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DOUBLE DIRICHLET AVERAGE OF GENERALIZED BESSEL-MAITLAND FUNCTION USING FRACTIONAL DERIVATIVE

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Abstract. The purpose of this paper is to establish the results of a double Dirichlet average of the generalized Bessel-Maitland function by using fractional derivative. We obtain the solution in the compact form of a double Dirichlet average of the generalized Bessel-Maitland function. Further, several special cases involving a number of well-known functions such as the Bessel-Maitland function, Mittag Leffler functions, Wright-hypergeometric functions and H-functions etc. have been established. Numerous expansions of the generalised Bessel-Maitland function, which reduces to the Mittag-Leffler function, have been studied and applied to solve a wide range of problems in physics, biology, chemistry and engineering. In the context of fractional calculus, Bessel-Maitland function, Mittag-Leffler functions, hypergeometric function and H-functions etc. have been widely studied. Since Carlson first introduced the concept of the Dirichlet average and its various forms. The Dirichlet average of elementary function like power function, exponential function etc is given by many notable mathematicians. These averages have been investigated and utilized in a number of different fields.

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1. INTRODUCTION

The Dirichlet averages of function are certain type of integral averages with respect to the Dirichlet measure. The concept of various type of Dirichlet average was introduced by Carlson [2-6]. Also, it has been studied by among others by Ali et al.[1], Deora and Banerji [8], Gupta and Agrawal [12-13], Gurjar [14], Khan et al. [15], Kilbas and Kattuveettil [16], Saxena et al.[25], Sharma and Jain [26-29], Sharma et al. [30] and Sharma and Sharma [31]. Carlson [5] has given a detailed and comprehensive account of various types of Dirichlet average in his monograph. In this paper, the authors investigate the double Dirichlet average of generalized Bessel-Maitland function.

2. PRELIMINARIES

In this section, we give some definitions, which will be helpful in our further investigations.

2.1. **The Generalized Bessel-Maitland function.**

Let $\mu, v, \gamma, \delta \in \mathbb{C}$ with $\Re(\mu) > 0$, $\Re(v) > -1$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$. Also, Let $p, q \in \mathbb{R}^+$ with q $< R(\mu)+p$. The generalized Bessel-Maitland function is introduced by Ghayasuddin and Khan [11] and defined as

$$J_{v,\gamma,\delta}^{\mu,q,p}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(-z)^n}{\Gamma\left(\mu n+v+1\right)(\delta)_{pn}} \tag{1}$$

where $(\gamma)_{qn} = \frac{\Gamma(\gamma+qn)}{\Gamma(\gamma)}$ denotes the generalized Pochhammer symbol (see [21]).

If we put $\delta = p = 1$ in equation (1), the function reduces to the generalization of Bessel-Maitland function given by Singh et al. [32] and defined as

$$J_{v,\gamma}^{\mu,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(-z)^n}{\Gamma\left(\mu n+v+1\right)n!} \tag{2}$$

where $\mu,\ v.\ \gamma \in \mathbb{C}$ with $\Re(\mu) > 0$, $\Re(v) > -1$, $\Re(\gamma) > 0$ and q$\in$ (0,1)$\cup$ N.

We put $\delta = p = 1$ and $q = 0$ in equation (1), the function reduces to the Bessel-Maitland function or the Wright generalized function given by Marichev [17] and defined as

$$J_{v}^{\mu}(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{\Gamma\left(\mu n+v+1\right)n!} = \phi(\mu, \nu+1; -z) \tag{3}$$

If we replace v by $v-1$ in (1), the function reduces to the known following relation with Mittag-Leffler function, which is given by Salim and Faraj [23] as

$$J^{\mu,q,p}_{v-1,\gamma,\delta}(-z) = E^{\gamma,\delta,q}_{\mu,v,p}(z). \tag{4}$$

