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GENERALIZED FRACTIONAL ECONOMIC MODELS BY MARKET EQUILIBRIUM

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Abstract. The main goal of this paper is to use non-local fractional operators, specifically the generalized proportional fractional Caputo derivatives, to analyze certain economic problems. The paper also compares the results obtained using these fractional operators with already established results using many other different kinds of the fractional derivatives. At the end, a more comprehensive view of considered economic problems in the market, which includes simulation analysis, is provided.

1. Introduction and preliminaries

The use of fractional calculus is highly significant in the modeling of complex systems across various scientific and engineering disciplines. Fractional differential equations, offer an alternative approach to modeling with ordinary differential equations. This methodology finds applications in a wide range of fields, including nonlinear oscillations, viscoelastic systems, dielectric polarization, electrode–electrolyte polarization, electromagnetic waves, earthquake modeling, study of phenomena in biology, physics, and engineering, such as seepage flow in porous media and fluid dynamic traffic models (see [11], [16]-[17], [20]-[22]). These applications highlight the importance and practical value of this theory in the analysis of dynamical systems.

Generalized proportional Caputo fractional derivatives are an important tool in the field of fractional calculus, which deals with derivatives and integrals of non-integer order. These derivatives generalize the classical Caputo fractional derivatives by allowing the order parameter to take on a wider range of values, including

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non-positive values. There are several generalizations of Caputo fractional derivatives; more information on this topic can be found in [5], [9], [13]-[15], [24] and [25]. It should be noted that these references do not cover all available literature on the subject. Moreover, some of the recent results in the asymptotic stability of generalized Caputo fractional differential systems can be found in [3], [12], [18] and [26].

In a market with a large number of economic agents competing with each other, the price is independent of individual behavior and satisfaction. There are various criteria used to differentiate market structures, such as the number of economic agents, suppliers, demanders, and turnover rates, among others. Each economic model has its own elements, such as goods, companies, and individuals, and aims to maximize utility for buyers and profit for sellers while maintaining pricing freedom within an equilibrium model. Economics deals with the interactions between price, supply, and demand, and how the equilibrium point is reached on supply and demand curves. Mathematical economics, on the other hand, aims to formulate economic processes mathematically using economic concepts, allowing for a better understanding of economic behavior and interactions.

The description of economic processes leans heavily on differentiation and integration, which are fundamental tools in constructing economic phenomena and models. In economic theory, the behavior of economic agents often depends on past fluctuations, making the utilization of fractional differentiation and integration, as opposed to integer order ones, crucial. This allows for the incorporation of memory effects, enabling a comprehensive observation of the economic history. The modern stage of mathematical economics, which aligns with existing economic principles, involves expressing economic concepts in terms of fractional operators, giving rise to fractional mathematical economics. Traditional derivatives have limitations in accurately describing economic concepts, thus constructing economic models using non-local fractional operators, which do not have restrictions within a small neighborhood, offers distinct advantages over models based on integer-order derivatives. Consequently, this study aims to leverage the memory effect of non-local operators to enhance the understanding and observation of past changes in the economy.

Our research paper centers on the investigation of economic models that incorporate the utilization of generalized proportional Caputo fractional derivatives. By harnessing the potential of the generalized proportional Caputo fractional derivative, scientists and economists can delve into the intricacies of economic models with greater flexibility. This approach empowers us to comprehensively examine diverse facets of economic behavior and analyze the repercussions of modifying parameters on the dynamics of the system. The capacity to closely approximate solutions derived from other fractional derivatives presents a valuable tool for comprehending the interconnected nature and resemblances between various economic phenomena.

The paper's organization can be succinctly described as follows. The preliminary subsection provides definitions of Mittag-Leffler functions, various fractional derivatives mentioned in this paper, and the generalized Laplace transform. Additionally, this section outlines the economic problems under investigation. Section 2 is dedicated to established economic models within different fractional derivative frameworks. The subsequent two sections present the primary results: first, obtaining solutions for the economic model employing generalized proportional Caputo fractional derivative, and then comparing these solutions with previously established ones for economic problems utilizing different types of fractional differential operators through graphical analysis. The paper concludes with a summary of the obtained results in the Conclusion section.

1.1. **Preliminaries.** In this subsection, we revisit some important notations and useful results from the theory of fractional differential equations.

We recall the definitions of one parameter and two parameters Mittag-Leffler functions. For more details, the interested reader is referred to [1] and [16].

The Mittag-Leffler function with one parameter α is given by the formula

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)},$$

for $\Re\alpha>0$, $z\in\mathbb{C}$.

The generalized Mittag-Leffler function with two parameters α, β is defined as

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},$$

where $\Re\alpha>0$, $\Re\beta>0$ and $z\in\mathbb{C}$. Note that $E_\alpha(z)=e^z$, for $\alpha=1$ and $E_{\alpha,1}(z)=E_\alpha$.
The Mittag-Leffler function, with two additional variable λ is defined as

$$E_\alpha(\lambda,z)=\sum_{k=0}^{\infty}\frac{\lambda^k z^{\alpha k}}{\Gamma(\alpha k+1)},$$

where $\lambda\in\mathbb{R}$, $\lambda\neq 0$, $z,\alpha\in\mathbb{C}$, $\Re\alpha>0$. Define

$$E_{\alpha,\beta}(\lambda,z)=\sum_{k=0}^{\infty}\frac{\lambda^k z^{\alpha k+\beta-1}}{\Gamma(\alpha k+\beta)},$$

for $\lambda\in\mathbb{R}$, $\lambda\neq 0$, $z,\alpha,\beta\in\mathbb{C}$, $\Re\alpha>0$.
Note that $E_{\alpha,1}(\lambda,z)=E_\alpha(\lambda,z)$. Moreover, the modified Mittag-Leffler function with three parameters of the special functions is defined as

$$E^{\rho}_{\alpha,\beta}(\lambda,z)=\sum_{k=0}^{\infty}\frac{\lambda^k z^{\alpha k+\beta-1}(\rho)_k}{\Gamma(\alpha k+\beta)k!},$$

for $\lambda\in\mathbb{R}$, $\lambda\neq 0$, $z,\alpha,\beta,\rho\in\mathbb{C}$, $\Re\alpha>0$, where $(\rho)_k$ is the Pochhammer symbol introduced by Prabhakar such that $(\rho)_k=\rho(\rho+1)\cdot\ldots\cdot(\rho+k-1)$ and $(1)_k=k!$. Note that $E^{1}_{\alpha,\beta}(\lambda,z)=E_{\alpha,\beta}(\lambda,z)$.

The Riemann-Liouville fractional integral of the function $f(t)$, of order $\alpha>0$, is defined by (see [16])

$$({}_{0^+}I^{\alpha}_{t}f)(t)=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{f(s)}{(t-s)^{1-\alpha}}\,ds,\quad t>0,$$

where $f(t)$ is an absolutely integrable function and $\Gamma(\cdot)$ is the Gamma function.

