a non-zero complex constant. Clearly, for the particular choice of the constant $c_0 = -1$, we get $L_q(\mathbb{f}) = \Delta_q(\mathbb{f})$. Hence the following corollaries are the easy consequences of the above four theorems.

Corollary 1.1. Let $\mathbb{f}(z)$ be a transcendental meromorphic function of zero order such that $L_q(\mathbb{f}) \not\equiv 0$ and $\varphi(z)$ be a small function with respect to $\mathbb{f}(z)$. If $n > k+7$, then the difference-differential polynomial $(\mathbb{f}(z)^n P(\mathbb{f}) L_q(\mathbb{f}))^{(k)} - \varphi(z)$ has infinitely many zeros.

Corollary 1.2. Let $\mathbb{f}(z)$ be a transcendental entire function with zero order, and let n, m, k be positive integers such that $L_q(\mathbb{f}) \not\equiv 0$ and $\varphi(z)$ be a small function with respect to $\mathbb{f}(z)$, then the difference-differential polynomial $(\mathbb{f}(z)^n P(\mathbb{f}) L_q(\mathbb{f}))^{(k)} - \varphi(z)$ has infinitely many zeros.

Corollary 1.3. Let $\mathbb{f}(z)$ and $\mathbb{g}(z)$ be transcendental meromorphic (resp. entire) functions of zero order, and n, k, m be three positive integers. Suppose that c is a non-zero complex constant such that $L_q(\mathbb{f}) \not\equiv 0$ and $L_q(\mathbb{g}) \not\equiv 0$. If $n > 2k+m+10$ (resp. $n > 2k+m+4$) and $(\mathbb{f}(z)^n P(\mathbb{f}) L_q(\mathbb{f}))^{(k)}$ and $(\mathbb{g}(z)^n P(\mathbb{g}) L_q(\mathbb{g}))^{(k)}$ share $\varphi(z), \infty$ CM, then one of the following two results holds:

(a) $\mathbb{f}(z) \equiv t\mathbb{g}(z)$, where t is a constant such that $t^d = 1$, $d = gcd(\Upsilon_0, \Upsilon_1, \ldots, \Upsilon_m)$, where $\Upsilon_i' s$ are defined by

$$\Upsilon_i = \begin{cases} n+i+1, & \text{if} \quad a_i \neq 0 \\ n+m+1, & \text{if} \quad a_i = 0 \end{cases} \quad i = 0, 1, \ldots, m.$$

(b) $\mathbb{f}$ and $\mathbb{g}$ satisfy the algebraic equation $R(\omega_1, \omega_2) = 0$, where $R(\omega_1, \omega_2)$ is given by

$$R(\omega_1, \omega_2) = \omega_1^n P(\omega_1) L_q(\omega_1) - \omega_2^n P(\omega_2) L_q(\omega_2).$$

Corollary 1.4. Let $\mathbb{f}(z)$ and $\mathbb{g}(z)$ be transcendental entire functions of zero order, and let n, k, m be three positive integers. Suppose that c is a non-zero complex constant such that $L_q(\mathbb{f}) \neq 0$ and $L_q(\mathbb{g}) \neq 0$. Let $(\mathbb{f}(z)^n P(\mathbb{f}) L_q(\mathbb{f}))^{(k)}$ and $(\mathbb{g}(z)^n P(\mathbb{g}) L_q(\mathbb{g}))^{(k)}$ share $(\varphi(z), \mathcal{L})$ and one of the following conditions hold:

(i) $\mathcal{L} \geq 2$ and $n > 2k+m+5$.
(ii) $\mathcal{L} = 1$ and $n > \frac{5k+4m}{2} + 6$.
(iii) $\mathcal{L} = 0$ and $n > 5k+4m+11$.

Then the conclusion of Corollary 1.3 holds:

2. Some Lemmas

For two non-constant meromorphic (entire) functions $\mathbb{F}$ and $\mathbb{G}$, what follows $\mathbb{H}$ represents the following function.

$$\mathbb{H} = \left(\frac{\mathbb{F}''}{\mathbb{F}'} - \frac{2\mathbb{F}'}{\mathbb{F}-1} \right) - \left(\frac{\mathbb{G}''}{\mathbb{G}'} - \frac{2\mathbb{G}''}{\mathbb{G}-1} \right) \tag{2.1}$$

Lemma 2.1 [20] Let $\mathbb{f}(z)$ be a meromorphic function of zero order. Then on a set of logarithmic density 1

$$m\left(r, \frac{\mathbb{f}(qz+c)}{\mathbb{f}(z)}\right) = o\left(T(r, \mathbb{f})\right).$$

Lemma 2.2 [29] Let $\mathbb{f}(z)$ be a non-constant meromorphic function, and $a_n(\neq 0), a_{n-1}, \dots, a_0$ be small functions with respect to $\mathbb{f}$. Then

$$T(r, a_n\mathbb{f}^n + a_{n-1}\mathbb{f}^{n-1} + \dots + a_1\mathbb{f} + a_0) = T(r, P(\mathbb{f})) = nT(r, \mathbb{f}) + S(r, \mathbb{f}).$$

Lemma 2.3 [27] Let $\mathbb{f}(z)$ be a non-constant meromorphic function of zero order, and let c and q be two non-zero complex numbers. Then

$$T(r, \mathbb{f}(qz+c)) = T(r, \mathbb{f}(z)) + S(r, \mathbb{f}),$$

on a set of logarithmic density 1.

Lemma 2.4 [28] Let $\mathbb{f}$ be a meromorphic function with zero order and c and q be two non-zero complex numbers. Then

$$\begin{aligned} N(r,0;\mathbb{f}(qz+c)) &\leq N(r,0;\mathbb{f}(z)) + S(r,\mathbb{f}),\\ N(r,\infty;\mathbb{f}(qz+c)) &\leq N(r,\infty;\mathbb{f}(z)) + S(r,\mathbb{f}),\\ \overline{N}(r,0;\mathbb{f}(qz+c)) &\leq N(r,0;\mathbb{f}(z)) + S(r,\mathbb{f}),\\ \overline{N}(r,\infty;\mathbb{f}(qz+c)) &\leq N(r,\infty;\mathbb{f}(z)) + S(r,\mathbb{f}), \end{aligned}$$

outside of a possible exceptional set E with finite logarithmic measure.

Lemma 2.5 [18] Let $\mathbb{f}(z)$ be a non-constant meromorphic function, and $p, k,$ be positive integers. Then

$$N_P\left(r, 1; \mathbb{f}^{(k)}\right) \leq T\left(r, \mathbb{f}^{(k)}\right) - T(r,\mathbb{f}) + N_{P+k}\left(r,1;\mathbb{f}\right) + S(r,\mathbb{f}), \tag{2.2}$$

$$N_P(r,1;\mathbb{f}^{(k)}) \leq k\overline{N}(r,\mathbb{f}) + N_{P+k}(r,1;\mathbb{f}). \tag{2.3}$$

Lemma 2.6 [15] If $N(r,0;\mathbb{f}^{(k)}|\mathbb{f}\neq 0)$ denotes the counting function of those zeros of $\mathbb{f}^{(k)}$ which are not the zeros of $\mathbb{f}$, where the zero of $\mathbb{f}^{(k)}$ is counted according to its multiplicity, then

$$N(r,0;\mathbb{f}^{(k)}|\mathbb{f}\neq 0) \leq k\overline{N}(r,\infty;\mathbb{f}) + N(r,0;\mathbb{f}|<k) + k\overline{N}(r,0;\mathbb{f}|\geq k) + S(r,\mathbb{f}).$$

Lemma 2.7 [5] Let $\mathbb{f}(z)$ be a meromorphic function of finite order σ, and let $c \in \mathbb{C}\backslash\{0\}$ be fixed. Then for each $\epsilon > 0$, we have

$$m\left(r, \frac{\mathbb{f}(qz+c)}{\mathbb{f}(z)}\right) + m\left(r, \frac{\mathbb{f}(z)}{\mathbb{f}(qz+c)}\right) = o(r^{\sigma-1+\epsilon}) = S(r,\mathbb{f}).$$

Lemma 2.8 Let $\mathbb{f}(z)$ be a transcendental entire function of zero order, $c \in \mathbb{C}-\{0\}$ be finite complex constants and $n \in \mathbb{N}$. Let $\mathbb{F}_1(z) = (\mathbb{f}^n P(\mathbb{f})\Delta_q(\mathbb{f}))$, where $\Delta_q(\mathbb{f}) \not\equiv 0$. Then we have

$$(n+m)T(r,\mathbb{f}) \leq T(r,\mathbb{F}_1(z)) - N(r,0;\Delta_q(\mathbb{f})) + S(r,\mathbb{f}).$$

Proof. Using the same arguments as in Lemma 2.7 [8], we can quickly obtain Lemma 2.8.