When $p = \delta = 1$ and replacing v by $v-1$, the function (1) yields the Mittag- Leffler function defined and studied by Shukla and Prajapati [24] as

$$J^{\mu,q,1}_{v-1,\gamma,1}(-z) = E^{\gamma,q}_{\mu,v}(z) \tag{5}$$

On setting $p = q = \delta = 1$ and replacing v by $v-1$ in (2.1), then it reduces to well-known Mittag- Leffler function given by Prabhakar [20] as

$$J^{\mu,1,1}_{v-1,\gamma,1}(-z) = E^{\gamma}_{\mu,v}(z). \tag{6}$$

Further, take $p = q = \delta = \gamma = 1$ and replacing v by $v-1$ (1), then we get a relation with Mittag- Leffler function defined by Wiman [34] as

$$J^{\mu,1,1}_{v-1,1,1}(-z) = E_{\mu,v}(z). \tag{7}$$

On setting $p = q = \delta = \gamma = 1$ and replacing $v = 0$ in (2.1), then it reduces to Mittag-Leffler function [19]. Its representation also reduces in the form of Wright hypergeometric functions [35] and Fox H-functions [33] as

$$J^{\mu,1,1}_{0,1,1}(-z) = E_{\mu}(z) = {}_1\Psi_1\left[\begin{matrix}(1,1)\\(1,\mu)\end{matrix};z\right] = H^{1,1}_{1,2}\left[\begin{matrix}(0,1)\\(0,1),(0,\mu)\end{matrix};-z\right] \tag{8}$$

For studying various types of generalizations of functions, their properties and applications readers may refer to the work done by Chandola et al. [7], Farid et al. [10], Mishra and Pandey [18], Rehman et al. [22] and others.

2.2. **The Euclidean Simplex.** The symbol E_n will denote the Euclidean simplex which is defined by Carlson ([5] pg. 62) and denoted by

$$E = E_n = \{(u_1, u_2, \ldots, u_n)\,; u_i \geq 0; i = 1, 2, \ldots, n : \ u_1 + u_2 + \ldots + u_n \leq 1\} \tag{9}$$

2.3. **Dirichlet Measure.** Let $b \in \mathbb{C}^n$, $n \geq 2$ and let $E = E_{n-1}$be the standard simplex in R^{n-1}, the complex measure μ_b, then the Dirichlet measure on E is denoted by $d\mu_b$ and defined as

$$d\mu_b(u) = \frac{1}{B(b)} u_1^{b_1-1} \ldots . u_{n-1}^{b_{n-1}-1} \left(1 - u_1 - \ldots - u_{n-1}\right)^{b_n-1} du_1 \ldots du_{n-1} \tag{10}$$

where $B(b)$ is the multivariable Beta function which is defined as

$$B(b)=B\left(b_1,\ldots,b_n\right)=\frac{\Gamma\left(b_1\right)\ldots\Gamma(b_n)}{\Gamma(b_1+\ldots+b_n)}:\left(\operatorname{Re}\left(b_j\right)>0:j=1,2,\ldots,n\right) \tag{11}$$

2.4. **Dirichlet Average.** The concept of Dirichlet average is introduced by Carlson [2-5]. Dirichlet averages have produced deep and interesting relations to special functions. Let Ω be a convex set in complex number $\mathbb{C}$ and let $z=(z_1,z_2,\ldots,z_n)\in\Omega^n,\ n\geq 2$. Let f be a measurable function on Ω. Then we have

$$F(b;z)=\int_E f(u\circ z)d\mu_b(u) \tag{12}$$

where $d\mu_b(u)$ is defined by equation (10) and
$u\circ z=\sum_{i=1}^n u_iz_i$ and $u_n=1-u_1-\ldots-u_{n-1}$

2.5. **Double Dirichlet Average.** The double Dirichlet average is equivalent to fractional derivative is discussed by Gupta and Agrawal [13]. Let z be a $k\times x$ matrix with complex element z_{ij}. Let $u=(u_1,\ldots,u_k)$ and $v=(v_1,\ldots,v_x)$ be an ordered k-tuple and x-tuple of real non-negative weights $\sum u_i=1$ and $\sum v_j=1$ respectively. Now, we define

$$u\circ z=\sum_{i=1}^k\sum_{j=1}^x u_iz_{\text{ij}}v_j$$

If z_{ij} is regarded as a point of the complex plane, all these convex combinations are points in the convex hull of $(z_{11},\ldots,z_{kx})$ denoted by $H(z)$.