The left–sided Riemann-Liouville fractional derivative of the function $f(t)$, of order $\alpha \in (0,1)$, is given by (see [16])

$$({}_{0^+}^{RL}D^{\alpha} f)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(s)}{(t-s)^{\alpha}}\, ds, \quad t > 0,$$

where $f(t)$ is an absolutely integrable function.

The left–sided Caputo fractional derivative of the function $f(t)$, of order $\alpha \in (0,1)$, with absolute integrable first derivative is given by (see [16])

$$({}_{0^+}^{C}D^{\alpha} f)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^{\alpha}}\, ds, \quad t > 0.$$

The left–sided Caputo-Fabrizio fractional derivative of $f(t)$ in the sense of Caputo approach (shortly (CFC)) is defined by (see [10])

$$({}_{0}^{CFC}D^{\alpha} f)(t) = \frac{M(\alpha)}{1-\alpha} \int_0^t e^{\lambda(t-s)} f'(s)\, ds,$$

where $\alpha \in (0,1)$, $M(\alpha)$ is a normalization function and $\lambda = -\frac{\alpha}{1-\alpha}$.

The left–sided Atangana–Baleanu fractional derivative of $f(t)$ in the sense of Caputo approach (shortly (ABC)) is presented as following (see [6]-[7])

$$({}_{0}^{ABC}D^{\alpha} f)(t) = \frac{B(\alpha)}{1-\alpha} \int_0^t E_{\alpha}\big(\lambda(t-s)^{\alpha}\big) f'(s)\, ds,$$

where $\alpha \in (0,1)$, $B(\alpha)$ is a normalization function and $\lambda = -\frac{\alpha}{1-\alpha}$.

The left–sided Atangana–Baleanu fractional derivative of $f(t)$, (shortly generalized (ABC)) with generalized Mittag–Leffler function $E^{\rho}_{\alpha,\beta}(\lambda t^{\alpha})$, such that $\rho \in \mathbb{R}$, $\Re\beta > 0$, $\alpha \in (0,1)$ and $\lambda = -\frac{\alpha}{1-\alpha}$ is defined as (see [1])

$$({}_{0}^{ABC}D^{\alpha,\beta,\rho} f)(t) = \frac{B(\alpha)}{1-\alpha} \int_0^t E^{\rho}_{\alpha,\beta}\big(\lambda(t-s)^{\alpha}\big) f'(s)\, ds.$$

The constant proportional Caputo (shortly (CPC)) derivative of $f(t)$ is given by (see [8])

$$({}_{0}^{CPC}D^{\alpha} f)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \big(k_1(\alpha) f(s) + k_0(\alpha) f'(s)\big)(x-s)^{-\alpha}\, ds,$$

where $\alpha \in (0,1)$ and k_0 and k_1 are functions of α such that $k_0(\alpha) \neq 0$, $k_1(\alpha) \neq 0$, for $\alpha \in (0,1)$, $\lim_{\alpha\to 0^+} k_0(\alpha) = 0$, $\lim_{\alpha\to 1^+} k_0(\alpha) = 1$, $\lim_{\alpha\to 0^+} k_1(\alpha) = 1$ and $\lim_{\alpha\to 1^-} k_1(\alpha) = 0$.

The basic definitions and properties of generalized proportional fractional Caputo derivatives and generalized Laplace transform will be recalled in the sequel (see [13]).

Here, we suppose that $g(t)$ is strictly increasing function with continuous derivative $g'(t)$ on $(0,\infty)$. The generalized Riemann-Liouville fractional integral of $f(t)$, with the respect to the function $g(t)$ of order $\alpha > 0$, is given by

$$({}_0I_t^{\alpha,g}f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (g(t) - g(s))^{\alpha-1} f(s)g'(s)\, ds.$$

We note that by putting $g(t) = t$ the above fractional integral becomes the classical Riemann-Liouville fractional integral and by putting $g(t) = \ln t$, the above fractional integral becomes Hadamarad fractional integral. The left Riemann-Liouville fractional derivative of function $f(t)$ of order $\alpha > 0$, with the respect to the function $g(t)$, is given by the following formula ([13])

$$({}_0^{RL}D_g^\alpha f)(t) = \frac{\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^n}{\Gamma(1-\alpha)} \int_0^t (g(t) - g(s))^{-\alpha} f(s)g'(s)\, ds,$$

where $n = [\alpha] + 1$, when α is not positive integer and $n = \alpha$, when $\alpha \in \mathbb{N}$, $g^{(i)} \neq 0$, $i = 2, 3, ..., n$. The generalized left–sided Caputo fractional derivative of the function $f(t)$ of order $\alpha > 0$, with the respect to the function $g(t)$, is given by ([5], [25])

$$({}_0^{C}D_g^\alpha f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (g(t) - g(s))^{n-\alpha-1} \Big(\frac{1}{g'(t)}\frac{d}{dt}\Big)^n f(s)g'(s)\, ds,$$

where $n = [\alpha] + 1$, when α is not positive integer and $n = \alpha$, in case when $\alpha \in \mathbb{N}$.

The generalized proportional Caputo fractional derivative is given by (see [14]–[15] and [23])

$$({}_0^{C}D^{\alpha,\xi,g} f)(t) = \frac{1}{\xi^{1-\alpha}\Gamma(1-\alpha)} \times \int_0^t e^{\frac{\xi-1}{\xi}(g(t)-g(0))} (g(t) - g(0))^{-\alpha} \Big((1-\xi)f(s) + \xi\frac{f'(s)}{g'(s)}\Big) g'(s)\, ds,$$

where $\alpha \in (0, 1]$, $g'(t) > 0$ for $t > 0$ and $\xi \in (0, 1]$. For $\xi = 1$ and $g(t) = t$ the generalized proportional Caputo fractional derivative is reduced to the classical Caputo fractional derivative.

For the sequel, we invoke the definition of the generalized convolution. Suppose that $u, g : [0, \infty) \to \mathbb{R}$, $g(t)$ be continuous function and $g'(t) > 0$ on $[0, \infty)$. The generalized Laplace transform of u (g-Laplace transform) is given by

$$\mathcal{L}_g(u(f))(s) = \int_0^\infty e^{-s(g(t)-g(0))} u(t)g'(t)\, dt,$$

if the integral for all values of s is valid.

