Journal of Fractional Calculus and Applications
Vol. 14(2) July 2023, No. 13
ISSN: 2090-5858.
http://jfca.journals.ekb.eg/

FRACTIONAL CALCULUS OF THE EXTENDED BESSEL-WRIGHT FUNCTIONS AND ITS APPLICATIONS TO FRACTIONAL KINETIC EQUATIONS

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Abstract. In this article, authors introduced the new $(p,q;\vartheta)$-extended Bessel-Wright function $J^{\sigma;\varsigma,\lambda}_{\omega;p,q}(z;\vartheta)$, some properties related to Marichev-Saigo-Maede fractional integral and derivative operators, and Caputo-type Marichev-Saigo-Maede fractional integral and derivative operators which are applied to the $(p,q;\vartheta)$-extended Bessel-Wright function. Some special cases such as Saigo, Riemann-Liouville and Erdeyi-Kober fractional integrals and derivative operators are obtained. In addition, applications of the new $(p,q;\vartheta)$-extended Bessel-Wright function $J^{\sigma;\varsigma,\lambda}_{\omega;p,q}(z;\vartheta)$ to the fractional kinetic equations is also discussed.

1. Introduction

Friedrich Wilhelm Bessel (1784-1846) was the first to introduced the Bessel's function, and later it was studied by Euler, Lagrange, Bernoulli, and many others. Jankov Masirevic et al. [16] introduced (p,q)-extended Bessel function $J_{\omega;p,q}$, (p,q)-extended modified Bessel function, $I_{\omega;p,q}$ of the first kind of order ω, (p,q)-extended Bessel-Struve function $H_{\omega;p,q}$, and (p,q)-extended modified Bessel-Struve function $L_{\omega;p,q}$ with their properties like integral formulas, complete monotonicity, Mellin transform, etc. (p,q)-extended modified Bessel-Struve function $M_{\omega;p,q}$ of the second kind and (p,q)-extended modified Bessel-Struve function $S_{\omega;p,q}$ and their integral formulas, Mellin transform, Laguerre polynomial representations have been studied by Parmar et al. [39]. Some of the properties of the Bessel-type family of functions such as fractional integration, fractional differentiation and their applications have been studied by Parmar and Choi [38], Choi and Parmar [11] and Habenom et al., [14]. Wright [50] intoduced and investigated the following Bessel-Wright function

2010 *Mathematics Subject Classification.* 33B15, 33C10.
Key words and phrases. Beta function, Bessel-Struve function, Bessel-Wright function, Fox-Wright function, Hadamard product, Hypergeometric functions.
Submitted Oct. 23, 2022. Revised Jan. 1, 2023.

$J_{\omega}^{\sigma}(z)$:

$$J_{\omega}^{\sigma}(z)=\sum_{\eta=0}^{\infty}\frac{1}{\Gamma(\sigma\eta+\omega+1)}\frac{(-z)^{\eta}}{\eta!},$$

equivalently,

$$J_{\omega}^{\sigma}(z)=\frac{\sqrt{\pi}}{\Gamma\left(\omega+1\right)}\sum_{\eta=0}^{\infty}\frac{B\left(\sigma\eta+\frac{1}{2},\omega+\frac{1}{2}\right)}{B\left(\frac{1}{2},\omega+\frac{1}{2}\right)\Gamma\left(\sigma\eta+\frac{1}{2}\right)}\frac{(-z)^{\eta}}{\eta!}. \tag{1}$$

Where $z,\omega\in\mathbb{C}$ and $\sigma>0$.

Recently, Srivastava et al. [48] investigated about the fractional behavior of (p,q)-extended Bessel-Wright function $J_{\omega;p,q}^{\sigma}(z)$:

$$J_{\omega,p,q}^{\sigma}(z)=\frac{1}{\Gamma\left(\omega+\frac{1}{2}\right)}\sum_{\eta=0}^{\infty}\frac{B_{p,q}\left(\sigma\eta+\frac{1}{2},\omega+\frac{1}{2}\right)}{\Gamma\left(\sigma\eta+\frac{1}{2}\right)}\frac{(-z)^{\eta}}{\eta!},$$

or

$$J_{\omega,p,q}^{\sigma}(z)=\frac{\sqrt{\pi}}{\Gamma\left(\omega+1\right)}\sum_{\eta=0}^{\infty}\frac{B_{p,q}\left(\sigma\eta+\frac{1}{2},\omega+\frac{1}{2}\right)}{B\left(\frac{1}{2},\omega+\frac{1}{2}\right)\Gamma\left(\sigma\eta+\frac{1}{2}\right)}\frac{(-z)^{\eta}}{\eta!}. \tag{2}$$

Where $\omega,z\in\mathbb{C}$, $\sigma>0$, $min\{Re(p),Re(q)\}>0$, $Re(\omega)>-1$, when $p=q=1$, and $B_{p,q}(\partial_1,\partial_2)$ is the extended beta function defined by Choi et al. [10] as:

$$B_{p,q}(\partial_1,\partial_2)=\int_0^1 t^{\partial_1-1}(1-t)^{\partial_2-1}e^{-\frac{p}{t}-\frac{q}{1-t}}\,dt$$

where $min\{Re(\partial_1),Re(\partial_2)\}>0$, $min\{Re(p),Re(q)\}>0$.

The well-known Riemann-Liouville fractional integral and derivative operators are shown in Kiryakova [25] and Yang et al. [51] as follows:

$$\left(I_{0^+}^{\gamma}f\right)(x)=\frac{1}{\Gamma(\gamma)}\int_0^x (x-t)^{\gamma-1}f(t)dt \tag{3}$$

$$\left(I_{-}^{\gamma}f\right)(x)=\frac{1}{\Gamma(\gamma)}\int_x^{\infty} (t-x)^{\gamma-1}f(t)dt \tag{4}$$

$$\left(D_{0^+}^{\gamma}f\right)(x)=\left(\frac{d}{dx}\right)^{\eta}\left(I_{0^+}^{\eta-\gamma}f\right)(x) \tag{5}$$

and

$$\left(D_{-}^{\gamma}f\right)(x)=\left(-\frac{d}{dx}\right)^{\eta}\left(I_{-}^{\eta-\gamma}f\right)(x). \tag{6}$$

where $\gamma\in\mathbb{C}$, $Re(\gamma)>0$, $\eta=1+[(\gamma)]$, and $x\in\mathbb{R}^+$.