Let $b=(b_1,\ldots,b_k)$ be an ordered k-tuple of complex numbers with positive real part Re(b)>0 and similarly for$\beta=(\beta_1,\ldots,\beta_x)$, then defined$d\mu_b(u)$ and $d\mu_\beta(v)$.

Let f be the holomorphic on a domain D in the complex plane, if $\operatorname{Re}(b)>0,\operatorname{Re}(\beta)>0$, and $H(z)\subset D$.

Then we have

$$F(b,z,\beta)=\iint f(u\circ z\circ v)d\mu_b(u)d\mu_\beta(v)$$

Double average for $(k=x=2)$ of $(u\circ z\circ v)^t$ is the S-function is given by Gupta and Agrawal [13] as follows:

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2)=\int_0^1\int_0^1(u\circ z\circ v)^t\mathrm{dm}_{\mu_1\mu_2}(u)_{\rho_1\rho_2}(v) \tag{13}$$

where

$$\operatorname{Re}\left(\mu_1\right)>0,\operatorname{Re}\left(\mu_2\right)>0,\operatorname{Re}\left(\rho_1\right)>0,\operatorname{Re}\left(\rho_2\right)>0$$

and

$$\begin{aligned} u \circ z \circ v &= \sum_{i=1}^{2} \sum_{j=1}^{2} \left(u_i \circ z_{\mathrm{ij}} \circ v_j\right) \\ &= \sum_{i=1}^{2} \left[u_1 \left(z_{i1} v_1 + z_{i2} v_2\right)\right] \\ &= \left[u_1 z_{11} v_1 + u_1 z_{12} v_2 + u_2 z_{21} v_1 + u_2 z_{22} v_2\right]. \end{aligned}$$

Let $z_{11} = a$, $z_{12} = b$, $z_{21} = c$, $z_{22} = d$ and

$$\left\{ \frac{u_1 = u u_2 = 1 - u}{v_1 = v v_2 = 1 - v} \right. .$$

Thus

$$z = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Therefore

$$\begin{aligned} u \circ z \circ v &= uva + ub(1 - v) + (1 - u)cv + (1 - u)d(1 - v) \\ &= uv(a - b - c + d) + u(b - d) + v(c - d) + d \qquad (14) \end{aligned}$$

$$dm_{\mu_1 \mu_2}(u) = \frac{\Gamma(\mu_1 + \mu_2)}{\Gamma\left(\mu_1\right) \Gamma(\mu_2)} u^{\mu_1 - 1} (1 - u)^{\mu_2 - 1} \mathrm{du} \qquad (15)$$

and

$$dm_{\rho_1 \rho_2}(v) = \frac{\Gamma(\rho_1 + \rho_2)}{\Gamma\left(\rho_1\right) \Gamma(\rho_2)} v^{\rho_1 - 1} (1 - v)^{\rho_2 - 1} \mathrm{dv}. \qquad (16)$$

Thus from equation (13), we obtain

$$\begin{aligned} S\left(\mu_1, \mu_2; z; \rho_1, \rho_2\right) &= \frac{\Gamma\left(\mu_1 + \mu_2\right)}{\Gamma\left(\mu_1\right) \Gamma\left(\mu_2\right)} \frac{\Gamma\left(\rho_1 + \rho_2\right)}{\Gamma\left(\rho_1\right) \Gamma\left(\rho_2\right)} \\ &\times \int_0^1 \int_0^1 \left[uv(a - b - c + d) + u(b - d) + v(c - d) + d\right]^t \\ &\times u^{\mu_1 - 1} (1 - u)^{\mu_2 - 1} du v^{\rho_1 - 1} (1 - v)^{\rho_2 - 1} dv. \end{aligned}$$

For more details of Dirichlet and double Dirichlet average, readers can refer to the work done by Ali et al. [1], Deora and Banerji [8], Gupta and Agrawal [12-13], Gurjar [14], Khan et al. [15] and others.