Under the above assumptions of the functions u and g, we have the following

Lemma 1.1. *([9], [13]-[15], [19]) Let $\alpha, \beta > 0$, A be an arbitrary square matrix of order m and I be the identity matrix of the same order as A.*

i) $\mathcal{L}_g\big(u(t)\big)(s) = \mathcal{L}\Big(u\big(g^{-1}(t + g(0))\big)\Big)(s)$, *where $\mathcal{L}$ is the usual Laplace transform of u;*

ii) *Suppose* a_1 *and* a_2 *are constants. Let* $u_1, u_2 : [0,\infty) \to \mathbb{R}$ *and the generalized Laplace transform exists for* u_1 *and* u_2*, for all* $s > c_1$ *and* $s > c_2$*, respectively. Then*

$$\mathcal{L}_g\Big(a_1u_1(t)+a_2u_2(t)\Big)(s) = a_1\mathcal{L}_g(u_1(t))(s) + a_2\mathcal{L}_g(u_2(t))(s), \text{ for } s > \max\{c_1, c_2\};$$

iii) $\mathcal{L}_g(1)(s) = \frac{1}{s}, \quad s > 0;$

iv) $\mathcal{L}_g\Big((g(t)-g(0))^\beta\Big)(s) = \frac{\Gamma(\beta+1)}{s^{\beta+1}}, \quad s, \beta > 0;$

v) $\mathcal{L}_g\big(e^{cg(t)}\big)(s) = \frac{e^{cg(0)}}{s-c}, \quad s > c;$

vi) $\mathcal{L}_g\Big(E_\alpha(A(g(t)-g(0))^\alpha)\Big) = s^{\alpha-1}(s^\alpha I - A)^{-1};$

vii) $\mathcal{L}_g\Big((g(t)-g(0))^{\beta-1}E_{\alpha,\beta}\big(A(g(t)-g(0))^\alpha\big)\Big)(s) = s^{\alpha-\beta}(s^\alpha I - A)^{-1};$

viii) $\mathcal{L}_g\Big({}_0I_g^\alpha u(t)\Big)(s) = \frac{\mathcal{L}_g(u(t))}{s^\alpha};$

ix) *Suppose that there exist constants* $K, C, T \geq 0$ *such that* $|u^{(i)}(t)| \leq Ke^{Cg(t)}$*, for all* $t \geq T$*,* $i = 0, 1, ..., n$*, where* $n = [\alpha]+1$*, when* α *is not positive integer and* $n = \alpha$*, for* $\alpha \in \mathbb{N}$*. We have*

$$\mathcal{L}_g\Big(\Big({}_0^CD_g^\alpha u\Big)(t)\Big)(s) = s^\alpha\Big(\mathcal{L}_g(u(t)) - \sum_{k=0}^{n-1} s^{-k-1}\big(u^(k)\big)(0^+)\Big);$$

x) $\mathcal{L}_g\Big({}_0^CD^{\alpha,\xi,g}f(t)\Big)(s) = (\xi s + 1 - \xi)^\alpha \mathcal{L}_g(f(t))(s) - \xi(\xi s + 1 - \xi)^{\alpha-1}$*, for* $\alpha \in (0,1)$*,* $\xi \in (0,1]$*,* $g'(t) > 0$ *for* $t > 0$*.*

Proof. We will prove only vi) and vii). The proofs of the other statements can be found in [9] and [13]. First we prove vi). Using the definition of the generalized Laplace transform, Mittag-Leffler function and iv), we obtain

$$\mathcal{L}_g\Big(E_\alpha(A(g(t)-g(0))^\alpha)\Big)(s) = \sum_{k=0}^\infty \frac{A^k}{\Gamma(k\alpha+1)}\mathcal{L}_g\Big((g(t)-g(0))^{k\alpha}\Big)(s)$$
$$= \sum_{k=0}^\infty \frac{A^k}{\Gamma(k\alpha+1)}\frac{\Gamma(k\alpha+1)}{s^{k\alpha+1}} = \frac{1}{s}\sum_{k=0}^\infty \frac{A^k}{s^{k\alpha}} = s^{\alpha-1}(s^\alpha I - A)^{-1}.$$

Now, we prove vii). We have

$$\mathcal{L}_g\Big((g(t)-g(0))^{\beta-1}E_{\alpha,\beta}\big(A(g(t)-g(0))^\alpha\big)\Big)(s) = \sum_{k=0}^\infty \frac{A^k\mathcal{L}_g\big((g(t)-g(0))^{k\alpha+\beta-1}\big)}{\Gamma(k\alpha+\beta)}$$
$$= \sum_{k=0}^\infty \frac{A^k}{\Gamma(k\alpha+\beta)}\frac{\Gamma(k\alpha+\beta)}{s^{k\alpha+\beta}} = \frac{1}{s^\beta}\sum_{k=0}^\infty \frac{A^k}{s^{k\alpha}} = s^{\alpha-\beta}(s^\alpha I - A)^{-1}.$$

□

The generalized convolution (g-convolution) of two functions u and v, which are piecewise continuous at each interval $[0,T]$ and of exponential order, is given by the following formula

$$(u*_gv)(t) = \int_0^t u(s)v\Big(g^{-1}\big(g(t)+g(0)-g(s)\big)\Big)g'(s)\,ds.$$

The generalized convolution of two functions is commutative and it holds (see [13])

$$\mathcal{L}_g(u *_g v) = \mathcal{L}_g(u)\mathcal{L}_g(v).$$

The rest of this subsection will be used for description of the economic models that we are going to investigate. A competitive market is closely linked to the idea of competitive equilibrium, which refers to a state where the amount of goods demanded by buyers is equal to the amount of goods supplied by sellers. This balance can be expressed using the demand function q_a and the supply function q_b, given by the following formulas:

$$q_a(t) = a_0 - a_1 p(t),$$

$$q_b(t) = -b_0 + b_1 p(t),$$

where p is the price of goods, a_0, a_1, b_0 and b_1 are positive constants, denoting the factors affecting the demanded and supplied quantity. When $q_a(t) = q_b(t)$, i.e. the demanded and supplied quantities are equal, the equilibrium price $p^* = \frac{a_0+b_0}{a_1+b_1}$ is obtained. This means that the price tends to stay stable and there is no shortage and surplus in economics. In [21], the following price adjustment

$$p'(t) = k(q_a - q_b),$$

where by $k > 0$ is denoted the speed of adjustment constant, is considered. Moreover, if we insert the upper equations for q_a and q_b in this differential equation, we obtain the following

$$p'(t) + k(a_1 + b_1)p(t) = k(a_0 + b_0),$$

having a solution given by

$$p(t) = \frac{a_0 + b_0}{a_1 + b_1} + \Big(p(0) - \frac{a_0 + b_0}{a_1 + b_1}\Big) e^{-k(a_1+b_1)t}.$$

Considering the expectation agents, the functions of demand and supply, involve additional factors a_2 and b_2, having the following form

$$q_a(t) = a_0 - a_1 p(t) + a_2 p'(t), \qquad q_b(t) = -b_0 + b_1 p(t) - b_2 p'(t),$$

and similar like above, we obtain the following differential equation

$$p'(t) - \frac{a_1 + b_1}{a_2 + b_2} p(t) = \frac{a_0 + b_0}{a_2 + b_2}.$$

The solution of this linear differential equation is given by

$$p(t) = \frac{a_0 + b_0}{a_1 + b_1} + \Big(p(0) - \frac{a_0 + b_0}{a_1 + b_1}\Big) e^{\frac{a_1+b_1}{a_2+b_2} t}.$$

When the prices of goods increase, buyers exhibit a higher demand to make purchases before prices escalate further, while sellers tend to offer fewer goods to capitalize on the anticipated higher prices in the future. The examined models incorporate the condition of $q_a(t) = q_b(t)$. Additionally, in a changing economy where $p'(t) = 0$ holds true for all $t > 0$, the market reaches a dynamic equilibrium state.