The Erdelyi-Kober fractional integral and derivative operators are given in Kilbas et al. [20], Samko et al. [45] and Kiryakova [21, 22]:

$$\left(I_{\kappa,\gamma}^{+}f\right)(x)=\frac{x^{-\kappa-\gamma}}{\Gamma(\gamma)}\int_0^x t^{\kappa}(x-t)^{\gamma-1}f(t)dt \tag{7}$$

$$\left(K_{-}^{-}f\right)(x)=\frac{x^{\kappa}}{\Gamma(\gamma)}\int_x^{\infty} t^{-\kappa-\gamma}(t-x)^{\gamma-1}f(t)dt \tag{8}$$

$$\left(D_{\kappa,\gamma}^{+}f\right)(x)=x^{-\kappa}\left(\frac{d}{dx}\right)^{\eta}\frac{1}{\Gamma(\eta-\gamma)}\int_0^x (x-t)^{\eta-\gamma-1}t^{\kappa+\gamma}f(t)dt \tag{9}$$

and

$$\left(D_{\kappa,\gamma}^{-}f\right)(x)=x^{\kappa+\gamma}\left(-\frac{d}{dx}\right)^{\eta}\frac{1}{\Gamma(\eta-\gamma)}\int_x^{\infty} (t-x)^{\eta-\gamma-1}t^{-\gamma}f(t)dt. \tag{10}$$

where $\gamma \in \mathbb{C}$, $Re(\gamma) > 0$, $\eta = 1 + [(\gamma)]$, and $x \in \mathbb{R}^+$.

Saigo [42, 43] presented the following fraction integral and derivative operator with the Gauss hypergeometric function as their kernel (see also Kiryakova [27]):

$$\left(I_{0^+}^{\gamma,\delta,\kappa} f\right)(x) = \frac{x^{-\gamma-\delta}}{\Gamma(\kappa)} \int_0^x (x-t)^{\gamma-1} f(t) {}_2F_1\left(\gamma+\delta, -\kappa, \gamma; 1 - \frac{t}{x}\right) dt \tag{11}$$

$$\left(I_{-}^{\gamma,\delta,\kappa} f\right)(x) = \frac{1}{\Gamma(\kappa)} \int_x^\infty (t-x)^{\gamma-1} t^{-\gamma-\delta} f(t) {}_2F_1\left(\gamma+\delta, -\kappa, \gamma; 1 - \frac{x}{t}\right) dt \tag{12}$$

$$\left(D_{0^+}^{\gamma,\delta,\kappa} f\right)(x) = \left(I_{0^+}^{-\gamma,-\delta,\gamma+\kappa} f\right)(x) \tag{13}$$

and

$$\left(D_{-}^{\gamma,\delta,\kappa} f\right)(x) = \left(I_{-}^{-\gamma,-\delta,\gamma+\kappa} f\right)(x). \tag{14}$$

where $\gamma, \delta, \kappa \in \mathbb{C}$, $Re(\kappa) > 0$, $x \in \mathbb{R}^+$, and ${}_2F_1(;)$ represent the Gauss hypergeometric function defined by Mathai et al. [29] as:

$${}_2F_1(\gamma, \delta; \kappa; x) = \sum_{\eta=0}^{\infty} \frac{(\gamma)_\eta (\delta)_\eta}{(\kappa)_\eta} \frac{x^\eta}{\eta!},$$

where $|z| < 1$, $\gamma, \delta, \kappa \in \mathbb{C}$.

Marichev [31] introduced the following generalized fractional calculus operators related to the Appell function:

$$\left(I_{0^+}^{\gamma,\gamma',\delta,\delta',\kappa} f\right)(x) = \frac{x^{-\gamma}}{\Gamma(\kappa)} \int_0^x (x-t)^{\kappa-1} t^{-\gamma'} f(t) F_3\left(\gamma, \gamma', \delta, \delta'; \kappa; 1 - \frac{t}{x}, 1 - \frac{x}{t}\right) dt \tag{15}$$

$$\left(I_{-}^{\gamma,\gamma',\delta,\delta',\kappa} f\right)(x) = \frac{x^{-\gamma'}}{\Gamma(\kappa)} \int_x^\infty (t-x)^{\kappa-1} t^{-\gamma} f(t) F_3\left(\gamma, \gamma', \delta, \delta'; \kappa; 1 - \frac{t}{x}, 1 - \frac{x}{t}\right) dt \tag{16}$$

$$\left(D_{0^+}^{\gamma,\gamma',\delta,\delta',\kappa} f\right)(x) = \left(I_{0^+}^{-\gamma,-\gamma',-\delta,-\delta',-\kappa} f\right)(x) \tag{17}$$

and

$$\left(D_{-}^{\gamma,\gamma',\delta,\delta',\kappa} f\right)(x) = \left(I_{-}^{-\gamma,-\gamma',-\delta,-\delta',-\kappa} f\right)(x). \tag{18}$$

where $\gamma \in \mathbb{C}$, $Re(\gamma) > 0$, $\eta = 1 + [(\gamma)]$, and $x \in \mathbb{R}^+$; and $F_3(;)$ represents the third Appell function (Horn's function) [46], which is stated as follows:

$$F_3(\gamma, \gamma', \delta, \delta'; \kappa; x, y) = \sum_{\eta,\varpi=0}^{\infty} \frac{(\gamma)_\eta (\gamma')_\varpi (\delta)_\eta (\delta')_\varpi}{(\kappa)_{\eta+\varpi}} \frac{x^\eta}{\eta!}, \ max\{|x|, |y|\} < 1,$$

where $\gamma, \delta, \kappa \in \mathbb{C}$, $Re(\kappa) > 0$, $x \in \mathbb{R}^+$.

Equations (15)-(18) were studied by Saigo and Maeda [44] and these fractional operators are called the Marichev-Saigo-Maeda operators (M-S-M operators).

In the present article, the following new $(p, q; \vartheta)$-extended Bessel-Wright function $J_{\omega;p,q}^{\sigma;\varsigma,\lambda}(z;\vartheta)$ will be studied, and some of its properties related to fractional calculus and its applications to fractional kinetic equations is also discussed:

$$J_{\omega;p,q}^{\sigma;\varsigma,\lambda}(z;\vartheta) = \frac{1}{\Gamma\left(\omega + \frac{1}{2}\right)} \sum_{\eta=0}^{\infty} \frac{B_{p,q}^{\varsigma,\lambda}\left(\sigma\eta + \frac{1}{2}, \omega + \frac{1}{2}; \vartheta\right)}{\Gamma\left(\sigma\eta + \frac{1}{2}\right)} \frac{(-z)^\eta}{\eta!},$$

or

$$J_{\omega,p,q}^{\sigma\varsigma,\lambda}(z;\vartheta) = \frac{\sqrt{\pi}}{\Gamma(\omega+1)} \sum_{\eta=0}^{\infty} \frac{B_{p,q}^{\varsigma,\lambda}\left(\sigma\eta + \frac{1}{2}, \omega + \frac{1}{2}; \vartheta\right)}{B\left(\frac{1}{2}, \omega + \frac{1}{2}\right)\Gamma\left(\sigma\eta + \frac{1}{2}\right)} \frac{(-z)^\eta}{\eta!}, \tag{19}$$

where $\omega, z \in \mathbb{C}$, $\sigma > 0$, $min\{Re(p), Re(q)\} > 0$, $min\{Re(\varsigma), Re(\lambda)\} > 0$, $\vartheta \in (0,\infty)\backslash\{1\}$, $Re(\omega) > -1$, when $p = q = 1$, and $B^{\varsigma,\lambda}_{p,q}(\partial_1, \partial_2; \vartheta)$ is the extended beta function defined by Abubakar et al. [1]:

$$B^{\varsigma,\lambda}_{p,q}(\partial_1, \partial_2; \vartheta) = \int_0^1 t^{\partial_1 - 1}(1-t)^{\partial_2 - 1}\vartheta^{-\frac{p}{t^{\varsigma}} - \frac{q}{(1-t)^{\lambda}}}\, dt,$$

where $min\{Re(\partial_1), Re(\partial_2)\} > 0$, $min\{Re(p), Re(q)\} > 0$, $min\{Re(\varsigma), Re(\lambda)\} > 0$, $\vartheta \in (0,\infty)\backslash\{1\}$.