2.6. **Fractional Derivative.** The Riemann-Liouville fractional integral of arbitrary order α, $\mathrm{Re}(\alpha) > 0$ is given by Erdélyi et al. [9] and defined as

$$\left(I_{a+}^{\alpha} F\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha - 1} F(t) dt x > a, a \in R.$$

The fractional derivative is obtained by proceeding via fractional integral. The Riemann- Liouvelle fractional differential operator $D_x^{\alpha}$of order α $(\alpha \in C)$is defined by Erdélyi et al. [9]

$$\left(D_x^{\alpha}F\right)(x)=\begin{cases}\frac{1}{\Gamma(-\alpha)}\int_a^x (x-t)^{-\alpha-1}F(t)dt\ Re(\alpha)<0.\\ \frac{d^n}{dx^n}\left\{D_x^{\alpha-n}\left(f(x)\right)\right\}\\ \\ ;n-1\leq \mathrm{Re}(\alpha)<n;n\in N\end{cases} \tag{17}$$

where $\mathrm{Re}(\alpha)<0$ and $F(x)$ is the form of $x^p f(x)$; $f(x)$ is analytic at $x=0$.

3. MAIN RESULT

Theorem 3.1. Let $\mu,\nu,\gamma,\ \delta\in\mathbb{C}$ with $\Re(\mu)>0$, $\Re(v)>-1$, $\Re(\gamma)>0$, $\Re(\delta)>0$. Also, Let $p,q\in\mathbb{R}^+$ with $q<R(\mu)+p$. Then the double Dirichlet average of generalized Bessel-Maitland function $J_{v,\gamma,\delta}^{\mu,q,p}(u\circ z\circ v)$ with the fractional derivative for $(k=x=2)$ is

$$S\left(\mu_1,\mu_2;z;\rho_1,\rho_2\right)=\frac{\Gamma\left(\rho_1+\rho_2\right)}{\Gamma\left(\rho_1\right)}(x-y)^{1-\rho_1-\rho_2}\left[D_{x-y}^{-\rho_2}\left\{t^{\rho_1-1}J_{v,\gamma,\delta}^{\mu,q,p}(y+t)\right\}\right](x-y).$$

Proof: Let us consider the double Dirichlet average for $(k=x=2)$ of generalized Bessel-Maitland function $J_{v,\gamma,\delta}^{\mu,q,p}(u\circ z\circ v)$ by using equation (13)

$$S\left(\mu_1,\mu_2;z;\rho_1,\rho_2\right)=\int_0^1\int_0^1 J_{v,\gamma,\delta}^{\mu,q,p}(u\circ z\circ v)\mathrm{dm}_{\mu_1\mu_2}(u)\ \mathrm{dm}_{\rho_1\rho_2}(v).$$

Applying an equation (1) and rearranging the order of integration and summation, we obtain

$$S\left(\mu_1,\mu_2;z;\rho_1,\rho_2\right)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-z)^n}{\Gamma\left(\mu n+v+1\right)(\delta)_{pn}}\int_0^1\int_0^1 (u\circ z\circ v)^n dm_{\mu_1\mu_2}(u)dm_{\rho_1\rho_2}(v).$$

Using (14), (15) and (16), we have

$$\begin{aligned}S(\mu_1,\mu_2;z;\rho_1,\rho_2)=&\frac{\Gamma(\mu_1+\mu_2)}{\Gamma\left(\mu_1\right)\Gamma(\mu_2)}\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)\Gamma(\rho_2)}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-1)^n}{\Gamma(\mu n+v+1)(\delta)_{pn}}\\&\times\int_0^1\int_0^1\left[uv(a-b-c+d)+u(b-d)+v(c-d)+d\right]^t\\&\times u^{\mu_1-1}(1-u)^{\mu_2-1}duv^{\rho_1-1}(1-v)^{\rho_2-1}dv.\end{aligned} \tag{18}$$