Lemma 2.9 [1] If $\mathbb{f}(z), \mathbb{g}(z)$ be two non-constant meromorphic functions such that they share (1,1). Then

$$2\overline{N}_L(r,1;\mathbb{f}) + 2\overline{N}_L(r,1;\mathbb{g}) + \overline{N}_E^{(2}(r,1;\mathbb{f}) - \overline{N}_{\mathbb{f}>2}(r,1;\mathbb{g}) \leq N(r,1;\mathbb{g}) - \overline{N}(r,1;\mathbb{g}).$$

Lemma 2.10 [2] Let $\mathbb{f}(z), \mathbb{g}(z)$ share (1,1). Then

$$\overline{N}_{\mathbb{f}>2}(r,1;\mathbb{g}) \leq \frac{1}{2}\overline{N}(r,0;\mathbb{f}) + \frac{1}{2}\overline{N}(r,\infty;\mathbb{f}) - \frac{1}{2}N_0(r,0;\mathbb{f}') + S(r,\mathbb{f}),$$

where $N_0(r,0;\mathbb{f}')$ is the counting function of those zeros of $\mathbb{f}'$ which are not the zeros of $\mathbb{f}(\mathbb{f}-1)$.

Lemma 2.11 [2] Let $\mathbb{f}(z)$ and $\mathbb{g}(z)$ be two non-constant meromorphic functions sharing (1,0). Then

$$\begin{aligned}&\overline{N}_L(r,1;\mathbb{f})+2\overline{N}_L(r,1;\mathbb{g})+\overline{N}_E^{(2}(r,1;\mathbb{f})-\overline{N}_{\mathbb{f}>1}(r,1;\mathbb{g})-\overline{N}_{\mathbb{g}>1}(r,1;\mathbb{f})\\&\leq N(r,1;\mathbb{g})-\overline{N}(r,1;\mathbb{g}).\end{aligned}$$

Lemma 2.12 [2] Let $\mathbb{f}(z),\mathbb{g}(z)$ share (1,0). Then

$$\overline{N}_L(r,1;\mathbb{f})\leq\overline{N}(r,0;\mathbb{f})+\overline{N}(r,\infty;\mathbb{f})+S(r,\mathbb{f}).$$

Lemma 2.13 [2] Let $\mathbb{f}(z),\mathbb{g}(z)$ share (1,0). Then

(i) $\overline{N}_{\mathbb{f}>1}(r,1;\mathbb{g})\leq\overline{N}(r,0;\mathbb{f})+\overline{N}(r,\infty;\mathbb{f})+N_0(r,0;\mathbb{f}')$.
(ii) $\overline{N}_{\mathbb{g}>1}(r,1;\mathbb{f})\leq\overline{N}(r,0;\mathbb{g})+\overline{N}(r,\infty;\mathbb{g})+N_0(r,0;\mathbb{g}')$.

Lemma 2.14 [11] Let $\mathbb{f}(z)$ be a nonconstant meromorphic function of zero order, and $q\in\mathbb{C}\backslash\{0\}$. Then

$$\begin{aligned}N(r,0;\mathbb{f}(qz))&\leq N(r,0;\mathbb{f}(z))+S(r,\mathbb{f}),\\N(r,\infty;\mathbb{f}(qz))&\leq N(r,\infty;\mathbb{f}(z))+S(r,\mathbb{f}),\\\overline{N}(r,0;\mathbb{f}(qz))&\leq N(r,0;\mathbb{f}(z))+S(r,\mathbb{f}),\\\overline{N}(r,\infty;\mathbb{f}(qz))&\leq N(r,\infty;\mathbb{f}(z))+S(r,\mathbb{f}),\end{aligned}$$

on a set of logarithmic density 1.

3. Proof of Theorems

Proof of Theorem 1.1. Let $\mathbb{F}(z)=\frac{(\mathbb{f}^nP(\mathbb{f})\Delta_q(\mathbb{f}))^{(k)}}{\varphi(z)}=\frac{(\mathbb{F}_1(z))^{(k)}}{\varphi(z)}$ where, $\mathbb{F}_1(z)=\mathbb{f}^nP(\mathbb{f})\Delta_q(\mathbb{f})$. Using the second fundamental theorem of Nevanlinna, we obtain

$$T(r,\mathbb{F})\leq\overline{N}(r,0;\mathbb{F})+\overline{N}(r,\infty;\mathbb{F})+\overline{N}(r,0;\mathbb{F}-\varphi(z))+S(r,\mathbb{F}).$$

From inequality (2.2) of Lemma 2.5, we get

$$\begin{aligned}T(r,\mathbb{F})\leq{}&\overline{N}(r,0;\mathbb{F}-\varphi(z))+T(r,\mathbb{F})-T(r,\mathbb{F}_1)+\overline{N}(r,\infty;\mathbb{F})+N_{k+1}(r,0;\mathbb{F}_1)\\&+S(r,\mathbb{F}).\end{aligned}$$

Thus the above inequality implies,

$$\begin{aligned}T(r,\mathbb{F}_1)&\leq\overline{N}(r,0;\mathbb{F}-\varphi(z))+N_{k+1}(r,0;\mathbb{F}_1)+\overline{N}(r,\infty;\mathbb{F})+S(r,\mathbb{F})\\&\leq\overline{N}(r,0;\mathbb{F}-\varphi(z))+N_{k+1}(r,0;\mathbb{f}^nP(\mathbb{f})\Delta_q(\mathbb{f}))\\&\quad+\overline{N}\left(r,\infty;(\mathbb{f}^nP(\mathbb{f})\Delta_q(\mathbb{f}))^{(k)}\right)+S(r,\mathbb{F})\\&\leq\overline{N}(r,0;\mathbb{F}-\varphi(z))+(k+1)\overline{N}(r,0;\mathbb{f})+N(r,0;P(\mathbb{f}))+N(r,0;\Delta_q(\mathbb{f}))\\&\quad+\overline{N}\left(r,\infty;[\mathbb{f}^nP(\mathbb{f})(\mathbb{f}(qz+c)-\mathbb{f}(qz))]^{(k)}\right)+S(r,\mathbb{f})\\&\leq\overline{N}(r,0;\mathbb{F}-\varphi(z))+(k+m+5)T(r,\mathbb{f})+S(r,\mathbb{f})\end{aligned}\tag{3.1}$$

On the other hand, from Lemma 2.1, we get

$$\begin{aligned}
(n+m+1)T(r,\mathbb{f}) &= T\left(r,\mathbb{f}^{n+m+1}\right) = m\left(r,\mathbb{f}^{n+m+1}\right) + N\left(r,\mathbb{f}^{n+m+1}\right)\\
&\leq m\left(r,\frac{\mathbb{f}^{n+m+1}\Delta_q(\mathbb{f})}{\Delta_q(\mathbb{f})}\right) + N\left(r,\frac{\mathbb{f}^{n+m+1}\Delta_q(\mathbb{f})}{\Delta_q(\mathbb{f})}\right) + S(r,\mathbb{f})\\
&\leq T(r,\mathbb{F}_1) + T\left(r,\frac{\Delta_q(\mathbb{f})}{\mathbb{f}(z)}\right) + S(r,\mathbb{f})\\
&\leq T(r,\mathbb{F}_1) + m\left(r,\frac{\Delta_q(\mathbb{f})}{\mathbb{f}(z)}\right) + N\left(r,\frac{\Delta_q(\mathbb{f})}{\mathbb{f}(z)}\right) + S(r,\mathbb{f})\\
&\leq T(r,\mathbb{F}_1) + 3T(r,\mathbb{f}) + S(r,\mathbb{f})\\
(n+m-2)T(r,\mathbb{f}) &\leq T(r,\mathbb{F}_1) + S(r,\mathbb{f}). \qquad (3.2)
\end{aligned}$$

From (3.1) and (3.2), we obtain

$$\begin{aligned}
(n+m-2)T(r,\mathbb{f}) &\leq \overline{N}(r,0;\mathbb{F}-\varphi(z)) + (k+m+5)T(r,\mathbb{f}) + S(r,\mathbb{f})\\
(n-k-7)T(r,\mathbb{f}) &\leq \overline{N}(r,0;\mathbb{F}-\varphi(z)) + S(r,\mathbb{f}).
\end{aligned}$$

Noting that $n > k+7$, we conclude that $\mathbb{F}-\varphi(z)$ has infinitely many zeros. This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. Let the functions $\mathbb{F}(z)$ and $\mathbb{F}_1(z)$ as in the proof of Theorem 1.1. Assume the opposite, that $\mathbb{F}(z)-\varphi(z)$ has only a finite number of zeros. Since, by assumption, $\mathbb{f}$ is a transcendental entire function with zero order, there exists a polynomial $P(z)$ such that $\mathbb{F}(z)-\varphi(z) = P(z)$. By integrating k times, we get from the above equation that $\mathbb{F}(z) = Q(z)$, where $Q(z)$ is a polynomial, given by $Q(z) = (\mathbb{f}^n P(\mathbb{f})\Delta_q(\mathbb{f}))^{(k)}$. Obviously, $Q(z) \not\equiv 0$. Hence, we can write

$$\begin{aligned}
(n+m+1)T(r,\mathbb{f}) = T\left(r,\mathbb{f}^{n+m+1}\right) = m\left(r,\mathbb{f}^{n+m+1}\right) &\leq m\left(r,\frac{\mathbb{f}^{n+m+1}\Delta_q(\mathbb{f})}{\Delta_q(\mathbb{f})}\right) + S(r,\mathbb{f})\\
&\leq T(r,\mathbb{F}_1) + m\left(r,\frac{\mathbb{f}(z)}{\Delta_q(\mathbb{f})}\right) + S(r,\mathbb{f})\\
&\leq T(r,\mathbb{F}_1) + S(r,\mathbb{f})\\
&\leq T(r,Q(z)) + S(r,\mathbb{f}), \qquad (3.3)
\end{aligned}$$

which is impossible. Therefore $\mathbb{F}(z)-\varphi(z)$ (or) $(\mathbb{F}_1(z))^{(k)}-\varphi(z)$ has infinitely many zeros. This completes the proof of Theorem 1.2.