2. Established economic models in fractional derivative framework

This section will showcase recent findings related to the discussed economic problems, utilizing various kinds of fractional operators. More precisely, we will recall results from [2] and [4] about the solutions of the mentioned economic problems in the previous section, by means of Caputo, Caputo-Fabrizio in the sense of Caputo approach (CFC), Atangana-Baleanu in the sense of Caputo approach (ABC), generalized Atangana-Baleanu in the sense of Caputo approach (ABC) including Mittag-Leffler function with three parameters, constant proportional Caputo and generalized Caputo proportional fractional derivatives.

The equation about the price adjustment, by means of Caputo fractional derivative, without considering the expectation of agents is given as following

$$ {}_0^C D^{\alpha} p(t) + k(a_1+b_1)p(t) = k(a_0+b_0), \quad \alpha \in (0,1). \tag{1}$$

The solution of (1) is given by

$$p(t) = p(0)E_{\alpha}(-k(a_1+b_1)t^{\alpha}) + \frac{a_0+b_0}{a_1+b_1}\Big(1 - E_{\alpha}(-k(a_1+b_1)t^{\alpha})\Big).$$

If we take into account the expectations of agents, the price adjustment equation incorporating the Caputo fractional derivative can be formulated as follows

$$ {}_0^C D^{\alpha} p(t) - \frac{a_1+b_1}{a_2+b_2}p(t) = -\frac{a_0+b_0}{a_2+b_2}. \tag{2}$$

The solution of (2) is given by the equation

$$p(t) = p(0)E_{\alpha}\Big(\frac{a_1+b_1}{a_2+b_2}t^{\alpha}\Big) - \frac{a_0+b_0}{a_1+b_1}\Big(1 - E_{\alpha}\Big(\frac{a_1+b_1}{a_2+b_2}t^{\alpha}\Big)\Big).$$

The price adjustment equation with the Caputo–Fabrizio fractional derivative in the Caputo sense, without accounting for the expectations of agents, can be expressed as follows

$$ {}_0^{CFC} D^{\alpha} p(t) + k(a_1+b_1)p(t) = k(a_0+b_0), \quad \alpha \in (0,1),$$

and its solution is given by

$$\begin{aligned} p(t) = {} & \frac{p(0)M(\alpha)e^{\frac{\alpha k(a_1+b_1)t}{-M(\alpha)+(\alpha-1)k(a_1+b_1)}}}{M(\alpha)-(\alpha-1)k(a_1+b_1)} \\ & - \frac{(a_0+b_0)(M(\alpha)(-1+e^{\frac{\alpha k(a_1+b_1)t}{-M(\alpha)+(\alpha-1)k(a_1+b_1)}}) + (\alpha-1)k(a_1+b_1))}{(a_1+b_1)(-M(\alpha)+(\alpha-1)k(a_1+b_1))}, \end{aligned}$$

where $p(0) = \frac{a_0+b_0}{a_1+b_1}$.

If we incorporate the expectations of agents, the price adjustment equation including the Caputo–Fabrizio fractional derivative in the Caputo sense in the following manner

$$ {}_0^{CFC} D^{\alpha} p(t) - \frac{a_1+b_1}{a_2+b_2}p(t) = -\frac{a_0+b_0}{a_2+b_2}. \tag{3}$$

The solution of (3) stated as follows

$$p(t)=\frac{p(0)M(\alpha)e^{\frac{\alpha(a_1+b_1)t}{(\alpha-1)a_1+(\alpha-1)b_1+M(\alpha)(a_2+b_2)}}(a_2+b_2)}{(\alpha-1)a_1+(\alpha-1)b_1+M(\alpha)(a_2+b_2)}$$
$$+\frac{(a_0+b_0)((\alpha-1)a_1+(\alpha-1)b_1-M(\alpha)(-1+e^{\frac{\alpha(a_1+b_1)t}{(\alpha-1)a_1+(\alpha-1)b_1+M(\alpha)(a_2+b_2)}}(a_2+b_2)))}{(a_1+b_1)((\alpha-1)a_1+(\alpha-1)b_1+M(\alpha)(a_2+b_2))},$$

where $p(0)=\frac{a_0+b_0}{a_1+b_1}$.

If we do not consider the expectations of agents, the price adjustment equation with the Atangana-Baleanu fractional derivative in the Caputo sense can be represented by the following equation

$$ {}_0^{ABC}D^{\alpha}p(t)+k(a_1+b_1)p(t)=k(a_0+b_0),\quad \alpha\in(0,1). \tag{4}$$

The solution of (4) is given by

$$\begin{aligned}p(t)=&\frac{p(0)B(\alpha)}{B(\alpha)+k(a_1+b_1)(1-\alpha)}E_\alpha\Big(-\frac{k(a_1+b_1)\alpha}{B(\alpha)+k(a_1+b_1)(1-\alpha)}t^\alpha\Big)\\&+\frac{k(a_0+b_0)(1-\alpha)}{B(\alpha)+k(a_1+b_1)(1-\alpha)}E_\alpha\Big(-\frac{k(a_1+b_1)\alpha}{B(\alpha)+k(a_1+b_1)(1-\alpha)}t^\alpha\Big)\\&+\frac{a_0+b_0}{a_1+b_1}\Big(1-E_\alpha\Big(-\frac{k(a_1+b_1)\alpha}{B(\alpha)+k(a_1+b_1)(1-\alpha)}t^\alpha\Big)\Big),\end{aligned}$$

where $p(0)=\frac{a_0+b_0}{a_1+b_1}$.

On the other hand, when we consider the expectations of agents, the price adjustment equation, by means of Atangana–Baleanu fractional derivative in the sense of Caputo approach, can be given as

$$ {}_0^{ABC}D^{\alpha}p(t)-\frac{a_1+b_1}{a_2+b_2}p(t)=-\frac{a_0+b_0}{a_2+b_2}. \tag{5}$$

The solution of (5) is given by

$$\begin{aligned}p(t)=&\frac{B(\alpha)p(0)(a_2+b_2)}{(\alpha-1)a_1+(\alpha-1)b_1+B(\alpha)(a_2+b_2)}\\&\times E_\alpha\Big(\frac{\alpha(a_1+b_1)}{(\alpha-1)a_1+(\alpha-1)b_1+B(\alpha)(a_2+b_2)}t^\alpha\Big)\\&-\frac{(\alpha-1)(a_0+b_0)}{(\alpha-1)a_1+(\alpha-1)b_1+B(\alpha)(a_2+b_2)}\\&\times E_\alpha\Big(\frac{\alpha(a_1+b_1)}{(\alpha-1)a_1+(\alpha-1)b_1+B(\alpha)(a_2+b_2)}t^\alpha\Big)+\frac{(\alpha-1)(a_0+b_0)}{\alpha(a_1+b_1)}\\&\times\Big(1-E_\alpha\Big(\frac{\alpha(a_1+b_1)}{(\alpha-1)a_1+(\alpha-1)b_1+B(\alpha)(a_2+b_2)}t^\alpha\Big)\Big),\end{aligned}$$

where $p(0)=\frac{a_0+b_0}{a_1+b_1}$.