We assume that $= c = x, \underline{=} d = y$, then we get

$$S\left(\mu_1,\mu_2;z;\rho_1,\rho_2\right)=\frac{\Gamma\left(\mu_1+\mu_2\right)}{\Gamma\left(\mu_1\right)\Gamma\left(\mu_2\right)}\frac{\Gamma\left(\rho_1+\rho_2\right)}{\Gamma\left(\rho_1\right)\Gamma\left(\rho_2\right)}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-1)^n}{\Gamma\left(\mu n+v+1\right)(\delta)_{pn}}$$
$$\times\int_0^1\int_0^1\left[v(x-y)+y\right]^n u^{\mu_1-1}(1-u)^{\mu_2-1}du\,v^{\rho_1-1}(1-v)^{\rho_2-1}dv.$$

Now, using the definition of Beta and Gamma functions, we obtain

$$S\left(\mu_1,\mu_2;z;\rho_1,\rho_2\right)=\frac{\Gamma\left(\rho_1+\rho_2\right)}{\Gamma\left(\rho_1\right)\Gamma\left(\rho_2\right)}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-1)^n}{\Gamma\left(\mu n+v+1\right)(\delta)_{pn}}$$
$$\times\int_0^1\left[v(x-y)+y\right]^n v^{\rho_1-1}(1-v)^{\rho_2-1}dv.$$

Let $v(x-y)=t$, then we have

$$S\left(\mu_1,\mu_2;z;\rho_1,\rho_2\right)=\frac{\Gamma\left(\rho_1+\rho_2\right)}{\Gamma\left(\rho_1\right)\Gamma\left(\rho_2\right)}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-z)^n}{\Gamma\left(\mu n+v+1\right)(\delta)_{pn}}$$
$$\times\int_0^{x-y}(y+t)^n\left(\frac{t}{x-y}\right)^{\rho_1-1}\left(1-\frac{t}{x-y}\right)^{\rho_2-1}\frac{dt}{x-y}$$
$$S\left(\mu_1,\mu_2;z;\rho_1,\rho_2\right)=\frac{\Gamma\left(\rho_1+\rho_2\right)}{\Gamma\left(\rho_1\right)\Gamma\left(\rho_2\right)}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-1)^n}{\Gamma\left(\mu n+v+1\right)(\delta)_{pn}}(x-y)^{1-\rho_1-\rho_2}$$
$$\times\int_0^{x-y}t^{\rho_1-1}(y+t)^n(x-y-t)^{\rho_2-1}dt$$
$$S\left(\mu_1,\mu_2;z;\rho_1,\rho_2\right)=\frac{\Gamma\left(\rho_1+\rho_2\right)}{\Gamma\left(\rho_1\right)\Gamma\left(\rho_2\right)}(x-y)^{1-\rho_1-\rho_2}\int_0^{x-y}(x-y-t)^{\rho_2-1}\left\{t^{\rho_1-1}J_{v,\gamma,\delta}^{\mu,q,p}(y+t)\right\}dt.$$

Using the definition of fractional derivative from (17), we get the required result.

Corollary 3.1. Let the conditions of **Theorem 3.1** are satisfied with $\delta = p = 1$, then the following result holds:

$$S\left(\mu_1,\mu_2;z;\rho_1,\rho_2\right)=\frac{\Gamma\left(\rho_1+\rho_2\right)}{\Gamma\left(\rho_1\right)}(x-y)^{1-\rho_1-\rho_2}\left[D_{x-y}^{-\rho_2}\left\{t^{\rho_1-1}J_{v,\gamma}^{\mu,q}(y+t)\right\}\right](x-y).$$

Corollary 3.2. Let the conditions of **Theorem 3.1** are satisfied with $\delta = p = 1$ and $q = 0$ then the following result holds:

$$S\left(\mu_1,\mu_2;z;\rho_1,\rho_2\right)=\frac{\Gamma\left(\rho_1+\rho_2\right)}{\Gamma\left(\rho_1\right)}(x-y)^{1-\rho_1-\rho_2}\left[D_{x-y}^{-\rho_2}\left\{t^{\rho_1-1}J_{v}^{\mu}(y+t)\right\}\right](x-y).$$

Corollary 3.3. Let the conditions of **Theorem 3.1** and $v = v - 1$, then we get the known result [15, p.79]:

$$S\left(\mu_1, \mu_2; z; \rho_1, \rho_2\right) = \frac{\Gamma\left(\rho_1+\rho_2\right)}{\Gamma\left(\rho_1\right)}(x-y)^{1-\rho_1-\rho_2}\left[D_{x-y}^{-\rho_2}\left\{t^{\rho_1-1}E_{\mu,v,p}^{\gamma,\delta,q}(y+t)\right\}\right](y-x).$$