Proof of Theorem 1.3. Let $\mathbb{F}(z) = \frac{(\mathbb{F}_1(z))^{(k)}}{\varphi(z)}$ and $\mathbb{G}(z) = \frac{(\mathbb{G}_1(z))^{(k)}}{\varphi(z)}$ where, $\mathbb{F}_1(z) = (\mathbb{f}^n P(\mathbb{f})\Delta_q(\mathbb{f}))$ and $\mathbb{G}_1(z) = (\mathbb{g}^n P(\mathbb{g})\Delta_q(\mathbb{g}))$. Since $\mathbb{F}(z)$ and $\mathbb{G}(z)$ share $\varphi(z), \infty$ CM, there exists an entire function $\alpha(z)$ such that

$$\frac{(\mathbb{f}^n P(\mathbb{f})\Delta_q(\mathbb{f}))^{(k)}/\varphi(z)-1}{(\mathbb{g}^n P(\mathbb{g})\Delta_q(\mathbb{g}))^{(k)}/\varphi(z)-1} = e^{\alpha(z)}. \qquad (3.4)$$

We denote that $e^{\alpha(z)} \equiv$ constant, say c, since $\mathbb{f}(z)$ and $\mathbb{g}(z)$ are both meromorphic functions of zero order. Rewriting (3.4), we obtain

$$c(\mathbb{g}^n P(\mathbb{g})\Delta_q(\mathbb{g}))^{(k)}/\varphi(z) = (\mathbb{f}^n P(\mathbb{f})\Delta_q(\mathbb{f}))^{(k)}/\varphi(z) - 1 + c. \qquad (3.5)$$

Therefore, from (3.5), we have

$$\mathbb{F}(z) - 1 = c(\mathbb{G}(z) - 1).$$

We assert that $c = 1$.
If $c \neq 1$, $\mathbb{f}(z)$ and $\mathbb{g}(z)$ are meromorphic functions of zero order, then we may apply the second fundamental theorem of Nevanlinna, Lemma 2.14, and (3.5) to obtain

$$\begin{aligned} T(r,\mathbb{F}) &\le \overline{N}(r,0;\mathbb{F}) + \overline{N}(r,\infty;\mathbb{F}) + \overline{N}(r,0;\mathbb{F}-1+c) + S(r,\mathbb{F}) \\ &\le \overline{N}(r,0;\mathbb{F}) + \overline{N}(r,\infty;\mathbb{F}) + \overline{N}(r,0;\mathbb{G}) + S(r,\mathbb{F}) + S(r,\mathbb{G}) \\ &\le T(r,\mathbb{F}) - T(r,\mathbb{F}_1) + N_{k+1}(r,0;\mathbb{F}_1) + N_{k+1}(r,0;\mathbb{G}_1) + \overline{N}(r,\infty;\mathbb{F}) \\ &\quad + S(r,\mathbb{F}) + S(r,\mathbb{G}), \end{aligned}$$

Thus,

$$\begin{aligned} T(r,\mathbb{F}_1) &\le N_{k+1}(r,0;\mathbb{F}_1) + N_{k+1}(r,0;\mathbb{G}_1) + \overline{N}(r,\infty;\mathbb{F}) + S(r,\mathbb{F}) + S(r,\mathbb{G}) \\ &\le (k+m+3)T(r,\mathbb{f}) + \overline{N}\left(r,\infty;[\mathbb{f}^n P(\mathbb{f})(\mathbb{f}(qz+c)-\mathbb{f}(qz))]^{(k)}\right) \\ &\quad + (k+m+3)T(r,\mathbb{g}) + S(r,\mathbb{f}) + S(r,\mathbb{g}) \\ &\le (k+m+5)T(r,\mathbb{f}) + (k+m+3)T(r,\mathbb{g}) + S(r,\mathbb{f}) + S(r,\mathbb{g}). \end{aligned} \tag{3.6}$$

Now, from inequality (3.2),

$$(n+m-2)T(r,\mathbb{f}) \le T(r,\mathbb{F}_1) + S(r,\mathbb{f}). \tag{3.7}$$

Substituting (3.7) into (3.6), we obtain that

$$(n+m-2)T(r,\mathbb{f}) \le (k+m+5)T(r,\mathbb{f}) + (k+m+3)T(r,\mathbb{g}) + S(r,\mathbb{f}) + S(r,\mathbb{g}). \tag{3.8}$$

Similarly, we have

$$(n+m-2)T(r,\mathbb{g}) \le (k+m+5)T(r,\mathbb{g}) + (k+m+3)T(r,\mathbb{f}) + S(r,\mathbb{f}) + S(r,\mathbb{g}). \tag{3.9}$$

By combining inequalities (3.8) and (3.9), we get

$$(n-m-2k-10)[T(r,\mathbb{f}) + T(r,\mathbb{g})] \le S(r,\mathbb{f}) + S(r,\mathbb{g}),$$

a contradiction, since $n > 2k+m+10$.
If $c \neq 1$, $\mathbb{f}(z)$ and $\mathbb{g}(z)$ are entire functions of zero order, then we may apply the second fundamental theorem of Nevanlinna, Lemma 2.14, and (3.5) to obtain

$$\begin{aligned} T(r,\mathbb{F}) &\le \overline{N}(r,0;\mathbb{F}) + \overline{N}(r,0;\mathbb{F}-1+c) + S(r,\mathbb{F}) \\ &\le \overline{N}(r,0;\mathbb{F}) + \overline{N}(r,0;\mathbb{G}) + S(r,\mathbb{F}) + S(r,\mathbb{G}) \\ &\le T(r,\mathbb{F}) - T(r,\mathbb{F}_1) + N_{k+1}(r,0;\mathbb{F}_1) + N_{k+1}(r,0;\mathbb{G}_1) + S(r,\mathbb{F}) + S(r,\mathbb{G}), \end{aligned}$$

Thus,

$$\begin{aligned} T(r,\mathbb{F}_1) &\le N_{k+1}(r,0;\mathbb{F}_1) + N_{k+1}(r,0;\mathbb{G}_1) + S(r,\mathbb{F}) + S(r,\mathbb{G}) \\ &\le (k+m+1)T(r,\mathbb{f}) + T(r,\Delta_q(\mathbb{f})) + (k+m+1)T(r,\mathbb{g}) + T(r,\Delta_q(\mathbb{g})) \\ &\quad + S(r,\mathbb{f}) + S(r,\mathbb{g}) \\ &\le (k+m+1)T(r,\mathbb{f}) + m\left(r,\frac{\Delta_q(\mathbb{f})}{\mathbb{f}}\right) + m(r,\mathbb{f}) + (k+m+1)T(r,\mathbb{g}) \\ &\quad + m\left(r,\frac{\Delta_q(\mathbb{g})}{\mathbb{g}}\right) + m(r,\mathbb{g}) + S(r,\mathbb{f}) + S(r,\mathbb{g}) \\ &\le (k+m+2)[T(r,\mathbb{f}) + T(r,\mathbb{g})] + S(r,\mathbb{f}) + S(r,\mathbb{g}). \end{aligned} \tag{3.10}$$

Taking use of the Valiron-Mohon'ko lemma, Lemma 2.1, Lemma 2.14, and above inequality, we deduce that

$$\begin{aligned}(n+m+1)T(r,\mathbb{f}) &= T(r,\mathbb{f}^{n+m}\mathbb{f}(qz))+S(r,\mathbb{f})\\ &= T\left(r,\frac{\mathbb{f}^{n+m}\Delta_q(\mathbb{f})\mathbb{f}(qz)}{\Delta_q(\mathbb{f})}\right)+S(r,\mathbb{f})\\ &\le T\left(r,\mathbb{f}^nP(\mathbb{f})\Delta_q(\mathbb{f})\right)+T\left(r,\frac{\mathbb{f}(qz)}{\Delta_q(\mathbb{f})}\right)+S(r,\mathbb{f})\\ &\le T\left(r,\mathbb{F}_1\right)+N\left(r,\frac{\Delta_q(\mathbb{f})}{\mathbb{f}(qz)}\right)+m\left(r,\frac{\Delta_q(\mathbb{f})}{\mathbb{f}(qz)}\right)+S(r,\mathbb{f})\\ &\le T\left(r,\mathbb{F}_1\right)+N\left(r,0;\mathbb{f}(qz)\right)+S(r,\mathbb{f})\\ &\le T\left(r,\mathbb{F}_1\right)+T\left(r,\mathbb{f}(qz)\right)+S(r,\mathbb{f})\\ (n+m+1)T(r,\mathbb{f}) &\le T\left(r,\mathbb{F}_1\right)+T\left(r,\mathbb{f}(z)\right)+S(r,\mathbb{f}).\end{aligned}$$