The following equation represents the price adjustment equation with the Atangana-Baleanu fractional derivative in the sense of Caputo approach, with generalized Mittag–Leffler function, when the expectations of agents are not taken into account:

$$ {}_0^{ABC}D^{\alpha,\beta,\rho}p(t)+k(a_1+b_1)p(t)=k(a_0+b_0),\quad \alpha\in(0,1). \tag{6}$$

The solution of (6) is given by

$$p(t) = p(0)\sum_{j=0}^{\infty}(-k(a_1+b_1))^j\Big(\frac{1-\alpha}{B(\alpha)}\Big)^j E^{-\rho j}_{\alpha,(1-\beta)j+1}(\lambda,t)$$
$$+ k(a_0+b_0)\sum_{j=0}^{\infty}(-k(a_1+b_1))^j\Big(\frac{1-\alpha}{B(\alpha)}\Big)^{j+1} E^{-\rho(j+1)}_{\alpha,(1-\beta)(j+1)+1}(\lambda,t).$$

If we take into account the expectations of the agents, we have

$$ {}^{ABC}_{0}D^{\alpha,\beta,\rho}p(t) - \frac{a_1+b_1}{a_2+b_2}p(t) = -\frac{a_0+b_0}{a_2+b_2}. \tag{7}$$

The solution of (7) is given by

$$p(t) = p(0)\sum_{j=1}^{\infty}\Big(-\frac{a_1+b_1}{a_2+b_2}\Big)^j\Big(\frac{1-\alpha}{B(\alpha)}\Big)^j E^{-\rho j}_{\alpha,(1-\beta)j+1}(\lambda,t)$$
$$-\frac{a_0+b_0}{a_2+b_2}\sum_{j=0}^{\infty}\Big(-\frac{a_1+b_1}{a_2+b_2}\Big)^j\Big(\frac{1-\alpha}{B(\alpha)}\Big)^{j+1} E^{-\rho(j+1)}_{\alpha,(1-\beta)(j+1)+1}(\lambda,t).$$

When the expectations of agents are not considered, the price adjustment equation incorporating the constant proportional Caputo fractional derivative, can be expressed by the following equation:

$$ {}^{CPC}_{0}D^{\alpha}p(t) + k(a_1+b_1)p(t) = k(a_0+b_0), \quad \alpha\in(0,1). \tag{8}$$

The solution of (8) is given by

$$p(t) = \frac{a_0+b_0}{a_1+b_1}E^{1}_{1-\alpha,-\alpha,1}\Big(\frac{-k_1(\alpha)}{k(a_0+b_0)}t^{1-\alpha}, \frac{-k_0(\alpha)}{k(a_0+b_0)}t^{-\alpha}\Big)$$
$$+ p(0)E^{1}_{1,\alpha,a}\Big(\frac{-k_1(\alpha)}{k_0(\alpha)}t, \frac{-k(a_1+b_1)}{k_0(\alpha)}t^{\alpha}\Big).$$

When we take into account the expectations of agents, the price adjustment equation, by means of the constant proportional Caputo fractional derivative, can be expressed as follows:

$$ {}^{CPC}_{0}D^{\alpha}p(t) - \frac{a_1+b_1}{a_2+b_2}p(t) = -\frac{a_0+b_0}{a_2+b_2}. \tag{9}$$

The solution of (9) is given by

$$p(t) = \frac{a_0+b_0}{a_1+b_1}E^{1}_{1-\alpha,-\alpha,1}\Big(\frac{a_2+b_2}{a_1+b_1}t^{1-\alpha}, \frac{(a_2+b_2)k_0(\alpha)}{a_1+b_1}t^{-\alpha}\Big)$$
$$+ p(0)E^{1}_{1,\alpha,1}\Big(\frac{-k_1(\alpha)}{k_0(\alpha)}t, \frac{a_1+b_1}{k_0(\alpha)(a_2+b_2)}t^{\alpha}\Big),$$

where k_0 and k_1 are functions of $\alpha\in[0,1]$, satisfying certain conditions given in the preliminaries.

3. Economic models in the framework of generalized proportional Caputo fractional derivatives

We will examine now the first model that incorporates the generalized proportional Caputo derivative, represented as follows:

$$ {}^{C}_{0}D^{\alpha,\xi,g}p(t) + k(a_1+b_1)p(t) = k(a_0+b_0), \tag{10}$$

where $\alpha \in (0,1)$ and $\xi \in (0,1]$. Taking the generalized Laplace transform on the both sides of (10), we have

$$\mathcal{L}_g\Big({}_0^C D^{\alpha,\xi,g}p(t)\Big)(s) + k(a_1+b_1)\mathcal{L}_g(p(t))(s) = \mathcal{L}_g\Big(k(a_0+b_0)\Big).$$

Using Lemma 1.1, we obtain

$$(\xi s+1-\xi)^\alpha \mathcal{L}_g(p(t))(s) - \xi(\xi s+1-\xi)^{\alpha-1}p(0) + k(a_1+b_1)\mathcal{L}_g(p(t))(s) = \frac{k(a_0+b_0)}{s}.$$

Now,

$$\mathcal{L}_g(p(t))(s)\Big((\xi s+1-\xi)^\alpha + k(a_1+b_1)\Big) = \xi(\xi s+1-\xi)^{\alpha-1}p(0) + \frac{k(a_0+b_0)}{s},$$

i.e.

$$\mathcal{L}_g(p(t))(s) = \frac{\xi(\xi s+1-\xi)^{\alpha-1}}{(\xi s+1-\xi)^\alpha + k(a_1+b_1)}p(0) + \frac{k(a_0+b_0)}{s\Big((\xi s+1-\xi)^\alpha + k(a_1+b_1)\Big)}.$$

Taking the inverse generalized Laplace transform, we obtain

$$\begin{aligned} p(t) = {} & \mathcal{L}_g^{-1}\left(\frac{\xi(\xi s+1-\xi)^{\alpha-1}}{(\xi s+1-\xi)^\alpha + k(a_1+b_1)}\right)p(0) \\ & + \frac{a_0+b_0}{a_1+b_1}\mathcal{L}_g^{-1}\left(\frac{k(a_0+b_0)}{s\Big((\xi s+1-\xi)^\alpha + k(a_1+b_1)\Big)}\right), \end{aligned}$$

or equivalently

$$\begin{aligned} p(t) = {} & E_\alpha\Big(\frac{-k(a_1+b_1)}{\xi^\alpha}(g(t)-g(0))^\alpha\Big)e^{\frac{\xi-1}{\xi}(g(t)-g(0))}p(0) \\ & + \frac{a_0+b_0}{a_1+b_1}\mathcal{L}_g^{-1}\left(\frac{A}{s} + \frac{B}{\Big(s-\frac{\xi-1}{\xi}\Big)^\alpha + \frac{k(a_1+b_1)}{\xi^\alpha}}\right), \end{aligned}$$