Corollary 3.4. Let the conditions of **Theorem 3.1** and $v = v - 1$ and $\delta = p = 1$, then we get the known result [15, p.79]:

$$S\left(\mu_1, \mu_2; z; \rho_1, \rho_2\right) = \frac{\Gamma\left(\rho_1+\rho_2\right)}{\Gamma\left(\rho_1\right)}(x-y)^{1-\rho_1-\rho_2}\left[D_{x-y}^{-\rho_2}\left\{t^{\rho_1-1}E_{\mu,v}^{\gamma,q}(y+t)\right\}\right](y-x).$$

Corollary 3.5. Let the conditions of **Theorem 3.1** with $v = v - 1$ and $\delta = p = q = 1$, then it reduces to known result [15, p.80] (see also [11]) in the following form:

$$S\left(\mu_1, \mu_2; z; \rho_1, \rho_2\right) = \frac{\Gamma\left(\rho_1+\rho_2\right)}{\Gamma\left(\rho_1\right)}(x-y)^{1-\rho_1-\rho_2}\left[D_{x-y}^{-\rho_2}\left\{t^{\rho_1-1}E_{\mu,v}^{\gamma}(y+t)\right\}\right](y-x).$$

Further, taking $\gamma = 1,\ v = 0$ in above **Corollary 3.5**, then we get a relation with Mittag- Leffler function [19]. Its representation also reduces in the form of Wright hypergeometric functions [35] and Fox H-functions [33] as

$$\begin{aligned}
(\mu_1, \mu_2; z; \rho_1, \rho_2) &= \frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)}(x-y)^{1-\rho_1-\rho_2}\left[D_{x-y}^{-\rho_2}\left\{t^{\rho_1-1}E_{\mu}(y+t)\right\}\right](y-x) \\
&= \frac{\Gamma\left(\rho_1+\rho_2\right)}{\Gamma\left(\rho_1\right)}(x-y)^{1-\rho_1-\rho_2}\left[D_{x-y}^{-\rho_2}\left\{t^{\rho_1-1}1\Psi1\left[\begin{matrix}(1,1)\\(1,\mu)\end{matrix};(y+t)\right]\right\}\right](y-x) \\
&= \frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)}(x-y)^{1-\rho_1-\rho_2}\left[D_{x-y}^{-\rho_2}\left\{t^{\rho_1-1}H_{1,2}^{1,1}\left[\begin{matrix}(0,1)\\(0,1),(0,\mu)\end{matrix};-(y+t)\right]\right\}\right](y-x).
\end{aligned}$$

Theorem 3.2. Let $\mu, \nu\gamma, \delta \in \mathbb{C}$ with $\Re(\mu) > 0$, $\Re(v) > -1$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$. Also, Let $p, q \in \mathbb{R}^+$ with $q < R(\mu) + p$. Then the double Dirichlet average of generalized Bessel-Maitland function $J_{v,\gamma,\delta}^{\mu,q,p}(\text{ u} \circ \text{z } \circ v)$ with the fractional derivative for $(k = x = 2)$ is

$$S\left(\mu_1, \mu_2; z; \rho_1, \rho_2\right) = \frac{\left(\mu_1\right)_n}{\left(\mu_1+\mu_2\right)_n}\frac{\Gamma\left(\rho_1+\rho_2\right)}{\Gamma\left(\rho_1\right)}(x-y)^{1-\rho_1-\rho_2}\left[D_{x-y}^{-\rho_2}\left\{t^{\rho_1-1}J_{v,\gamma,\delta}^{\mu,q,p}(y+t)\right\}\right](x-y).$$