Definition 1: The $(p, q; \vartheta)$-extended σ-Gauss hypergeoemtric function is defined in Abubakar [3]

$$R^{\sigma;\varsigma,\lambda}_{p,q}(a, b; c; z; \vartheta) = \sum_{\eta=0}^{\infty}(a)_\eta \frac{B^{\varsigma,\lambda}_{p,q}(b + \eta\sigma, c - b; \vartheta)}{B(b, c - b)}\frac{z^\eta}{\eta!}, \tag{20}$$

where $min\{Re(p), Re(q)\} > 0$, $min\{Re(\varsigma), Re(\lambda)\} > 0$, $\sigma \geq 0$, $\vartheta \in (0,\infty)\backslash\{1\}$, $Re(a) > 0$, $Re(c) > Re(b) > 0$.

Definition 2: The generalized Wright function is defined in Kiryakova [21, 24] as

$${}_p\Psi_q\left[\begin{array}{c|c} (\Upsilon_i, y_i)_{1,p} & \\ & z \\ (H_j, h_j)_{1,q} & \end{array}\right] .= \sum_{\eta=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(\Upsilon_i + \eta y_i)}{\prod_{j=1}^{q}\Gamma(H_j + \eta h_j)}\frac{z^\eta}{\eta!}, \tag{21}$$

where the coefficient $y_i, h_i \in \mathbb{R}^+, i \in \mathbb{N}$ with $1 + \sum_{i=1}^{q} h_i - \sum_{j=1}^{p} y_j \geq 0$.

Definition 3: The Hadamard product for the power series $f(z) = \sum_{\eta=1}^{\infty} a_\eta z^\eta$ and $g(z) = \sum_{\eta=1}^{\infty} b_\eta z^\eta$ is defined in Pohlen [40], Nadir and Khan [34] and also Hubenom et al. [14] as:

$$(f * g)(z) = \sum_{\eta=1}^{\infty} a_\eta b_\eta z^\eta = (f \cdot g)(z) \tag{22}$$

where $|z| < R$, and is defined as

$$R = \lim_{\eta\to\infty}\left|\frac{a_\eta b_\eta}{a_{\eta+1}b_{\eta+1}}\right| = \left(\lim_{\eta\to\infty}\left|\frac{a_\eta}{a_{\eta+1}}\right|\right)\left(\lim_{\eta\to\infty}\left|\frac{b_\eta}{b_{\eta+1}}\right|\right) = R_f R_g.$$

where R_f and R_g are the convergence radii of $f(z)$ and $g(z)$ respectively, and $R \geq R_f \cdot R_g$.

2. Marichev-Saigo-Maeda fractional integral of $J^{\sigma;\varsigma,\lambda}_{\omega;p,q}(z;\vartheta)$

This section covers the left- and Right-sided Marichev-Saigo-Maeda fractional integral operators of the new $(p, q; \vartheta)$-extended Bessel-Wright function $J^{\sigma;\varsigma,\lambda}_{\omega;p,q}(z;\vartheta)$.

2.1. Left-sided Marichev-Saigo-Maeda fractional integral of $J^{\sigma;\varsigma,\lambda}_{\omega;p,q}(z;\vartheta)$.

First of all we have gothrough following results, which are recorded in Marichev [31], Saigo and Maeda [44], see also Manzoor et al. [30] and Kilbas and Sebastian [19]:

Lemma 4: If $\gamma, \gamma', \delta, \delta', \kappa \in \mathbb{C}$ $x > 0$, such that $Re(\kappa) > 0$, $Re(\varrho) > max\{0, Re(\gamma - \gamma' - \delta - \kappa), Re(\gamma' - \delta')\}$, then

$$\left(I^{\gamma,\gamma',\delta,\delta',\kappa}_{0^+} t^{\varrho-1}\right)(x) \tag{23}$$

$$= x^{\varrho-\gamma-\gamma'+\kappa-1}\frac{\Gamma(\varrho)\Gamma(\varrho + \kappa - \gamma - \gamma' - \delta)\Gamma(\varrho - \delta' - \gamma')}{\Gamma(\varrho + \delta')\Gamma(\varrho + \kappa - \gamma - \gamma')\Gamma(\varrho + \kappa - \gamma' - \delta)} \tag{24}$$

In particular,

$$\left(I_{0^+}^{\gamma,\delta,\kappa} t^{\varrho-1}\right)(x) = x^{\varrho-\delta-1}\frac{\Gamma(\varrho)\Gamma(\varrho+\kappa-\delta)}{\Gamma(\varrho-\delta)\Gamma(\varrho+\gamma+\kappa)} \tag{25}$$

where $Re(\gamma) > 0$, $Re(\varrho) > max\{0, Re(\delta-\kappa)$.

$$\left(I_{0^+}^{\gamma} t^{\varrho-1}\right)(x) = x^{\varrho+\gamma-1}\frac{\Gamma(\varrho)}{\Gamma(\varrho+\gamma)} \tag{26}$$

where $min\{Re(\gamma), Re(\varrho)\} > 0$.
and

$$\left(I_{\kappa,\gamma}^{+} t^{\varrho-1}\right)(x) = x^{\varrho-1}\frac{\Gamma(\varrho+\kappa)}{\Gamma(\varrho+\gamma)} \tag{27}$$

Theorem 5: The following equality hold true:

$$\begin{aligned}
&\left(I_{0^+}^{\gamma,\gamma',\delta,\delta',\kappa}\left[t^{\varrho-1}J_{\omega;p,q}^{\sigma;\varsigma,\lambda}(\tau t;\vartheta)\right]\right)(x)\\
&= \frac{\sqrt{\pi}}{\Gamma(\omega+1)}x^{\varrho-\gamma-\gamma'+\kappa-1}R_{p,q}^{\sigma;\varsigma,\lambda}\left(1,\frac{1}{2};\omega+1;-\tau x;\vartheta\right)\\
&* {}_3\Psi_4\left[\begin{matrix}(\varrho,1),(\varrho+\kappa-\gamma-\gamma'-\delta,1),(\varrho+\delta'-\gamma',1)\\ \left(\frac{1}{2},\sigma\right),(\varrho+\delta',1),(\varrho+\kappa-\gamma-\gamma',1),(\varrho+\kappa-\gamma'-\delta,1)\end{matrix}\middle|-\tau x\right],
\end{aligned} \tag{28}$$

where $Re(\varrho) > max\{0, Re(\gamma-\gamma'-\delta-\kappa), Re(\gamma'-\delta')\}$, $min\{Re(p), Re(q)\} > 0$, $min\{Re(\varsigma), Re(\lambda)\} > 0$, $\vartheta \in (0,\infty)\backslash\{1\}$, $Re(\omega) > -1$, when $p = q = 1$, $Re(\kappa) > 0$, $Re(\sigma) > 0$.