Therefore,

$$(n+m)T(r,\mathbb{f})\le T(r,\mathbb{F}_1)+S(r,\mathbb{f}). \tag{3.11}$$

Substituting (3.10) into (3.11), we conclude that

$$(n+m)T(r,\mathbb{f})\le (k+m+2)[T(r,\mathbb{f})+T(r,\mathbb{g})]+S(r,\mathbb{f})+S(r,\mathbb{g}). \tag{3.12}$$

Similarly, we have

$$(n+m)T(r,\mathbb{g})\le (k+m+2)[T(r,\mathbb{f})+T(r,\mathbb{g})]+S(r,\mathbb{f})+S(r,\mathbb{g}). \tag{3.13}$$

By combining the above two equations (3.12) and (3.13), we get

$$(n-m-2k-4)[T(r,\mathbb{f})+T(r,\mathbb{g})]\le S(r,\mathbb{f})+S(r,\mathbb{g}),$$

a contradiction, since $n>2k+m+4$.
Thus, $c=1$ and (3.5) turn into

$$\left(\mathbb{f}^nP(\mathbb{f})\Delta_q(\mathbb{f})\right)^{(k)}=\left(\mathbb{g}^nP(\mathbb{g})\Delta_q(\mathbb{g})\right)^{(k)}.$$

By integrating the above inequality, we have

$$\left(\mathbb{f}^nP(\mathbb{f})\Delta_q(\mathbb{f})\right)^{(k-1)}=\left(\mathbb{g}^nP(\mathbb{g})\Delta_q(\mathbb{g})\right)^{(k-1)}+\rho(z),$$

where $\rho(z)$ is a polynomial of degree at most $(k-1)$. If $\rho(z)\not\equiv 0$, then from the second fundamental theorem of Nevanlinna for the small function and (3.7), we get

$$\begin{aligned}(n+m-2)T(r,\mathbb{f}) &\le T(r,\mathbb{F}_1)+S(r,\mathbb{F}_1)\\ &\le \overline{N}(r,0;\mathbb{F})+\overline{N}(r,0;\mathbb{G})+S(r,\mathbb{F})+S(r,\mathbb{G})\\ (n+m-2)T(r,\mathbb{f}) &\le (k+m+2)[T(r,\mathbb{f})+T(r,\mathbb{g})]+S(r,\mathbb{g}).\end{aligned}$$

Similarly, we have

$$(n+m-2)T(r,\mathbb{g})\le (k+m+2)[T(r,\mathbb{f})+T(r,\mathbb{g})]+S(r,\mathbb{g}).$$

Combining the above two, we can get

$$(n-m-2k-6)[T(r,\mathbb{f})+T(r,\mathbb{g})]\le S(r,\mathbb{f})+S(r,\mathbb{g}),$$

a contradiction, since $n>2k+m+6$.
Thus $\rho(z)\equiv 0$, which implies

$$\mathbb{f}^nP(\mathbb{f})\Delta_q(\mathbb{f})=\mathbb{g}^nP(\mathbb{g})\Delta_q(\mathbb{g}).$$

i.e.,

$$\begin{aligned}&\mathbb{f}^n(a_m\mathbb{f}^m + a_{m-1}\mathbb{f}^{m-1} + \ldots + a_1\mathbb{f} + a_0)(\mathbb{f}(qz+c) - \mathbb{f}(qz))\\&\equiv \mathbb{g}^n(a_m\mathbb{g}^m + a_{m-1}\mathbb{g}^{m-1} + \ldots + a_1\mathbb{g} + a_0)(\mathbb{g}(qz+c) - \mathbb{g}(qz)).\end{aligned} \tag{3.14}$$

Let $h = \frac{\mathbb{f}}{\mathbb{g}}$. Then, the above inequality (3.14) can be written as

$$\begin{aligned}&[a_m\mathbb{g}^m(h^{n+m}h(qz+c) - 1) + a_{m-1}\mathbb{g}^{m-1}(h^{n+m-1}h(qz+c) - 1) + \ldots +\\&a_0(h^n h(qz+c) - 1)]\mathbb{g}(qz+c) + [a_m\mathbb{g}^m(h^{n+m+1} - 1) + a_{m-1}\mathbb{g}^{m-1}(h^{n+m} - 1)\\&+ \ldots + a_0(h^{n+1} - 1)]\mathbb{g}(qz) \equiv 0.\end{aligned}$$

If 'h' is constant, then the above inequality can be written as

$$[a_m\mathbb{g}^m(h^{n+m+1} - 1) + a_{m-1}\mathbb{g}^{m-1}(h^{n+m} - 1) + \ldots + a_0(h^{n+1} - 1)]\Delta_q(\mathbb{g}) \equiv 0.$$

Since $\Delta_q(\mathbb{g}) \not\equiv 0$, we must have

$$a_m(h^{n+m+1} - 1)\mathbb{g}^m + a_{m-1}(h^{n+m} - 1)\mathbb{g}^{m-1} + \ldots + a_0(h^{n+1} - 1) = 0.$$

Then by a similar argument as in Case 2 in the proof of Theorem 11 [29], we obtain $\mathbb{f}(z) \equiv t\mathbb{g}(z)$, where $'t'$ is a constant such that $t^d = 1$, $d = gcd(\Upsilon_0, \Upsilon_1, \ldots, \Upsilon_m)$, where $\Upsilon_i's$ are defined by

$$\Upsilon_i = \begin{cases} n+i+1, & \text{if} \quad a_i \neq 0 \\ n+m+1, & \text{if} \quad a_i = 0 \end{cases} \quad i = 0, 1, \ldots, m.$$

If 'h' is not a constant, then it follows that $\mathbb{f}$ and $\mathbb{g}$ satisfy the algebraic equation $R(\omega_1, \omega_2) = 0$, where $R(\omega_1, \omega_2)$ is given by

$$R(\omega_1, \omega_2) = \omega_1^n P(\omega_1)\Delta_q(\omega_1) - \omega_2^n P(\omega_2)\Delta_q(\omega_2).$$

Proof of Theorem 1.4. Let $\mathbb{F}(z) = \frac{(\mathbb{F}_1(z))^{(k)}}{\varphi(z)}$ and $\mathbb{G}(z) = \frac{(\mathbb{G}_1(z))^{(k)}}{\varphi(z)}$. Where, $\mathbb{F}_1(z) = (\mathbb{f}^n P(\mathbb{f})\Delta_q(\mathbb{f}))$ and $\mathbb{G}_1(z) = (\mathbb{g}^n P(\mathbb{g})\Delta_q(\mathbb{g}))$. Then $\mathbb{F}(z)$ and $\mathbb{G}(z)$ are two transcendental meromorphic functions that share $(\varphi(z), \mathcal{L})$ except the zeros and poles of $\varphi(z)$.

Case 1. Let $\mathbb{H} \not\equiv 0$.

Subcase 1.1. $\mathcal{L} \geq 1$. From (2.1), it can be easily calculated that the possible poles of $\mathbb{H}$ occur at (i) multiple zeros of $\mathbb{F}(z)$ and $\mathbb{G}(z)$, (ii) those 1-points of $\mathbb{F}(z)$ and $\mathbb{G}(z)$ whose multiplicities are different, (iii) zeros of $\mathbb{F}'(\mathbb{G}')$ which are not the zeros of $\mathbb{F}(\mathbb{F}-1)(\mathbb{G}(\mathbb{G}-1))$.

Since $\mathbb{H}$ has only simple poles, we get

$$\begin{aligned}N(r, \infty; \mathbb{H}) \leq\ & \overline{N}_*(r, 1; \mathbb{F}, \mathbb{G}) + \overline{N}(r, 0; \mathbb{F}| \geq 2) + \overline{N}(r, 0; \mathbb{G}| \geq 2) + \overline{N}_0(r, 0; \mathbb{F}')\\&+ \overline{N}_0(r, 0; \mathbb{G}'),\end{aligned} \tag{3.15}$$

where $\overline{N}_0(r, 0; \mathbb{F}')$ is the reduced counting function of those zeros of $\mathbb{F}'$ which are not the zeros of $\mathbb{F}(\mathbb{F}-1)$ and $\overline{N}_0(r, 0; \mathbb{G}')$ is similarly defined.