where $A = \frac{\xi^\alpha k(a_1+b_1)}{(1-\xi)^\alpha + k(a_1+b_1)}$ and $B = \frac{\xi k(a_1+b_1)}{(\xi-1)(-k(a_1+b_1))^{\frac{1}{\alpha}}}$. Hence,

$$\begin{aligned} p(t) = {} & E_\alpha\Big(\frac{-k(a_1+b_1)}{\xi^\alpha}(g(t)-g(0))^\alpha\Big)e^{\frac{\xi-1}{\xi}(g(t)-g(0))}p(0) \\ & + \frac{a_0+b_0}{a_1+b_1}\Big(\frac{\xi^\alpha k(a_1+b_1)}{(1-\xi)^\alpha + k(a_1+b_1)} \\ & + \frac{\xi k(a_1+b_1)}{(\xi-1)(-k(a_1+b_1))^{\frac{1}{\alpha}}}\big(g(t)-g(0)\big)^{\alpha-1}e^{\frac{\xi-1}{\xi}(g(t)-g(0))} \\ & \times E_{\alpha,\alpha}\Big(-\frac{k(a_1+b_1)}{\xi^\alpha}(g(t)-g(0))^\alpha\Big)\Big). \end{aligned}$$

Next, we consider the following equation about the price adjustment equation with generalized proportional Caputo fractional derivative, employing the expectation of agents,

$${}_0^C D^{\alpha,\xi,g}p(t) - \frac{a_1+b_1}{a_2+b_2}p(t) = -\frac{a_0+b_0}{a_2+b_2}. \tag{11}$$

Applying the generalized Laplace transform on the both sides of the equation, we have

$$\mathcal{L}_g({}_0^C D^{\alpha,\xi,g} p(t))(s) - \frac{a_1+b_1}{a_2+b_2}\mathcal{L}_g(p(t))(s) = -\mathcal{L}_g\Big(\frac{a_0+b_0}{a_2+b_2}\Big).$$

Hence,

$$(\xi s+1-\xi)^{\alpha}\mathcal{L}_g(p(t))(s) - \xi(\xi+1-\xi)^{\alpha-1}p(0) - \frac{a_1+b_1}{a_2+b_2}\mathcal{L}_g(p(t))(s) = -\frac{a_0+b_0}{s(a_2+b_2)},$$

$$\mathcal{L}_g(p(t))(s)\Big((\xi s+1-\xi)^{\alpha} - \frac{a_1+b_1}{a_2+b_2}\Big) = \xi(\xi s+1-\xi)^{\alpha-1}p(0) - \frac{a_0+b_0}{s(a_2+b_2)}$$

and

$$\mathcal{L}_g(p(t))(s) = \frac{\xi(\xi s+1-\xi)^{\alpha-1}p(0)}{(\xi s+1-\xi)^{\alpha} - \frac{a_1+b_1}{a_2+b_2}} - \frac{a_0+b_0}{a_2+b_2}\cdot\frac{1}{s\Big((\xi s+1-\xi)^{\alpha} - \frac{a_1+b_1}{a_2+b_2}\Big)}.$$

Taking the generalized inverse Laplace transform of the both sides of the last equation, we obtain

$$p(t) = \mathcal{L}_g^{-1}\Big(\frac{\xi(\xi s+1-\xi)^{\alpha-1}}{(\xi s+1-\xi)^{\alpha} - \frac{a_1+b_1}{a_2+b_2}}\Big)p(0) - \frac{a_0+b_0}{a_2+b_2}\mathcal{L}_g^{-1}\Big(\frac{1}{s\Big((\xi s+1-\xi)^{\alpha} - \frac{a_1+b_1}{a_2+b_2}\Big)}\Big)$$

$$\begin{aligned} p(t) &= \mathcal{L}_g^{-1}\left(\frac{\left(s-\frac{\xi-1}{\xi}\right)^{\alpha-1}}{\left(s-\frac{\xi-1}{\xi}\right)^{\alpha} - \frac{a_1+b_1}{\xi^{\alpha}(a_2+b_2)}}\right)p(0) \\ &- \frac{a_0+b_0}{a_2+b_2}\mathcal{L}_g^{-1}\left(\frac{1}{\xi^{\alpha}s\Big(\left(s-\frac{\xi-1}{\xi}\right)^{\alpha} - \frac{a_1+b_1}{\xi^{\alpha}(a_2+b_2)}\Big)}\right). \end{aligned}$$

We have

$$\begin{aligned} p(t) &= E_{\alpha}\Big(\frac{a_1+b_1}{\xi^{\alpha}(a_2+b_2)}(g(t)-g(0))^{\alpha}\Big)e^{\frac{\xi-1}{\xi}(g(t)-g(0))}p(0) \\ &- \frac{a_0+b_0}{\xi^{\alpha}(a_2+b_2)}\mathcal{L}_g^{-1}\left(\frac{A}{s} + \frac{B}{\left(s-\frac{\xi-1}{\xi}\right)^{\alpha} - \frac{a_1+b_1}{\xi^{\alpha}(a_2+b_2)}}\right), \end{aligned}$$

where $A = \frac{1}{\left(\frac{1-\xi}{\xi}\right)^{\alpha} - \frac{a_1+b_1}{\xi^{\alpha}(a_2+b_2)}}$ and $B = \frac{1}{\frac{\xi-1}{\xi} + \left(\frac{a_1+b_1}{\xi^{\alpha}(a_2+b_2)}\right)^{\frac{1}{\alpha}}}$. Finally,

$$\begin{aligned} p(t) &= E_{\alpha}\Big(\frac{a_1+b_1}{\xi^{\alpha}(a_2+b_2)}(g(t)-g(0))^{\alpha}\Big)e^{\frac{\xi-1}{\xi}(g(t)-g(0))}p(0) \\ &- \frac{a_0+b_0}{\xi^{\alpha}(a_2+b_2)}\left(\frac{1}{\left(\frac{1-\xi}{\xi}\right)^{\alpha} - \frac{a_1+b_1}{\xi^{\alpha}(a_2+b_2)}}\right. \\ &+ \frac{1}{\frac{\xi-1}{\xi} + \left(\frac{a_1+b_1}{\xi^{\alpha}(a_2+b_2)}\right)^{\frac{1}{\alpha}}}\big(g(t)-g(0)\big)^{\alpha-1}e^{\frac{\xi-1}{\xi}(g(t)-g(0))} \\ &\left.\times E_{\alpha,\alpha}\Big(\frac{(a_1+b_1)}{\xi^{\alpha}(a_2+b_2)}(g(t)-g(0))^{\alpha}\Big)\right). \end{aligned}$$

4. Comparison of the different frameworks of the economic models

This section focuses on conducting simulation analysis of six significant non-local fractional operators: Caputo, Caputo–Fabrizio, Atangana–Baleanu in the sense of Caputo approach, generalized Atangana–Baleanu in the sense of Caputo approach, constant proportional Caputo and generalized Caputo proportional. A comparative assessment is performed between these fractional derivatives and the classical derivative. Through illustrative examples, we demonstrate the actions of these operators for various values of α, and for generalized Atangana–Baleanu in Caputo sense, different values of β are also utilized when $\rho = 1$. Furthermore, by assigning specific values to constants a_0, b_0, a_1, b_1, a_2 and b_2, which influence market equilibrium, we take into account the expectations of agents. The graphics exhibit profound changes in curve behavior as arbitrary orders are altered.