Proof: With the help of the equation (19), we have:

$$\begin{aligned}
&\left(I_{0^+}^{\gamma,\gamma',\delta,\delta',\kappa}\left[t^{\varrho-1}J_{\omega;p,q}^{\sigma;\varsigma,\lambda}(\tau t;\vartheta)\right]\right)(x)\\
&= \left(I_{0^+}^{\gamma,\gamma',\delta,\delta',\kappa}\left[t^{\varrho-1}\left\{\frac{\sqrt{\pi}}{\Gamma(\omega+1)}\sum_{\eta=0}^{\infty}\frac{B_{p,q}^{\varsigma,\lambda}\left(\sigma\eta+\frac{1}{2},\omega+\frac{1}{2};\vartheta\right)}{B\left(\frac{1}{2},\omega+\frac{1}{2}\right)\Gamma\left(\sigma\eta+\frac{1}{2}\right)}\frac{(-\tau t)^\eta}{\eta!}\right\}\right]\right)(x).
\end{aligned} \tag{29}$$

by changing the order of summation and integral operator, we have:

$$\begin{aligned}
&\left(I_{0^+}^{\gamma,\gamma',\delta,\delta',\kappa}\left[t^{\varrho-1}J_{\omega;p,q}^{\sigma;\varsigma,\lambda}(\tau t;\vartheta)\right]\right)(x)\\
&= \frac{\sqrt{\pi}}{\Gamma(\omega+1)}\sum_{\eta=0}^{\infty}\frac{B_{p,q}^{\varsigma,\lambda}\left(\sigma\eta+\frac{1}{2},\omega+\frac{1}{2};\vartheta\right)}{B\left(\frac{1}{2},\omega+\frac{1}{2}\right)\Gamma\left(\sigma\eta+\frac{1}{2}\right)}\frac{(-\tau)^\eta}{\eta!}\left(I_{0^+}^{\gamma,\gamma',\delta,\delta',\kappa}\left[t^{\varrho+\eta-1}\right]\right)(x).
\end{aligned} \tag{30}$$

Now applying (23) into above equations and after little algebra, we obtain:

$$\begin{aligned}
&\left(I_{0^+}^{\gamma,\gamma',\delta,\delta',\kappa}\left[t^{\varrho-1}J_{\omega;p,q}^{\sigma;\varsigma,\lambda}(\tau t;\vartheta)\right]\right)(x)\\
&= \frac{\sqrt{\pi}}{\Gamma(\omega+1)}x^{\varrho-\gamma-\gamma'+\kappa-1}\sum_{\eta=0}^{\infty}(1)_\eta\frac{B_{p,q}^{\sigma;\varsigma,\lambda}\left(\frac{1}{2}+\eta\sigma,\frac{1}{2}+\omega;\vartheta\right)}{B\left(\frac{1}{2},\frac{1}{2}+\omega\right)\eta!}\\
&\times\frac{\Gamma(\varrho+\eta)\Gamma(\varrho+\kappa-\gamma-\gamma'-\delta+\eta)\Gamma(\varrho+\delta'-\gamma+\eta)}{\Gamma\left(\sigma\eta+\frac{1}{2}\right)\Gamma(\varrho+\delta'+\eta)\Gamma(\varrho+\kappa-\gamma'-\delta+\eta)\eta!}(-\tau x)^\eta.
\end{aligned}$$

With the help of $(p,q;\vartheta)$-extended σ-Gauss hypergeometric function (20), the generalized Wright function (21), the Hadamard product (convolution) (22), we obtain required the result (28). The proof of Theorem **??** is complete.
Furthermore, we noticed that the following consequences arise from using equations

(25)-(27) and (20)-(22).

Corollary 6: The following result holds:

$$\left(I_{0^+}^{\gamma,\delta,\kappa}\left[t^{\varrho-1}J_{\omega;p,q}^{\sigma;\varsigma,\lambda}(\tau t;\vartheta)\right]\right)(x)=\frac{\sqrt{\pi}}{\Gamma(\omega+1)}x^{\varrho-\delta-1}R_{p,q}^{\sigma;\varsigma,\lambda}\left(1,\frac{1}{2};\omega+1;-\tau x;\vartheta\right)$$
$$*\,{}_2\Psi_3\left[\begin{array}{c}(\varrho,1),(\varrho+\kappa-\delta,1)\\ \left(\frac{1}{2},\sigma\right),(\varrho-\delta',1),(\varrho+\kappa+\gamma,1)\end{array}\middle|-\tau x\right],$$

where $Re(\gamma)>0$, $Re(\varrho)>max\{0,Re(\delta-\kappa)\}$, $\vartheta\in(0,\infty)\backslash\{1\}$, $min\{Re(p),Re(q)\}>0$, $min\{Re(\varsigma),Re(\lambda)\}>0$, $Re(\omega)>-1$, when $p=q=1$, $Re(\kappa)>0$, $Re(\sigma)>0$.

Corollary 7: The following equation is true:

$$\left(I_{0^+}^{\gamma}\left[t^{\varrho-1}J_{\omega;p,q}^{\sigma;\varsigma,\lambda}(\tau t;\vartheta)\right]\right)(x)=\frac{\sqrt{\pi}}{\Gamma(\omega+1)}x^{\varrho-\delta-1}R_{p,q}^{\sigma;\varsigma,\lambda}\left(1,\frac{1}{2};\omega+1;-\tau x;\vartheta\right)$$
$$*\,{}_1\Psi_2\left[\begin{array}{c}(\varrho,1)\\ \left(\frac{1}{2},\sigma\right),(\varrho+\gamma,1)\end{array}\middle|-\tau x\right],$$

where $min\{Re(\gamma),Re(\varrho)\}>0$, $min\{Re(p),Re(q)\}>0$, $min\{Re(\varsigma),Re(\lambda)\}>0$, $\vartheta\in(0,\infty)\backslash\{1\}$, $Re(\omega)>-1$, when $p=q=1$, $Re(\kappa)>0$, $Re(\sigma)>0$.

Corollary 8: The following equality is valid:

$$\left(I_{\gamma,\kappa}^{+}\left[t^{\varrho-1}J_{\omega;p,q}^{\sigma;\varsigma,\lambda}(\tau t;\vartheta)\right]\right)(x)=\frac{\sqrt{\pi}}{\Gamma(\omega+1)}x^{\varrho-1}R_{p,q}^{\sigma;\varsigma,\lambda}\left(1,\frac{1}{2};\omega+1;-\tau x;\vartheta\right)$$
$$*\,{}_1\Psi_2\left[\begin{array}{c}(\varrho+\kappa,1)\\ \left(\frac{1}{2},\sigma\right),(\varrho+\gamma,1)\end{array}\middle|-\tau x\right],$$

where $Re(\gamma)>0$, $Re(\varrho)>-Re(\kappa)$, $min\{Re(p),Re(q)\}>0$, $min\{Re(\varsigma),Re(\lambda)\}>0$, $\vartheta\in(0,\infty)\backslash\{1\}$, $Re(\omega)>-1$, when $p=q=1$, $Re(\kappa)>0$, $Re(\sigma)>0$.