Proof: Using (18), we have

$$\begin{aligned}
S\left(\mu_1, \mu_2; z; \rho_1, \rho_2\right) = {}& \frac{\Gamma\left(\mu_1+\mu_2\right)}{\Gamma\left(\mu_1\right)\Gamma\left(\mu_2\right)}\frac{\Gamma\left(\rho_1+\rho_2\right)}{\Gamma\left(\rho_1\right)\Gamma\left(\rho_2\right)}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-1)^n}{\Gamma(\mu n+v+1)(\delta)_{pn}} \\
&\times \int_0^1\int_0^1 \left[\text{uv}(a-b-c+d)+u(b-d)+v(c-d)+d\right]^n \\
&\times u^{\mu_1-1}(1-u)^{\mu_2-1}\ du v^{\rho_1-1}(1-v)^{\rho_2-1}dv.
\end{aligned}$$

We assume that $a = x, \underline{=}y,\ c = d = 0$, then we get

$$S\left(\mu_1, \mu_2; z; \rho_1, \rho_2\right) = \frac{\Gamma\left(\mu_1+\mu_2\right)}{\Gamma\left(\mu_1\right)\Gamma\left(\mu_2\right)}\frac{\Gamma\left(\rho_1+\rho_2\right)}{\Gamma\left(\rho_1\right)\Gamma\left(\rho_2\right)}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-1)^n}{\Gamma(\mu n+v+1)(\delta)_{pn}}$$
$$\times\int_0^1\int_0^1 [\mathrm{uv}(x-y)+uy]^n u^{\mu_1-1}(1-u)^{\mu_2-1}v^{\rho_1-1}(1-v)^{\rho_2-1}du\ dv$$
$$S\left(\mu_1, \mu_2; z; \rho_1, \rho_2\right) = \frac{\Gamma\left(\mu_1+\mu_2\right)}{\Gamma\left(\mu_1\right)\Gamma\left(\mu_2\right)}\frac{\Gamma\left(\rho_1+\rho_2\right)}{\Gamma\left(\rho_1\right)\Gamma\left(\rho_2\right)}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-1)^n}{\Gamma(\mu n+v+1)(\delta)_{pn}}$$
$$\times\int_0^1\int_0^1 [vx+y(1-v)]^n u^{\mu_1+n-1}(1-u)^{\mu_2-1}v^{\rho_1-1}(1-v)^{\rho_2-1}du\ dv\ .$$

Now, using the definition of Beta and Gamma functions, we obtain

$$S\left(\mu_1, \mu_2; z; \rho_1, \rho_2\right) = \frac{\Gamma\left(\mu_1+\mu_2\right)}{\Gamma\left(\mu_1\right)\Gamma\left(\mu_2\right)}\frac{\Gamma\left(\rho_1+\rho_2\right)\Gamma\left(\mu_1+n\right)\Gamma(\mu_2)}{\Gamma\left(\rho_1\right)\Gamma\left(\rho_2\right)\Gamma(\mu_1+\mu_2+n)}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-1)^n}{\Gamma(\mu n+v+1)(\delta)_{pn}}$$
$$\times\int_0^1 [v(x-y)+y]^n\, v^{\rho_1-1}(1-v)^{\rho_2-1}dv.$$

On putting $v(x-y) = t$ in above equation,then we get

$$S\left(\mu_1, \mu_2; z; \rho_1, \rho_2\right) = \frac{\Gamma\left(\rho_1+\rho_2\right)\left(\mu_1\right)_n}{\Gamma\left(\rho_1\right)\Gamma\left(\rho_2\right)\left(\mu_1+\mu_2\right)_n}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-1)^n}{\Gamma(\mu n+v+1)(\delta)_{pn}}$$
$$\times\int_0^{x-y}(y+t)^n\left(\frac{t}{x-y}\right)^{\rho_1-1}\left(1-\frac{t}{x-y}\right)^{\rho_2-1}\frac{dt}{x-y}.$$

This yields the assertion of Theorem 3.2 after applying the definition of fractional derivative from (17).

Similarly, a number of several new and known results of Theorem 3.2 can also be obtained.

4. CONCLUSION

We conclude that every analytic function can be averaged with respect to Dirichlet average and associated with fractional derivative. In this paper, we investigated the double Dirichlet average of generalized Bessel-Maitland function by using a fractional derivative. Also, we obtained several special cases with some new and known results.

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