Let z_0 be a simple zero of $\mathbb{F} - 1$. Then z_0 is a simple zero of $\mathbb{G} - 1$ and a zero of $\mathbb{H}$. So

$$N(r, 1; \mathbb{F}| = 1) \leq N(r, 0; \mathbb{H}) + N(r, \infty; \mathbb{H}) + S(r, \mathbb{F}) + S(r, \mathbb{G}). \tag{3.16}$$

While $\mathcal{L} \geq 2$, using (3.15) and (3.16), we get

$$\begin{aligned}\overline{N}(r,1;\mathbb{F}) &\leq N(r,1;\mathbb{F}|=1) + \overline{N}(r,0;\mathbb{F}|\geq 2)\\ &\leq \overline{N}(r,0;\mathbb{F}|\geq 2) + \overline{N}(r,0;\mathbb{G}|\geq 2) + \overline{N}_*(r,1;\mathbb{F},\mathbb{G}) + \overline{N}_0(r,0;\mathbb{F}')\\ &\quad + \overline{N}_0(r,0;\mathbb{G}') + S(r,\mathbb{f}) + S(r,\mathbb{g}). \end{aligned} \tag{3.17}$$

Now in view of Lemma 2.6, we get

$$\begin{aligned}&\overline{N}_0(r,0;\mathbb{G}') + \overline{N}(r,0;\mathbb{F}|\geq 2) + \overline{N}_*(r,1;\mathbb{F},\mathbb{G})\\ &\leq \overline{N}_0(r,0;\mathbb{G}') + \overline{N}(r,0;\mathbb{F}|\geq 2) + \overline{N}(r,0;\mathbb{F}|\geq 3)\\ &= \overline{N}_0(r,0;\mathbb{G}') + \overline{N}(r,0;\mathbb{G}|\geq 2) + \overline{N}(r,0;\mathbb{G}|\geq 3)\\ &\leq \overline{N}_0(r,0;\mathbb{G}') + N(r,1;\mathbb{G}) - \overline{N}(r,1;\mathbb{G})\\ &\leq N_0(r,0;\mathbb{G}'|\mathbb{G}\neq 0) \leq \overline{N}(r,1;\mathbb{G}) + S(r,\mathbb{g}). \end{aligned} \tag{3.18}$$

Hence using (3.17), (3.18), Lemmas 2.5, 2.7, 2.8, and 2.14, we get that from the second fundamental theorem of Nevanlinna,

$$\begin{aligned}(n+m)T(r,\mathbb{f}) &\leq T(r,\mathbb{F}_1) - N(r,0;\Delta_q(\mathbb{f})) + S(r,\mathbb{f})\\ &\leq T(r,\mathbb{F}) + N_{k+2}(r,0;\mathbb{F}_1) - N_2(r,0;\mathbb{F}) - N(r,0;\Delta_q(\mathbb{f})) + S(r,\mathbb{f})\\ &\leq \overline{N}(r,0;\mathbb{F}) + \overline{N}(r,0;\mathbb{F}_1) + N_{k+2}(r,0;\mathbb{F}_1) - N_2(r,0;\mathbb{F})\\ &\quad - N(r,0;\Delta_q(\mathbb{f})) - \overline{N}_0(r,0;\mathbb{F}') + S(r,\mathbb{f})\\ &\leq \overline{N}(r,0;\mathbb{F}) + N_{k+2}(r,0;\mathbb{F}_1) + \overline{N}(r,0;\mathbb{F}|\geq 2) + \overline{N}(r,0;\mathbb{G}|\geq 2)\\ &\quad + \overline{N}(r,1;\mathbb{F}|\geq 2) + \overline{N}_*(r,1;\mathbb{F},\mathbb{G}) + \overline{N}_0(r,0;\mathbb{G}') - N(r,0;\Delta_q(\mathbb{f}))\\ &\quad - N_2(r,0;\mathbb{F}) + S(r,\mathbb{f}) + S(r,\mathbb{g})\\ &\leq N_{k+2}(r,0;\mathbb{F}_1) + N_2(r,0;\mathbb{G}) - N(r,0;\Delta_q(\mathbb{f})) + S(r,\mathbb{f}) + S(r,\mathbb{g})\\ &\leq N_{k+2}(r,0;\mathbb{F}_1) + N_{k+2}(r,0;\mathbb{G}_1) - N(r,0;\Delta_q(\mathbb{f})) + S(r,\mathbb{f}) + S(r,\mathbb{g})\\ &\leq (k+2)\overline{N}(r,0;\mathbb{f}) + N(r,0;P(\mathbb{f})) + N(r,0;\Delta_q(\mathbb{f}))\\ &\quad + (k+2)\overline{N}(r,0;\mathbb{g}) + N(r,0;P(\mathbb{g})) + N(r,0;\Delta_q(\mathbb{g}))\\ &\quad - N(r,0;\Delta_q(\mathbb{f})) + S(r,\mathbb{f}) + S(r,\mathbb{g})\\ &\leq (k+m+2)T(r,\mathbb{f}) + (k+m+2)T(r,\mathbb{g}) + T(r,\Delta_q(\mathbb{g})) + S(r,\mathbb{f})\\ &\quad + S(r,\mathbb{g})\\ &\leq (k+m+2)T(r,\mathbb{f}) + (k+m+2)T(r,\mathbb{g}) + m(r,\Delta_q(\mathbb{g})) + S(r,\mathbb{f})\\ &\quad + S(r,\mathbb{g})\\ &\leq (k+m+2)T(r,\mathbb{f}) + (k+m+2)T(r,\mathbb{g}) + m\left(r,\frac{\Delta_q(\mathbb{g})}{\mathbb{g}(qz)}\right)\\ &\quad + m(r,\mathbb{g}(qz)) + S(r,\mathbb{f}) + S(r,\mathbb{g})\\ (n+m)T(r,\mathbb{f}) &\leq (k+m+2)T(r,\mathbb{f}) + (k+m+3)T(r,\mathbb{g}) + S(r,\mathbb{f}) + S(r,\mathbb{g}). \end{aligned} \tag{3.19}$$

In a similar way, we can obtain

$$(n+m)T(r,\mathbb{g}) \leq (k+m+2)T(r,\mathbb{g}) + (k+m+3)T(r,\mathbb{f}) + S(r,\mathbb{f}) + S(r,\mathbb{g}). \tag{3.20}$$

Combining (3.19) and (3.20), we see that

$$(n-2k-m-5)[T(r,\mathbb{f})+T(r,\mathbb{g})] \le S(r,\mathbb{f})+S(r,\mathbb{g}),$$

a contradiction, since $n > 2k+m+5$.
While $\mathcal{L}=1$, using (3.15), (3.16), and Lemmas 2.6, 2.9, 2.10, we get

$$\begin{aligned}
\overline{N}(r,1;\mathbb{F}) &\le N(r,1;\mathbb{F}|=1)+\overline{N}_L(r,1;\mathbb{F})+\overline{N}_L(r,1;\mathbb{G})+\overline{N}^{(2}_E(r,1;\mathbb{F})\\
&\le \overline{N}(r,0;\mathbb{F}|\ge 2)+\overline{N}(r,0;\mathbb{G}|\ge 2)+\overline{N}_*(r,1;\mathbb{F},\mathbb{G})+\overline{N}_L(r,1;\mathbb{F})\\
&\quad+\overline{N}_L(r,1;\mathbb{G})+\overline{N}^{(2}_E(r,1;\mathbb{F})+\overline{N}_0(r,0;\mathbb{F}')+\overline{N}_0(r,0;\mathbb{G}')+S(r,\mathbb{f})\\
&\quad+S(r,\mathbb{g})\\
&\le \overline{N}(r,0;\mathbb{F}|\ge 2)+\overline{N}(r,0;\mathbb{G}|\ge 2)+2\overline{N}_L(r,1;\mathbb{F})+2\overline{N}_L(r,1;\mathbb{G})\\
&\quad+\overline{N}^{(2}_E(r,1;\mathbb{F})+\overline{N}_0(r,0;\mathbb{F}')+\overline{N}_0(r,0;\mathbb{G}')+S(r,\mathbb{f})+S(r,\mathbb{g})\\
&\le \overline{N}(r,0;\mathbb{F}|\ge 2)+\overline{N}(r,0;\mathbb{G}|\ge 2)+2\overline{N}_{\mathbb{F}>2}(r,1;\mathbb{G})+N(r,1;\mathbb{G})\\
&\quad-\overline{N}(r,1;\mathbb{G})+\overline{N}_0(r,0;\mathbb{F}')+\overline{N}_0(r,0;\mathbb{G}')+S(r,\mathbb{f})+S(r,\mathbb{g})\\
&\le \overline{N}(r,0;\mathbb{F}|\ge 2)+\frac{1}{2}\overline{N}(r,0;\mathbb{F})+\overline{N}(r,0;\mathbb{G}|\ge 2)+N(r,1;\mathbb{G})\\
&\quad-\overline{N}(r,1;\mathbb{G})+\overline{N}_0(r,0;\mathbb{F}')+\overline{N}_0(r,0;\mathbb{G}')+S(r,\mathbb{f})+S(r,\mathbb{g})\\
&\le \overline{N}(r,0;\mathbb{F}|\ge 2)+\frac{1}{2}\overline{N}(r,0;\mathbb{F})+\overline{N}(r,0;\mathbb{G}|\ge 2)+N(r,0;\mathbb{G}'|\mathbb{G}\ne 0)\\
&\quad+\overline{N}_0(r,0;\mathbb{F}')+S(r,\mathbb{f})+S(r,\mathbb{g})\\
&\le \overline{N}(r,0;\mathbb{F}|\ge 2)+\frac{1}{2}\overline{N}(r,0;\mathbb{F})+N_2(r,0;\mathbb{G})+\overline{N}_0(r,0;\mathbb{F}')+S(r,\mathbb{f})\\
&\quad+S(r,\mathbb{g}).
\end{aligned}\tag{3.21}$$