2.2. **Right-sided Marichev-Saigo-Maeda fractional integral of $J_{\omega;p,q}^{\sigma;\varsigma,\lambda}(z;\vartheta)$.** We have to go through the following results, which are recorded by Katarian and Velaisamy [35], and Nisar [18]:

Lemma 9: If $\gamma,\gamma',\delta,\delta',\kappa\in\mathbb{C}$, $x>0$, and there exist the following relations $Re(\kappa)>0$, $Re(\varrho)<1+min\{Re(-\delta),Re(\gamma+\gamma'-\kappa),Re(\gamma'+\delta'-\kappa)\}$, then we have:

$$\left(I_{-}^{\gamma,\gamma',\delta,\delta',\kappa}t^{\varrho-1}\right)(x)$$
$$=x^{\varrho-\gamma-\gamma'+\kappa-1}\frac{\Gamma(1-\varrho-\delta)\Gamma(1-\varrho-\kappa+\gamma+\gamma')\Gamma(1-\varrho+\gamma+\delta'-\kappa)}{\Gamma(1-\varrho)\Gamma(1-\varrho+\gamma+\gamma'+\delta'-\kappa)\Gamma(1-\varrho+\gamma'-\delta)}\tag{31}$$

In particular,

$$\left(I_{-}^{\gamma,\delta,\kappa}t^{\varrho-1}\right)(x)=x^{\varrho-\delta-1}\frac{\Gamma(1-\varrho+\delta)\Gamma(1-\varrho+\kappa)}{\Gamma(1-\varrho)\Gamma(1-\varrho+\gamma+\delta+\kappa)}\tag{32}$$

where $Re(\gamma)>0$, $Re(\varrho)>1+min\{Re(\delta),Re(\kappa)$,

$$\left(I_{-}^{\gamma}t^{\varrho-1}\right)=x^{\varrho-1}\frac{\Gamma(1-\varrho-\gamma)}{\Gamma(1-\varrho)}\tag{33}$$

where $0 < (\gamma) < 1 - Re(\varrho) > 0$, $x > 0$ and

$$\left(K^{-}_{\kappa,\gamma} t^{\varrho-1}\right) = x^{\varrho-1} \frac{\Gamma(1-\varrho+\kappa)}{\Gamma(1-\varrho+\gamma+\kappa)} \tag{34}$$

where $Re(\gamma) > 0$, $Re(\varrho) > -Re(\kappa)$.

Theorem 10: The following result is true:

$$\begin{aligned}
&\left(I_{-}^{\gamma,\gamma',\delta,\delta',\kappa}\left[t^{\varrho-1} J^{\sigma;\varsigma,\lambda}_{\omega;p,q}(\tau t^{-1};\vartheta)\right]\right)(x) \\
&= \frac{\sqrt{\pi}}{\Gamma(\omega+1)} x^{\varrho-\gamma-\gamma'+\kappa-1} R^{\sigma;\varsigma,\lambda}_{p,q}\left(1, \frac{1}{2}; \omega+1; -\tau x; \vartheta\right) \\
&* {}_3\Psi_4 \left[\begin{matrix} (1-\varrho-\delta,1),(1-\varrho-\kappa+\gamma-\gamma',1),(1-\varrho+\gamma+\delta'-\kappa,1) \\ \left(\frac{1}{2},\sigma\right),(1-\varrho,1),(1-\varrho+\gamma+\gamma'-\kappa,1),(1-\varrho+\gamma-\delta,1) \end{matrix} \middle| -\tau x^{-1} \right]
\end{aligned} \tag{35}$$

where $Re(\varrho) < 1 + min\{Re(-\delta), Re(\gamma+\gamma'-\kappa), Re(\gamma+\delta'-\kappa)\}$, $min\{Re(\varsigma), Re(\lambda)\} > 0$,$min\{Re(p), Re(q)\} > 0$, $\vartheta \in (0,\infty)\backslash\{1\}$, $Re(\omega) > -1$, when $p = q = 1$, $Re(\kappa) > 0$, $Re(\sigma) > 0$.

Proof: The proof is easy, and the required result (35) can be obtain with the help of (28), (31) and (19). the desired preferred end result in (35) is obtained.

Further by considering equations (32)-(34) and (19) the following results are obtained.

Corollary 11: The following result holds:

$$\begin{aligned}
&\left(I_{0^+}^{\gamma,\delta,\kappa}\left[t^{\varrho-1} J^{\sigma;\varsigma,\lambda}_{\omega;p,q}(\tau t^{-1};\vartheta)\right]\right)(x) \\
&= \frac{\sqrt{\pi}}{\Gamma(\omega+1)} x^{\varrho-\delta-1} R^{\sigma;\varsigma,\lambda}_{p,q}\left(1, \frac{1}{2}; \omega+1; -x\tau; \vartheta\right) \\
&* {}_2\Psi_3 \left[\begin{matrix} (1-\varrho+\delta,1),(1-\varrho+\kappa,1) \\ \left(\frac{1}{2},\sigma\right),(1-\varrho,1),(1-\varrho+\gamma+\kappa,1) \end{matrix} \middle| \tau x^{-1} \right],
\end{aligned}$$

where $Re(\varrho) > max\{Re(\delta), Re(\kappa)\}$, $min\{Re(p), Re(q)\} > 0$, $min\{Re(\varsigma), Re(\lambda)\} > 0$, $\vartheta \in (0,\infty)\backslash\{1\}$, $Re(\omega) > -1$, $Re(\gamma) > 0$, when $p = q = 1$, $Re(\kappa) > 0$, $Re(\sigma) > 0$.

Corollary 12: The following equation is true:

$$\begin{aligned}
\left(I_{-}^{\gamma}\left[t^{\varrho-1} J^{\sigma;\varsigma,\lambda}_{\omega;p,q}(\tau t^{-1};\vartheta)\right]\right)(x) =& \frac{\sqrt{\pi}}{\Gamma(\omega+1)} x^{\varrho+\gamma-1} R^{\sigma;\varsigma,\lambda}_{p,q}\left(1, \frac{1}{2}; \omega+1; -\tau x^{-1}; \vartheta\right) \\
&* {}_1\Psi_2 \left[\begin{matrix} (1-\varrho-\gamma,1) \\ \left(\frac{1}{2},\sigma\right),(1-\varrho,1) \end{matrix} \middle| \tau x^{-1} \right],
\end{aligned}$$

where $0 < Re(\gamma), 1 - Re(\varrho) > 0$, $min\{Re(p), Re(q)\} > 0$, $min\{Re(\varsigma), Re(\lambda)\} > 0$, $\vartheta \in (0,\infty)\backslash\{1\}$, $Re(\omega) > -1$, when $p = q = 1$, $Re(\kappa) > 0$, $Re(\sigma) > 0$.