Here we put $a_0 = 10$, $b_0 = 100$, $a_1 = 14$, $b_1 = 97$, $a_2 = 18$ and $b_2 = 94$. The solution curves of price adjustment equation, in the case of consideration of expectation of agents are given in the following ten figures. Given our primary focus on economic models utilizing generalized proportional Caputo fractional derivatives, as well as the versatility observed in the cases involving constant proportional Caputo fractional operators, which are determined by the functions $k_0(\alpha)$ and $k_1(\alpha)$, we have chosen not to include graphical representations of these specific solution types in this paper. In Figure 1, expectation curves for generalized proportional Caputo fractional operator for fixed $\alpha = 0.93$, $\xi = 0.7$ and the cases of $g(t) = t$, $g(t) = t^2$, $g(t) = e^t$, $g(t) = \sqrt{t}$, $g(t) = t^{\frac{1}{3}}$ in are showed. As it can been seen from Figure 1, and also intuitively by the obtained formulas, the biggest growth in the price adjustment equations has in the case when $g(t) = e^t$. The Figure 2 shows how the solutions of (11), varied over the value of ξ, for fixed values of α and the function $g(t)$. The next figure considers the variation of the graphs of the solutions of (11), for $g(t) = t$, $\xi = 0.5$ and various values of α. On figures 4-6, comparisons between the solutions in the case of generalized proportional Caputo fractional derivative for given $g(t)$, ξ and α, Caputo fractional, Caputo–Fabrizio fractional in the sense of Caputo, Atangana–Baleanu fractional in the Caputo sense and classical derivative, are given. On Figures 7-8, comparisons between the solutions in the case of generalized proportional Caputo fractional derivative for given $g(t) = e^t$, $\xi = 0.7$ and $\alpha = 0.8$ and previously mentioned cases of fractional and classical differential operators are showcased. The last two figures are displaying the comparisons in the cases when the solutions of (11) in the case when $g(t) = \sqrt{t}$, $\alpha = 0.97$, $\xi = 0.6$, with the cases of the solutions with considered differential operators.

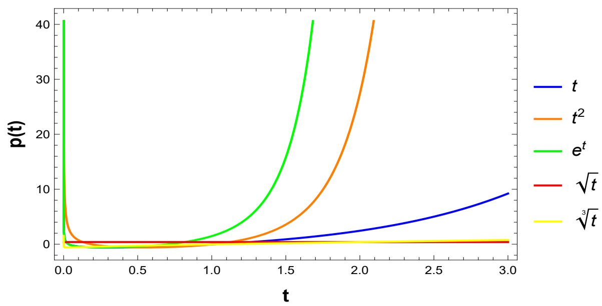

FIGURE 1. Comparison for generalized proportional Caputo fractional derivative for $\alpha = 0.93$, $\xi = 0.7$ and cases of $g(t) = t$, $g(t) = t^2$, $g(t) = e^t$, $g(t) = \sqrt{t}$, $g(t) = t^{\frac{1}{3}}$.

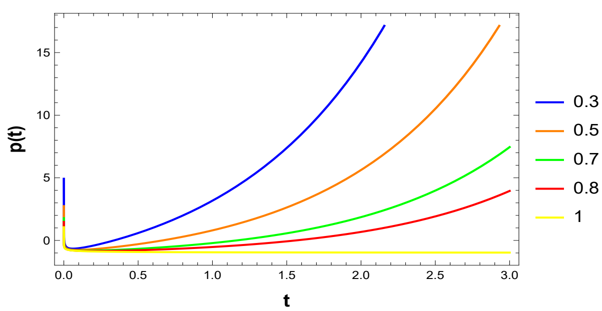

FIGURE 2. Comparison for generalized proportional Caputo fractional derivative for $g(t) = t$, $\alpha = 0.93$ and cases of $\xi = 0.3$, $\xi = 0.5$, $\xi = 0.7$, $\xi = 0.8$, $\xi = 1$.

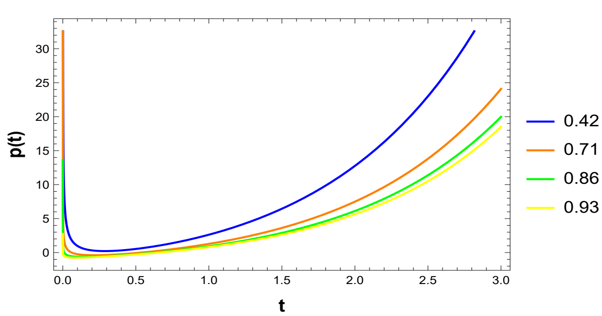

FIGURE 3. Comparison for generalized proportional Caputo fractional derivative for $g(t) = t$, $\xi = 0.5$ and cases of $\alpha = 0.42$, $\alpha = 0.71$, $\alpha = 0.86$, $\alpha = 0.93$.

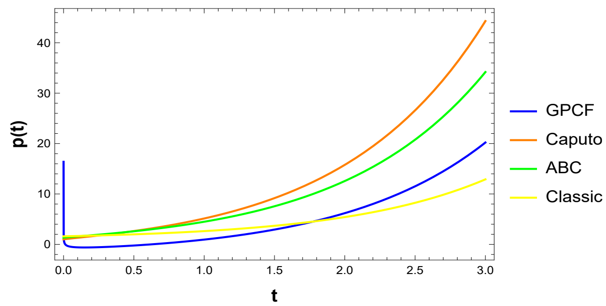

FIGURE 4. Comparison for generalized proportional Caputo fractional derivative for $g(t) = t$, $\alpha = 0.85$, $\xi = 0.5$, Caputo fractional, Atangana–Baleanu fractional in the sense of Caputo approach and classical derivative.

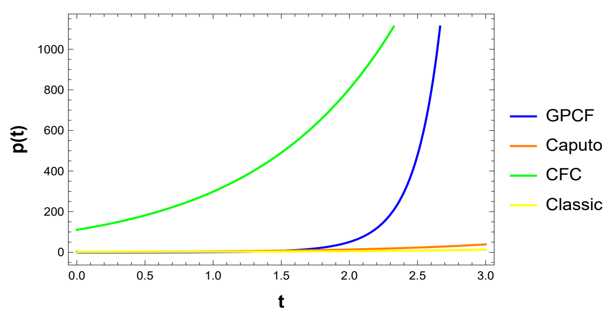

FIGURE 5. Comparison for generalized proportional Caputo fractional derivative for $g(t) = t^2$, $\alpha = 0.97$, $\xi = 0.5$, Caputo fractional, Caputo–Fabrizio fractional in the sense of Caputo approach and classical derivative.