Hence using (3.21), Lemmas 2.5, 2.7, 2.8, 2.14, and from the second fundamental theorem of Nevanlinna, we get

$$\begin{aligned}
(n+m)T(r,\mathbb{f}) &\le T(r,\mathbb{F}_1)-N(r,0;\Delta_q(\mathbb{f}))+S(r,\mathbb{f})\\
&\le T(r,\mathbb{F})+N_{k+2}(r,0;\mathbb{F}_1)-N_2(r,0;\mathbb{F})-N(r,0;\Delta_q(\mathbb{f}))+S(r,\mathbb{f})\\
&\le \overline{N}(r,0;\mathbb{F})+\overline{N}(r,1;\mathbb{F})+N_{k+2}(r,0;\mathbb{F}_1)-N_2(r,0;\mathbb{F})-\overline{N}_0(r,0;\mathbb{F}')\\
&\quad-N(r,0;\Delta_q(\mathbb{f}))+S(r,\mathbb{f})\\
&\le N_2(r,0;\mathbb{F})+\frac{1}{2}\overline{N}(r,0;\mathbb{F})+N_{k+2}(r,0;\mathbb{F}_1)+N_2(r,0;\mathbb{G})\\
&\quad-N_2(r,0;\mathbb{F})-N(r,0;\Delta_q(\mathbb{f}))+S(r,\mathbb{f})+S(r,\mathbb{g})\\
&\le N_{k+2}(r,0;\mathbb{F}_1)+\frac{1}{2}\overline{N}(r,0;\mathbb{F})+N_2(r,0;\mathbb{G})-N(r,0;\Delta_q(\mathbb{f}))\\
&\quad+S(r,\mathbb{f})+S(r,\mathbb{g})\\
&\le N_{k+2}(r,0;\mathbb{F}_1)+N_{k+2}(r,0;\mathbb{G}_1)+\frac{1}{2}N_{k+1}(r,0;\mathbb{F}_1)-N(r,0;\Delta_q(\mathbb{f}))\\
&\quad+S(r,\mathbb{f})+S(r,\mathbb{g})
\end{aligned}$$

$$
\begin{aligned}
&\leq (k+2)\overline{N}(r,0;\mathbb{f}) + N(r,0;P(\mathbb{f})) + N(r,0;\Delta_q(\mathbb{f}))\\
&\quad + \frac{(k+1)}{2}\overline{N}(r,0;\mathbb{f}) + \frac{1}{2}N(r,0;P(\mathbb{f})) + \frac{1}{2}N(r,0;\Delta_q(\mathbb{f}))\\
&\quad + (k+2)\overline{N}(r,0;\mathbb{g}) + N(r,0;P(\mathbb{g})) + N(r,0;\Delta_q(\mathbb{g}))\\
&\quad - N(r,0;\Delta_q(\mathbb{f})) + S(r,\mathbb{f}) + S(r,\mathbb{g})\\
&\leq \left(\frac{3k+3m+5}{2}\right)T(r,\mathbb{f}) + (k+m+2)T(r,\mathbb{g}) + \frac{1}{2}T(r,\Delta_q(\mathbb{f}))\\
&\quad + T(r,\Delta_q(\mathbb{g})) + S(r,\mathbb{f}) + S(r,\mathbb{g})\\
&\leq \left(\frac{3k+3m+5}{2}\right)T(r,\mathbb{f}) + (k+m+2)T(r,\mathbb{g}) + \frac{1}{2}m\left(r,\frac{\Delta_q(\mathbb{f})}{\mathbb{f}(qz)}\right)\\
&\quad + \frac{1}{2}m(r,\mathbb{f}(qz)) + m\left(r,\frac{\Delta_q(\mathbb{g})}{\mathbb{g}(qz)}\right) + m(r,\mathbb{g}(qz)) + S(r,\mathbb{f}) + S(r,\mathbb{g})
\end{aligned}
$$

$$
(n+m)T(r,\mathbb{f}) \leq \left(\frac{3k+3m+6}{2}\right)T(r,\mathbb{f}) + (k+m+3)T(r,\mathbb{g}) + S(r,\mathbb{f}) + S(r,\mathbb{g}) \tag{3.22}
$$

In a similar way, we can obtain

$$
(n+m)T(r,\mathbb{g}) \leq \left(\frac{3k+3m+6}{2}\right)T(r,\mathbb{g}) + (k+m+3)T(r,\mathbb{f}) + S(r,\mathbb{f}) + S(r,\mathbb{g}) \tag{3.23}
$$

Combining (3.22) and (3.23), we see that

$$
\left(n - \left(\frac{5k+4m}{2}\right) - 6\right)[T(r,\mathbb{f}) + T(r,\mathbb{g})] \leq S(r,\mathbb{f}) + S(r,\mathbb{g}),
$$

a contradiction, since $n > \frac{5k+4m}{2} + 6$.

Subcase 1.2. While $\mathcal{L} = 0$. Here (3.16) changes to

$$
\overline{N}_E^{1)}(r,1;\mathbb{F}|=1) \leq N(r,0;\mathbb{H}) + N(r,\infty;\mathbb{H}) + S(r,\mathbb{F}) + S(r,\mathbb{G}) \tag{3.24}
$$

Using (3.15), (3.24), Lemmas 2.6, 2.11, 2.12, and 2.13, we get

$$
\begin{aligned}
\overline{N}(r,1;\mathbb{F}) &\leq \overline{N}_E^{1)}(r,1;\mathbb{F}) + \overline{N}_L(r,1;\mathbb{F}) + \overline{N}_L(r,1;\mathbb{G}) + \overline{N}_E^{(2}(r,1;\mathbb{F})\\
&\leq \overline{N}(r,0;\mathbb{F}|\geq 2) + \overline{N}(r,0;\mathbb{G}|\geq 2) + \overline{N}_*(r,1;\mathbb{F},\mathbb{G}) + \overline{N}_L(r,1;\mathbb{F})\\
&\quad + \overline{N}_L(r,1;\mathbb{G}) + \overline{N}_E^{(2}(r,1;\mathbb{F}) + \overline{N}_0(r,0;\mathbb{F}') + \overline{N}_0(r,0;\mathbb{G}') + S(r,\mathbb{f})\\
&\quad + S(r,\mathbb{g})\\
&\leq \overline{N}(r,0;\mathbb{F}|\geq 2) + \overline{N}(r,0;\mathbb{G}|\geq 2) + 2\overline{N}_L(r,1;\mathbb{F}) + 2\overline{N}_L(r,1;\mathbb{G})\\
&\quad + \overline{N}_E^{(2}(r,1;\mathbb{F}) + \overline{N}_0(r,0;\mathbb{F}') + \overline{N}_0(r,0;\mathbb{G}') + S(r,\mathbb{f}) + S(r,\mathbb{g})\\
&\leq \overline{N}(r,0;\mathbb{F}|\geq 2) + \overline{N}(r,0;\mathbb{G}|\geq 2) + \overline{N}_{\mathbb{F}>1}(r,1;\mathbb{G}) + \overline{N}_{\mathbb{G}>1}(r,1;\mathbb{F})\\
&\quad + \overline{N}_L(r,1;\mathbb{F}) + N(r,1;\mathbb{G}) - \overline{N}(r,1;\mathbb{G}) + \overline{N}_0(r,0;\mathbb{F}') + \overline{N}_0(r,0;\mathbb{G}')\\
&\quad + S(r,\mathbb{f}) + S(r,\mathbb{g})\\
&\leq N_2(r,0;\mathbb{F}) + \overline{N}(r,0;\mathbb{F}) + N_2(r,0;\mathbb{G}) + N(r,1;\mathbb{G}) - \overline{N}(r,1;\mathbb{G})\\
&\quad + \overline{N}_0(r,0;\mathbb{F}') + \overline{N}_0(r,0;\mathbb{G}') + S(r,\mathbb{f}) + S(r,\mathbb{g})
\end{aligned}
$$

$$
\begin{aligned}
&\leq N_2(r,0;\mathbb{F})+\overline{N}(r,0;\mathbb{F})+N_2(r,0;\mathbb{G})+N(r,0;\mathbb{G}'|\mathbb{G}\neq 0)+\overline{N}_0(r,0;\mathbb{F}') \\
&\quad +S(r,\mathbb{f})+S(r,\mathbb{g}) \\
\overline{N}(r,1;\mathbb{F}) &\leq N_2(r,0;\mathbb{F})+\overline{N}(r,0;\mathbb{F})+N_2(r,0;\mathbb{G})+\overline{N}(r,0;\mathbb{G})+\overline{N}_0(r,0;\mathbb{F}') \\
&\quad +S(r,\mathbb{f})+S(r,\mathbb{g}). \qquad (3.25)
\end{aligned}
$$