Corollary 13: The following equality is valid:

$$\begin{aligned}
\left(K^{-}_{\gamma,\kappa}\left[t^{\varrho-1} J^{\sigma;\varsigma,\lambda}_{\omega;p,q}(\tau t^{-1};\vartheta)\right]\right)(x) =& \frac{\sqrt{\pi}}{\Gamma(\omega+1)} x^{\varrho-1} R^{\sigma;\varsigma,\lambda}_{p,q}\left(1, \frac{1}{2}; \omega+1; -x\tau; \vartheta\right) \\
&* {}_1\Psi_2 \left[\begin{matrix} (1-\varrho+\kappa,1) \\ \left(\frac{1}{2},\sigma\right),(1-\varrho+\gamma+\kappa,1) \end{matrix} \middle| \tau x^{-1} \right],
\end{aligned}$$

where $Re(\gamma) > 1 + Re(\kappa)$, $min\{Re(p), Re(q)\} > 0$, $min\{Re(\varsigma), Re(\lambda)\} > 0$, $\vartheta \in (0,\infty)\backslash\{1\}$, $Re(\omega) > -1$, when $p = q = 1$, $Re(\kappa) > 0$, $Re(\sigma) > 0$.

3. Marichev-Saigo-Maeda fractional derivative of $J^{\sigma;\varsigma,\lambda}_{\omega;p,q}(z;\vartheta)$

This section consist of the left-sided and right sided, Marichev-Saigo-Maeda fractional derivative operators of the new $(p, q; \vartheta)$-extended Bessel-Wright function $J^{\sigma;\varsigma,\lambda}_{\omega;p,q}(z;\vartheta)$.

3.1. Left-sided Marichev-Saigo-Maeda fractional derivative of $J^{\sigma;\varsigma,\lambda}_{\omega;p,q}(z;\vartheta)$.

The following lemmas are recorded by Marichev [31] and Saigo and Maeda [44], see also Abubakar in [2]

Lemma 14: If $\gamma, \gamma', \delta, \delta', \kappa \in \mathbb{C}$, $x > 0$; $Re(\kappa) > 0$, $Re(\varrho) > max\{0, Re(\kappa - \gamma - \gamma' - \delta), Re(\delta - \gamma)\}$, then we have:

$$\left(D^{\gamma,\gamma',\delta,\delta',\kappa}_{0^+} t^{\varrho-1}\right)(x) = x^{\varrho+\gamma+\gamma'-\kappa-1} \frac{\Gamma(\varrho)\Gamma(\varrho-\kappa+\gamma+\gamma'+\delta')\Gamma(\varrho-\delta+\gamma)}{\Gamma(\varrho-\delta)\Gamma(\varrho-\kappa+\gamma+\gamma')\Gamma(\varrho-\kappa+\gamma+\delta')} \tag{36}$$

In particular,

$$\left(D^{\gamma,\delta,\kappa}_{0^+} t^{\varrho-1}\right)(x) = x^{\varrho+\delta-1} \frac{\Gamma(\varrho)\Gamma(\varrho+\kappa+\gamma+\delta)}{\Gamma(\varrho+\delta)\Gamma(\varrho+\kappa)}. \tag{37}$$

where $Re(\gamma) > 0$, $Re(\varrho) > -min\{0, Re(\gamma+\delta+\kappa)$.

$$\left(D^{\gamma}_{0^+} t^{\varrho-1}\right)(x) = x^{\varrho-\gamma-1} \frac{\Gamma(\varrho)}{\Gamma(\varrho+\gamma)} \tag{38}$$

where $min\{Re(\gamma), Re(\varrho)\} > 0$, and

$$\left(D^{+}_{\kappa,\gamma} t^{\varrho-1}\right)(x) = x^{\varrho-1} \frac{\Gamma(\varrho+\gamma+\kappa)}{\Gamma(\varrho-\gamma)} \tag{39}$$

where $Re(\gamma) > 0$, $Re(\varrho) > -Re(\gamma+\kappa)$.

Theorem 15: The following results holds true:

$$\begin{aligned} &\left(D^{\gamma,\gamma',\delta,\delta',\kappa}_{0^+}\left[t^{\varrho-1} J^{\sigma;\varsigma,\lambda}_{\omega;p,q}(\tau t;\vartheta)\right]\right)(x) \\ &= \frac{\sqrt{\pi}}{\Gamma(\omega+1)} x^{\varrho-\gamma-\gamma'+\kappa-1} R^{\sigma;\varsigma,\lambda}_{p,q}\left(1, \frac{1}{2}; \omega+1; -x\tau; \vartheta\right) \\ &* {}_3\Psi_4 \left[\begin{matrix} (\varrho,1),(\varrho+\gamma+\gamma'+\delta'-\kappa,1),(\varrho+\gamma-\delta,1) \\ \left(\frac{1}{2},\sigma\right),(\varrho-\delta,1),(\varrho-\kappa+\gamma+\gamma',1),(\varrho-\kappa+\gamma+\delta'-,1) \end{matrix} \middle| -\tau x \right], \end{aligned} \tag{40}$$

where $Re(\kappa) > 0$, $Re(\varrho) > max\{0, Re(\kappa-\gamma-\gamma'-\delta), Re(\delta-\gamma)\}$, $min\{Re(p), Re(q)\} > 0$, $min\{Re(\varsigma), Re(\lambda)\} > 0$, $\vartheta \in (0,\infty)\backslash\{1\}$, $Re(\omega) > -1$, when $p = q = 1$, $Re(\kappa) > 0$, $Re(\sigma) > 0$.

Proof: Using equation (19), we have

$$\begin{aligned} &\left(D^{\gamma,\gamma',\delta,\delta',\kappa}_{0^+}\left[t^{\varrho-1} J^{\sigma;\varsigma,\lambda}_{\omega;p,q}(\tau t;\vartheta)\right]\right)(x) \\ &= \left(D^{\gamma,\gamma',\delta,\delta',\kappa}_{0^+}\left[t^{\varrho-1}\left\{\frac{\sqrt{\pi}}{\Gamma(\omega+1)} \sum_{\eta=0}^{\infty} \frac{B^{\varsigma,\lambda}_{p,q}\left(\sigma\eta+\frac{1}{2},\omega+\frac{1}{2};\vartheta\right)}{B\left(\frac{1}{2},\omega+\frac{1}{2}\right)\Gamma\left(\sigma\eta+\frac{1}{2}\right)} \frac{(-\tau t)^{\eta}}{\eta!}\right\}\right]\right)(x) \end{aligned} \tag{41}$$

by changing the order of summation and integral operator, we have:

$$\begin{aligned}&\left(D_{0^+}^{\gamma,\gamma',\delta,\delta',\kappa}\left[t^{\varrho-1}J_{\omega;p,q}^{\sigma;\varsigma,\lambda}(\tau t;\vartheta)\right]\right)(x)\\&=\frac{\sqrt{\pi}}{\Gamma(\omega+1)}\sum_{\eta=0}^{\infty}\frac{B_{p,q}^{\varsigma,\lambda}\left(\sigma\eta+\frac{1}{2},\omega+\frac{1}{2};\vartheta\right)}{B\left(\frac{1}{2},\omega+\frac{1}{2}\right)\Gamma\left(\sigma\eta+\frac{1}{2}\right)}\frac{(-\tau)^{\eta}}{\eta!}\left(D_{0^+}^{\gamma,\gamma',\delta,\delta',\kappa}\left[t^{\varrho+\eta-1}\right]\right)(x).\end{aligned}\tag{42}$$