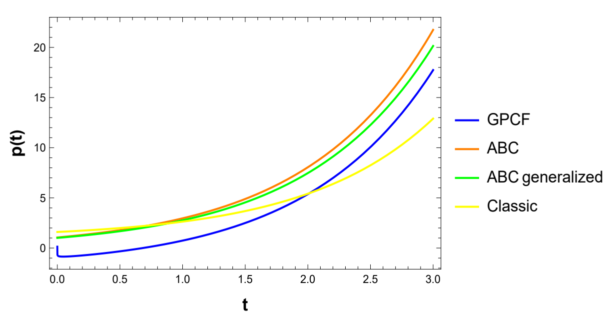

FIGURE 6. Comparison for generalized proportional Caputo fractional derivative for $g(t) = t$, $\alpha = 0.97$, $\xi = 0.5$, Atangana–Baleanu fractional in the sense of Caputo approach, Atangana–Baleanu fractional in the sense of Caputo approach with generalized Mittag–Leffler function with $\beta = 0.95$, $\rho = 1$ and classical derivative.

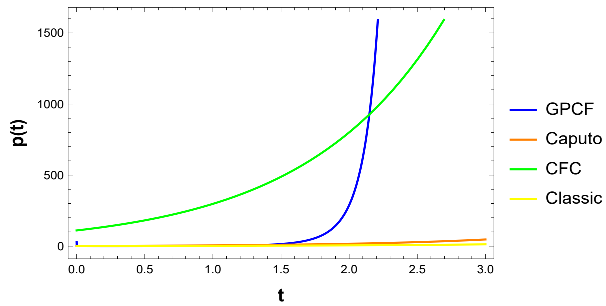

FIGURE 7. Comparison for generalized proportional Caputo fractional derivative for $g(t) = e^t$, $\alpha = 0.8$, $\xi = 0.7$, Caputo fractional, Caputo–Fabrizio fractional in the sense of Caputo approach and classical derivative.

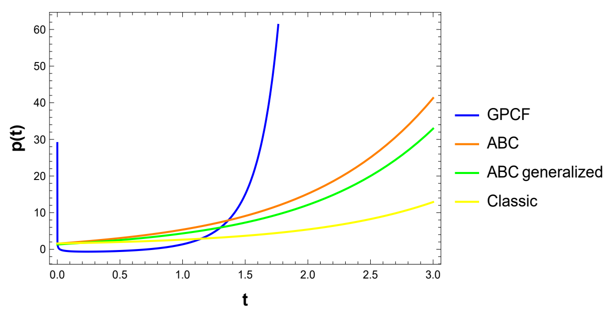

FIGURE 8. Comparison for generalized proportional Caputo fractional derivative for $g(t) = e^t$, $\alpha = 0.8$, $\xi = 0.7$, Atangana–Baleanu fractionalin the sense of Caputo approach, Atangana–Baleanu fractional in the sense of Caputo approach with generalized Mittag–Leffler function with $\beta = 0.95$, $\rho = 1$ and classical derivative.

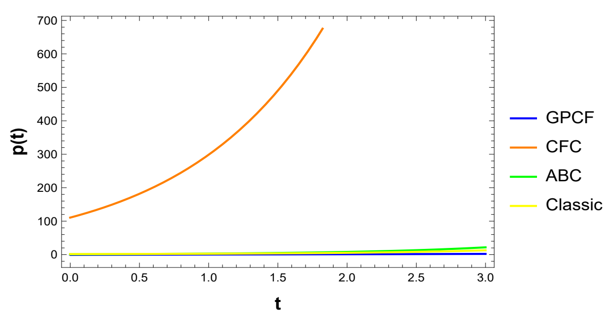

FIGURE 9. Comparison for generalized proportional Caputo fractional derivative for $g(t) = \sqrt{t}$, $\alpha = 0.97$, $\xi = 0.6$, Caputo–Fabrizio fractional in the sense of Caputo approach, Atangana–Baleanu fractional in the sense of Caputo approach and classical derivative.

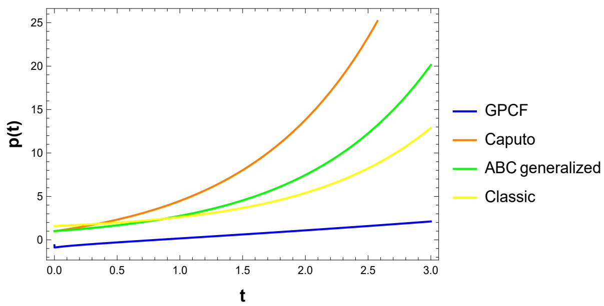

FIGURE 10. Comparison for generalized proportional Caputo fractional derivative for $g(t) = \sqrt{t}$, $\alpha = 0.97$, $\xi = 0.6$, Caputo fractional, Atangana–Baleanu fractional in the sense of Caputo approach with generalized Mittag–Leffler function with $\beta = 0.95$, $\rho = 1$ and classical derivative.

5. Conclusion

In this paper using non-local fractional operators, specifically generalized proportional Caputo fractional derivatives, we investigate certain economic problems and compare the obtained results with those ones using other fractional operators such as Caputo, Caputo-Fabrizio in the sense of Caputo approach, Caputo type Atangana-Baleanu, Caputo type Atangana–Baleanu with generalized Mittag-Leffler kernel and constant proportional in the sense of Caputo approach.

These fractional operators are applied beyond traditional differentiation and integration to improve our understanding of supply, demand, and price interactions in an equilibrium market. To gain deeper insights into the controversial issues in the market, we also conduct simulation analysis. Our study yields several findings. First, we present the economic models using non-local fractional operators such as Caputo, Caputo-Fabrizio in the sense of Caputo, Caputo type Atangana-Baleanu, Caputo type Atangana–Baleanu with generalized Mittag-Leffler kernel and constant proportional in Caputo sense, with or without expectation of agents. Second, we solve the price adjustment equation, which is crucial for achieving market equilibrium, using generalized proportional Caputo fractional derivatives while considering and not considering agents' expectations. Thus, we obtain two separate solutions for each fractional operator, which are quite general comparing to the previous consideration in this types of economic models. Third, this types of models are important, since they capture past fluctuations in the economy, and provide a more detailed representation of the interactions between the factors that strongly influence market equilibrium.

The constant proportional Caputo fractional derivative plays a significant role in analyzing various economic models. The functions $k_0(\alpha)$ and $k_1(\alpha)$ introduce distinct cases in the solutions of these models, leading to different economic phenomena. These functions, dependent on the fractional order α, influence the behavior and dynamics of the solutions, providing insights into the underlying economic processes.

In the realm of the generalized proportional Caputo fractional derivative, we encounter the most comprehensive scenario. With adjustments to the parameters ξ and $g(t)$, the solution graphs can closely resemble those obtained from employing other types of fractional derivatives. This flexibility allows for a wide range of modeling possibilities and facilitates the comparison and analysis of various economic systems.

By leveraging the generalized proportional Caputo fractional derivative, researchers can explore the dynamics of economic models in a more versatile manner. This approach enables us to capture different aspects of economic behavior and investigate the impacts of varying parameters on the system's dynamics. The ability to closely approximate solutions obtained from other fractional derivatives provides a valuable tool for understanding the interconnectedness and similarities among different economic phenomena.

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