Hence using (3.25), Lemmas 2.5, 2.7, 2.8, and 2.14, we get from the second fundamental theorem that

$$
\begin{aligned}
(n+m)T(r,\mathbb{f}) &\leq T(r,\mathbb{F}_1)-N(r,0;\Delta_q(\mathbb{f}))+S(r,\mathbb{f}) \\
&\leq T(r,\mathbb{F})+N_{k+2}(r,0;\mathbb{F}_1)-N_2(r,0;\mathbb{F})-N(r,0;\Delta_q(\mathbb{f}))+S(r,\mathbb{f}) \\
&\leq \overline{N}(r,0;\mathbb{F})+N_{k+2}(r,0;\mathbb{F}_1)+\overline{N}(r,1;\mathbb{F})-N_2(r,0;\mathbb{F})+\overline{N}(r,0;\mathbb{G}) \\
&\quad -\overline{N}_0(r,0;\mathbb{F}')-N(r,0;\Delta_q(\mathbb{f}))+S(r,\mathbb{f}) \\
&\leq N_2(r,0;\mathbb{F})+2\overline{N}(r,0;\mathbb{F})+N_{k+2}(r,0;\mathbb{F}_1)+N_2(r,0;\mathbb{G})+\overline{N}(r,0;\mathbb{G}) \\
&\quad -N_2(r,0;\mathbb{F})-N(r,0;\Delta_q(\mathbb{f}))+S(r,\mathbb{f})+S(r,\mathbb{g}) \\
&\leq N_{k+2}(r,0;\mathbb{F}_1)+2\overline{N}(r,0;\mathbb{F})+N_2(r,0;\mathbb{G})+\overline{N}(r,0;\mathbb{G}) \\
&\quad -N(r,0;\Delta_q(\mathbb{f}))+S(r,\mathbb{f}) \\
&\leq N_{k+2}(r,0;\mathbb{F}_1)+2N_{k+1}(r,0;\mathbb{F}_1)+N_{k+2}(r,0;\mathbb{G}_1)+N_{k+1}(r,0;\mathbb{G}_1) \\
&\quad -N(r,0;\Delta_q(\mathbb{f}))+S(r,\mathbb{f})+S(r,\mathbb{g}) \\
&\leq (k+2)\overline{N}(r,0;\mathbb{f})+N(r,0;P(\mathbb{f}))+N(r,0;\Delta_q(\mathbb{f})) \\
&\quad +2(k+1)\overline{N}(r,0;\mathbb{f})+2N(r,0;P(\mathbb{f}))+2N(r,0;\Delta_q(\mathbb{f})) \\
&\quad +(k+2)\overline{N}(r,0;\mathbb{g})+N(r,0;P(\mathbb{g}))+N(r,0;\Delta_q(\mathbb{g})) \\
&\quad +(k+1)\overline{N}(r,0;\mathbb{g})+N(r,0;P(\mathbb{g}))+N(r,0;\Delta_q(\mathbb{g})) \\
&\quad -N(r,0;\Delta_q(\mathbb{f}))+S(r,\mathbb{f})+S(r,\mathbb{g}) \\
&\leq (3k+3m+4)\,T(r,\mathbb{f})+2N(r,0;\Delta_q(\mathbb{f})) \\
&\quad +(2k+2m+3)T(r,\mathbb{g})+2N(r,0;\Delta_q(\mathbb{g}))+S(r,\mathbb{f}) \\
&\quad +S(r,\mathbb{g}) \\
&\leq (3k+3m+4)\,T(r,\mathbb{f})+(2k+2m+3)T(r,\mathbb{g})+2T(r,\Delta_q(\mathbb{f})) \\
&\quad +2T(r,\Delta_q(\mathbb{g}))+S(r,\mathbb{f})+S(r,\mathbb{g}) \\
&\leq (3k+3m+4)\,T(r,\mathbb{f})+(2k+2m+3)T(r,\mathbb{g})+2m\left(r,\frac{\Delta_q(\mathbb{f})}{\mathbb{f}(qz)}\right) \\
&\quad +2m(r,\mathbb{f}(qz))+2m\left(r,\frac{\Delta_q(\mathbb{g})}{\mathbb{g}(qz)}\right)+2m(r,\mathbb{g}(qz))+S(r,\mathbb{f}) \\
&\quad +S(r,\mathbb{g}) \\
&\leq (3k+3m+6)\,T(r,\mathbb{f})+(2k+2m+5)T(r,\mathbb{g})+S(r,\mathbb{f})+S(r,\mathbb{g}). \qquad (3.26)
\end{aligned}
$$

In a similar way, we can obtain

$$
(n+m)T(r,\mathbb{g}) \leq (3k+3m+6)\,T(r,\mathbb{g})+(2k+2m+5)T(r,\mathbb{f})+S(r,\mathbb{f})+S(r,\mathbb{g}). \qquad (3.27)
$$

Combining (3.26) and (3.27), we see that

$$(n-5k-4m-11)[T(r,\mathbb{f})+T(r,\mathbb{g})]\leq S(r,\mathbb{f})+S(r,\mathbb{g}),$$

a contradiction, since $n>5k+4m+11$.

Case 2. We now assume that $\mathbb{H}\equiv 0$, integrating (2.1), we get

$$\frac{1}{\mathbb{F}-1}=\frac{b\mathbb{G}+a-b}{\mathbb{G}-1}, \tag{3.28}$$

where $a(\neq 0)$, b are constants. From (3.28), it is clear that $\mathbb{F}$ and $\mathbb{G}$ share $(1,\infty)$. We now discuss the following three subcases separately.

Subcase 2.1. Suppose that $b\neq 0$ and $a\neq b$. If $b=-1$, then from (3.28), we have

$$\mathbb{F}\equiv\frac{-a}{\mathbb{G}-a-1}.$$

Therefore,

$$\overline{N}(r,a+1;\mathbb{G})=\overline{N}(r,\infty;\mathbb{F})=S(r,\mathbb{F}).$$

So in view of Lemmas 2.5, 2.8, 2.14, and the second fundamental theorem of Nevanlinna, we get

$$\begin{aligned}(n+m)T(r,\mathbb{g})&\leq T(r,\mathbb{G}_1)-N(r,0;\Delta_q(\mathbb{g}))+S(r,\mathbb{g})\\&\leq T(r,\mathbb{G})+N_{k+1}(r,0;\mathbb{G}_1)-\overline{N}(r,0;\mathbb{G})-N(r,0;\Delta_q(\mathbb{g}))+S(r,\mathbb{g})\\&\leq \overline{N}(r,0;\mathbb{G})+\overline{N}(r,a+1;\mathbb{G})+N_{k+1}(r,0;\mathbb{G}_1)-\overline{N}(r,0;\mathbb{G})\\&\quad-N(r,0;\Delta_q(\mathbb{g}))+S(r,\mathbb{g})\\&\leq N_{k+1}(r,0;\mathbb{G}_1)-\overline{N}(r,0;\mathbb{G})-N(r,0;\Delta_q(\mathbb{g}))+S(r,\mathbb{g})\\&\leq (k+1)\overline{N}(r,0;\mathbb{g}^n)+N(r,0;P(\mathbb{g}))+N(r,0;\Delta_q(\mathbb{g}))\\&\quad-N(r,0;\Delta_q(\mathbb{g})+S(r,\mathbb{g})\\&\leq (k+m+1)T(r,\mathbb{g})+S(r,\mathbb{g}),\end{aligned}$$

a contradiction, since $n>k+1$.

If $b\neq -1$, from (3.28), we obtain that

$$\mathbb{F}-\left(1+\frac{1}{b}\right)\equiv\frac{-a}{b^2\left(\mathbb{G}+\frac{a-b}{b}\right)}$$

So,

$$\overline{N}\left(r,\frac{b-a}{b};\mathbb{G}\right)=\overline{N}(r,\infty;\mathbb{F}).$$

Using Lemmas 2.5, 2.8 and with the same arguments as used in the Case for $b=-1$, we can get a contradiction.