Applying equation (36) into above equation, we obtain:

$$\begin{aligned}&\left(D_{0^+}^{\gamma,\gamma',\delta,\delta',\kappa}\left[t^{\varrho-1}J_{\omega;p,q}^{\sigma;\varsigma,\lambda}(\tau t;\vartheta)\right]\right)(x)\\&=\frac{\sqrt{\pi}}{\Gamma(\omega+1)}x^{\varrho-\gamma-\gamma'+\kappa-1}\sum_{\eta=0}^{\infty}(1)_{\eta}\frac{B_{p,q}^{\sigma;\varsigma,\lambda}\left(\frac{1}{2}+\eta\sigma,\frac{1}{2}+\omega;\vartheta\right)}{B\left(\frac{1}{2},\frac{1}{2}+\omega\right)\eta!}\\&\times\frac{\Gamma(\varrho+\eta)\Gamma(\varrho+\gamma+\gamma'+\delta'-\kappa+\eta)\Gamma(\varrho+\gamma+\delta+\eta)}{\Gamma\left(\sigma\eta+\frac{1}{2}\right)\Gamma(\varrho-\delta+\eta)\Gamma(\varrho-\kappa+\gamma+\gamma'+\eta)\Gamma(\varrho-\kappa+\gamma+\delta+\eta)\eta!}(-\tau x)^{\eta}.\end{aligned}\tag{43}$$

With the help of $(p,q;\vartheta)$-extended σ-Gauss hypergeometric function in (20), the generalized Wright function in (21) and the Hadamard product (convolution) in (22), we obtain the required result (40).
Further,the following results are obtained with the help of (19) and (37)-(39).
Corollary 16: The following equation is true:

$$\begin{aligned}\left(D_{0^+}^{\gamma,\delta,\kappa}\left[t^{\varrho-1}J_{\omega;p,q}^{\sigma;\varsigma,\lambda}(\tau t;\vartheta)\right]\right)(x)=&\frac{\sqrt{\pi}}{\Gamma(\omega+1)}x^{\varrho-\delta-1}R_{p,q}^{\sigma;\varsigma,\lambda}\left(1,\frac{1}{2};\omega+1;-\tau x;\vartheta\right)\\&*{}_2\Psi_3\left[\begin{array}{c|c}(\varrho,1),(\varrho+\gamma+\delta+\kappa,1)&\\&-\tau x\\\left(\frac{1}{2},\sigma\right),(\varrho+\delta,1),(\varrho+\kappa,1)&\end{array}\right],\end{aligned}$$

where $Re(\varrho)>-min\{0,Re(\kappa+\gamma+\delta)\}$, $min\{Re(p),Re(q)\}>0$, $min\{Re(\varsigma),Re(\lambda)\}>0$, $\vartheta\in(0,\infty)\backslash\{1\}$, $Re(\omega)>-1$, $Re(\gamma)>0$, when $p=q=1$, $Re(\kappa)>0$, $Re(\sigma)>0$.
Corollary 17: The following equality is true:

$$\begin{aligned}\left(D_{0^+}^{\gamma}\left[t^{\varrho-1}J_{\omega;p,q}^{\sigma;\varsigma,\lambda}(t\tau;\vartheta)\right]\right)(x)=&\frac{\sqrt{\pi}}{\Gamma(\omega+1)}x^{\varrho-\delta-1}R_{p,q}^{\sigma;\varsigma,\lambda}\left(1,\frac{1}{2};\omega+1;-x\tau;\vartheta\right)\\&*{}_1\Psi_2\left[\begin{array}{c|c}(\varrho,1)&\\&-\tau x\\\left(\frac{1}{2},\sigma\right),(\varrho-\gamma,1)&\end{array}\right],\end{aligned}$$

where $min\{Re(\gamma),Re(\varrho)\}>0$, $min\{Re(p),Re(q)\}>0$, $min\{Re(\varsigma),Re(\lambda)\}>0$, $\vartheta\in(0,\infty)\backslash\{1\}$, $Re(\omega)>-1$, when $p=q=1$, $Re(\kappa)>0$, $Re(\sigma)>0$.
Corollary 18: The following equality is valid:

$$\begin{aligned}\left(D_{\gamma,\kappa}^{+}\left[t^{\varrho-1}J_{\omega;p,q}^{\sigma;\varsigma,\lambda}(\tau t;\vartheta)\right]\right)(x)=&\frac{\sqrt{\pi}}{\Gamma(\omega+1)}x^{\varrho-1}R_{p,q}^{\sigma;\varsigma,\lambda}\left(1,\frac{1}{2};\omega+1;-x\tau;\vartheta\right)\\&*{}_1\psi_2\left[\begin{array}{c|c}(\varrho+\gamma+\kappa,1)&\\&-\tau x\\\left(\frac{1}{2},\sigma\right),(\varrho+\kappa,1)&\end{array}\right],\end{aligned}$$

where $Re(\gamma)>0$, $Re(\varrho)>-Re(\gamma+\kappa)$, $min\{Re(p),Re(q)\}>0$, $min\{Re(\varsigma),Re(\lambda)\}>0$, $\vartheta\in(0,\infty)\backslash\{1\}$, $Re(\omega)>-1$, when $p=q=1$, $Re(\kappa)>0$, $Re(\sigma)>0$.

3.2. **Right-sided Marichev-Saigo-Maeda fractional derivative of $J^{\sigma;\varsigma,\lambda}_{\omega;p,q}(z;\vartheta)$.** The following result holds true and established by Marichev [31], Saigo and Maeda [44] and Nisar et al. in [36].