Subcase 2.2. Let $b\neq 0$ and $a=b$. If $b=-1$, then from (3.28), we have

$$\mathbb{F}.\mathbb{G}\equiv 1,$$

i.e.,

$$\left(\mathbb{f}^nP(\mathbb{f})\Delta_q(\mathbb{f})\right)^{(k)}\left(\mathbb{g}^nP(\mathbb{g})\Delta_q(\mathbb{g})\right)^{(k)}\equiv\varphi^2(z).$$

If $b\neq -1$, from (3.28), we have

$$\frac{1}{\mathbb{F}}\equiv\frac{b\mathbb{G}}{(1+b)\mathbb{G}-1}.$$

Therefore,

$$\overline{N}\left(r, \frac{1}{1+b}; \mathbb{G}\right) = \overline{N}(r, 0; \mathbb{F}).$$

So in view of Lemma 2.5, 2.7, 2.8, 2.14, and using the second fundamental theorem of Nevanlinna, we get

$$\begin{aligned}
(n+m)T(r,\mathbb{g}) &\leq T(r,\mathbb{G}_1) - N(r,0;\Delta_q(\mathbb{g})) + S(r,\mathbb{g}) \\
&\leq T(r,\mathbb{G}) + N_{k+1}(r,0;\mathbb{G}_1) - \overline{N}(r,0;\mathbb{G}) - N(r,0;\Delta_q(\mathbb{g})) + S(r,\mathbb{g}) \\
&\leq \overline{N}(r,0;\mathbb{G}) + \overline{N}\left(r, \frac{1}{1+b}; \mathbb{G}\right) + N_{k+1}(r,0;\mathbb{G}_1) - \overline{N}(r,0;\mathbb{G}) \\
&\quad - N(r,0;\Delta_q(\mathbb{g})) + S(r,\mathbb{g}) \\
&\leq \overline{N}(r,0;\mathbb{F}) + N_{k+1}(r,0;\mathbb{G}_1) - N(r,0;\Delta_q(\mathbb{g})) + S(r,\mathbb{g}) \\
&\leq N_{k+1}(r,0;\mathbb{F}_1) + N_{k+1}(r,0;\mathbb{G}_1) - N(r,0;\Delta_q(\mathbb{g})) + S(r,\mathbb{g}) \\
&\leq (k+1)\overline{N}(r,0;\mathbb{f}^n) + N(r,0;P(\mathbb{f})) + N(r,0;\Delta_q(\mathbb{f})) \\
&\quad + (k+1)\overline{N}(r,0;\mathbb{g}^n) + N(r,0;P(\mathbb{g})) + N(r,0;\Delta_q(\mathbb{g})) \\
&\quad - N(r,0;\Delta_q(\mathbb{g}) + S(r,\mathbb{g}) \\
&\leq (k+m+1)T(r,\mathbb{f}) + (k+m+1)T(r,\mathbb{g}) + T(r,\Delta_q(\mathbb{f})) + S(r,\mathbb{f}) \\
&\quad + S(r,\mathbb{g}) \\
&\leq (k+m+1)T(r,\mathbb{f}) + (k+m+1)T(r,\mathbb{g}) + m\left(r, \frac{\Delta_q(\mathbb{f})}{\mathbb{f}}\right) \\
&\quad + m(r,\mathbb{f}) + S(r,\mathbb{f}) + S(r,\mathbb{g}) \\
(n+m)T(r,\mathbb{g}) &\leq (k+m+2)T(r,\mathbb{f}) + (k+m+1)T(r,\mathbb{g}) + S(r,\mathbb{f}) + S(r,\mathbb{g}).
\end{aligned}$$

In a similar way, we can obtain

$$(n+m)T(r,\mathbb{f}) \leq (k+m+2)T(r,\mathbb{g}) + (k+m+1)T(r,\mathbb{f}) + S(r,\mathbb{f}) + S(r,\mathbb{g})$$

Adding the above two inequalities, we get

$$(n-2k-m-3)[T(r,\mathbb{f}) + T(r,\mathbb{g})] \leq S(r,\mathbb{f}) + S(r,\mathbb{g}),$$

a contradiction, since $n > 2k+m+3$.

Subcase 2.3. Let $b = 0$, from (3.28), we have

$$\mathbb{F} \equiv \frac{\mathbb{G} + a - 1}{a}. \tag{3.29}$$

If $a \neq 1$, then from (3.29), we obtain

$$\overline{N}(r, 1-a; \mathbb{G}) = \overline{N}(r, 0; \mathbb{G}).$$

So, using the same arguments used in case 1.2. for $b \neq -1$, we can similarly deduce a contradiction. Therefore $a = 1$ and from (3.29), we obtain $\mathbb{F} \equiv \mathbb{G}$,
i.e.,

$$\left(\mathbb{f}^n P(\mathbb{f}) \Delta_q(\mathbb{f})\right)^{(k)} \equiv \left(\mathbb{g}^n P(\mathbb{g}) \Delta_q(\mathbb{g})\right)^{(k)}.$$

Integrating, we have

$$\mathbb{f}^n P(\mathbb{f}) \Delta_q(\mathbb{f}) \equiv \mathbb{g}^n P(\mathbb{g}) \Delta_q(\mathbb{g}) + \rho(z),$$

where $\rho(z)$ is a polynomial of degree at most $(k-1)$.
If $\rho(z) \not\equiv 0$, now in view of Lemmas 2.7, 2.8, 2.14, and the second fundamental theorem of Nevanlinna, we have

$$\begin{aligned}
(n+m)T(r,\mathbb{f}) &\le T(r,\mathbb{F}_1) - N(r,0;\Delta_q(\mathbb{f})) + S(r,\mathbb{f})\\
&\le \overline{N}(r,0;\mathbb{F}_1) + \overline{N}(r,\infty;\mathbb{F}_1) + \overline{N}(r,1;\mathbb{F}_1) - N(r,0;\Delta_q(\mathbb{f})) + S(r,\mathbb{g})\\
&\le \overline{N}(r,0;\mathbb{f}) + \overline{N}(r,0;\Delta_q(\mathbb{f})) + \overline{N}(r,0;\mathbb{g}) + \overline{N}(r,0;\Delta_q(\mathbb{g}))\\
&\quad - N(r,0;\Delta_q(\mathbb{f})) + S(r,\mathbb{f}) + S(r,\mathbb{g})\\
&\le \overline{N}(r,0;\mathbb{f}) + \overline{N}(r,0;\mathbb{g}) + \overline{N}(r,0;\Delta_q(\mathbb{g})) + S(r,\mathbb{f}) + S(r,\mathbb{g})\\
&\le T(r,\mathbb{f}) + T(r,\mathbb{g}) + T(r,\Delta_q(\mathbb{g})) + S(r,\mathbb{f}) + S(r,\mathbb{g})\\
&\le T(r,\mathbb{f}) + T(r,\mathbb{g}) + m\left(r,\frac{\Delta_q(\mathbb{g})}{\mathbb{g}}\right) + m(r,\mathbb{g}) + S(r,\mathbb{f}) + S(r,\mathbb{g})\\
(n+m)T(r,\mathbb{f}) &\le T(r,\mathbb{f}) + 2T(r,\mathbb{g}) + S(r,\mathbb{f}) + S(r,\mathbb{g}).
\end{aligned}$$

Similarly, we have

$$(n+m)T(r,\mathbb{g}) \le T(r,\mathbb{g}) + 2T(r,\mathbb{f}) + S(r,\mathbb{f}) + S(r,\mathbb{g}).$$

Combining the above two inequalities, we can get

$$(n+m-3)[T(r,\mathbb{f}) + T(r,\mathbb{g})] \le S(r,\mathbb{f}) + S(r,\mathbb{g}),$$

a contradiction, since $n > 3-m$.
Thus $\rho(z) \equiv 0$, which implies

$$\mathbb{f}^n P(\mathbb{f})\Delta_q(\mathbb{f}) \equiv \mathbb{g}^n P(\mathbb{g})\Delta_q(\mathbb{g})$$

i.e.,

$$\begin{aligned}
&\mathbb{f}^n(a_m\mathbb{f}^m + a_{m-1}\mathbb{f}^{m-1} + \ldots + a_1\mathbb{f} + a_0)(\mathbb{f}(qz+c) - \mathbb{f}(qz))\\
&\equiv \mathbb{g}^n(a_m\mathbb{g}^m + a_{m-1}\mathbb{g}^{m-1} + \ldots + a_1\mathbb{g} + a_0)(\mathbb{g}(qz+c) - \mathbb{g}(qz)).
\end{aligned}$$

By using similar arguments as in Theorem 1.3 (from inequality (3.14) onwards), we can easily prove Theorem 1.4.

Using similar arguments as in Theorem 1.1-1.4, we can quickly obtain Corollary 1.1-1.4, respectively.

Conclusion: In this paper, we investigate the uniqueness problems and distribution of zeroes of q-shift difference-differential polynomials of meromorphic (entire) functions having zero order. Also, by using the concept of weighted sharing, we investigate the uniqueness problem of q-shift difference-differential polynomials of meromorphic (entire) functions having zero order, sharing a small function with finite weight. The results of this paper are helpful in investigating the behavior of meromorphic (entire) functions in different contexts. Some of the applications of such results can be seen in signal processing, communication networks, design of filters and controllers for systems with complex dynamics. Also, understanding the properties of these functions is essential for solving difference-differential equations, analyzing complex systems, and studying mathematical physics phenomena.

Continuing further research, we can pose the following open questions.

Open questions:

(1) Can the condition for the lower bound n in Theorems 1.1-1.4 be reduced any further?

(2) What happens to Theorems 1.1-1.4, if we replace the q-difference operator $\Delta_q \mathbb{f}$ by the product of q-difference $\prod_{j=1}^{l}(\Delta_{q,c}^{u}(f))^{\mu_j}$, where u, μ_j are positive integers and q, c are non-zero complex constants?

(3) What happens to Theorems 1.1-1.4, if we replace the q-difference operator $\Delta_q(\mathbb{f})$ by the linear q-difference polynomial $L_k(f, E_q)$ and its q-difference operator of the form $L_k(f, \Delta)$ as in [23]?

(4) What happens to Theorems 1.1-1.4 if we study them using the concepts of weakly weighted sharing, and truncated sharing, which are weaker than weighted sharing?

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