Lemma 19: If $\gamma,\gamma',\delta,\delta',\kappa\in\mathbb{C}$, $x>0$ and $Re(\kappa)>0$, $Re(\varrho)<1+min\{Re(\delta),Re(\kappa-\gamma+\gamma'),Re(\kappa-\gamma-\delta)\}$, then

$$\left(D_{-}^{\gamma,\gamma',\delta,\delta',\kappa}t^{\varrho-1}\right)$$
$$=x^{\varrho+\gamma+\gamma'-\kappa-1}\frac{\Gamma(1-\varrho+\delta')\Gamma(1-\varrho+\kappa-\gamma-\gamma')\Gamma(1-\varrho-\gamma'-\delta+\gamma)}{\Gamma(1-\varrho)\Gamma(1-\varrho-\delta-\gamma-\gamma'+\kappa)\Gamma(1-\varrho-\gamma'+\delta')} \tag{44}$$

In particular,

$$\left(D_{-}^{\gamma,\delta,\kappa}t^{\varrho-1}\right)=x^{\varrho+\delta-1}\frac{\Gamma(1-\varrho-\delta)\Gamma(1-\varrho+\kappa+\gamma)}{\Gamma(1-\varrho)\Gamma(1-\varrho-\delta+\kappa)} \tag{45}$$

where $Re(\gamma)>0$, $Re(\varrho)<1+min\{Re(-\delta-\Lambda),Re(\gamma+\kappa)\}$, $\Lambda=1+Re[(\gamma)]$.

$$\left(D_{-}^{\gamma}t^{\varrho-1}\right)=x^{\varrho-\gamma-1}\frac{\Gamma(1-\varrho+\gamma)}{\Gamma(1-\varrho)} \tag{46}$$

where $Re()>0$, $Re(\varrho)<1+Re(\gamma)-\Lambda$, and

$$\left(D_{\kappa,\gamma}^{-}t^{\varrho-1}\right)=x^{\varrho-1}\frac{\Gamma(1-\varrho+\gamma+\kappa)}{\Gamma(1-\varrho-\kappa)} \tag{47}$$

where $Re(\gamma)>0$, $Re(\varrho)<1+Re(\gamma-\kappa)-\Lambda$.

Theorem 20: The following formula is true

$$\left(D_{-}^{\gamma,\gamma',\delta,\delta',\kappa}\left[t^{\varrho-1}J^{\sigma;\varsigma,\lambda}_{\omega;p,q}(\tau t^{-1};\vartheta)\right]\right)(x)$$
$$=\frac{\sqrt{\pi}}{\Gamma(\omega+1)}x^{\varrho+\gamma+\gamma'-\kappa-1}R^{\sigma;\varsigma,\lambda}_{p,q}\left(1,\frac{1}{2};\omega+1;-x\tau;\vartheta\right)$$
$$*\,{}_3\Psi_4\left[\begin{array}{c|c}(1-\varrho+\delta',1),(1-\varrho+\kappa-\gamma-\gamma',1),(1-\varrho-\gamma-\delta+\gamma,1) & \\ & -\tau x^{-1}\\ \left(\frac{1}{2},\sigma\right),(1-\varrho,1),(1-\varrho-\gamma-\gamma'+\kappa,1),(1-\varrho-\gamma'+\delta',1) & \end{array}\right], \tag{48}$$

where $Re(\varrho)<1+min\{Re(\delta),Re(\kappa-\gamma+\gamma'),Re(\kappa-\gamma-\delta)\}$, $min\{Re(p),Re(q)\}>0$, $min\{Re(\varsigma),Re(\lambda)\}>0$, $\vartheta\in(0,\infty)\backslash\{1\}$, $Re(\omega)>-1$, when $p=q=1$, $Re(\kappa)>0$, $Re(\sigma)>0$.

Proof: Proof is easy, by joining (19) and (44), and further utilizing (45)-(47), after little algebra, we obtain required result (48).

Further, we have following interesting results.

Corollary 21: The following result holds:

$$\left(D_{-}^{\gamma,\delta,\kappa}\left[t^{\varrho-1}J^{\sigma;\varsigma,\lambda}_{\omega;p,q}(\tau t;\vartheta)\right]\right)(x)$$
$$=\frac{\sqrt{\pi}}{\Gamma(\omega+1)}x^{\varrho+\delta-1}R^{\sigma;\varsigma,\lambda}_{p,q}\left(1,\frac{1}{2};\omega+1;-x\tau;\vartheta\right)$$
$$*\,{}_2\Psi_3\left[\begin{array}{c|c}(1-\varrho-\delta,1),(1-\varrho+\kappa+\gamma,1) & \\ & \tau x^{-1}\\ \left(\frac{1}{2},\sigma\right),(1-\varrho+\delta',1),(1-\varrho-\delta+\kappa,1) & \end{array}\right],$$

where $Re(\varrho)<1+min\{Re(-\delta-\Lambda),Re(\kappa+\gamma)\}$, $\Lambda=1+Re(\gamma)$, $min\{Re(p),Re(q)\}>0$, $min\{Re(\varsigma),Re(\lambda)\}>0$, $\vartheta\in(0,\infty)\backslash\{1\}$, $Re(\gamma)>0$, $Re(\omega)>-1$, when

$p=q=1$, $Re(\kappa)>0$, $Re(\sigma)>0$.

Corollary 22: The following equation is true:

$$\left(D_{-}^{\gamma}\left[t^{\varrho-1}J_{\omega;p,q}^{\sigma;\varsigma,\lambda}(\tau t^{-1};\vartheta)\right]\right)(x)=\frac{\sqrt{\pi}}{\Gamma(\omega+1)}x^{\varrho-\gamma-1}R_{p,q}^{\sigma;\varsigma,\lambda}\left(1,\frac{1}{2};\omega+1;-\tau x^{-1};\vartheta\right) * {}_1\Psi_2\left[\begin{array}{c|c}(1-\varrho+\gamma,1) & \\ & -\tau x^{-1} \\ \left(\frac{1}{2},\sigma\right),(1-\varrho,1) & \end{array}\right],$$

where $Re(\gamma)>0$, $Re(\varrho)<1+Re(\gamma)-\Lambda$, $min\{Re(p),Re(q)\}>0$, $min\{Re(\varsigma),Re(\lambda)\}>0$, $\vartheta\in(0,\infty)\backslash\{1\}$, $Re(\omega)>-1$, when $p=q=1$,$Re(\kappa)>0$, $Re(\sigma)>0$.

Corollary 23: The following equality is valid:

$$\left(D_{\gamma,\kappa}^{-}\left[t^{\varrho-1}J_{\omega;p,q}^{\sigma;\varsigma,\lambda}(\tau t^{-1};\vartheta)\right]\right)(x)=\frac{\sqrt{\pi}}{\Gamma(\omega+1)}x^{\varrho-1}R_{p,q}^{\sigma;\varsigma,\lambda}\left(1,\frac{1}{2};\omega+1;-x\tau;\vartheta\right) * {}_1\Psi_2\left[\begin{array}{c|c}(1-\varrho+\gamma+\kappa,1) & \\ & -\tau x^{-1} \\ \left(\frac{1}{2},\sigma\right),(1-\varrho-\kappa,1) & \end{array}\right],$$

where $Re(\gamma)>0$, $Re(\varrho)<Re(\gamma-\kappa)-[Re(\Lambda)]$, $min\{Re(p),Re(q)\}>0$, $\vartheta\in(0,\infty)\backslash\{1\}$, $min\{Re(\varsigma),Re(\lambda)\}>0$, $Re(\omega)>-1$, when $p=q=1$, $Re(\kappa)>0$, $Re(\sigma)>0$.

4. Caputo Marichev-Saigo-Maeda fractional derivative of $J_{\omega;p,q}^{\sigma;\varsigma,\lambda}(z;\vartheta)$

We have to record following results which are obtained by Araci et al. in [4]:

Lemma 24: If $\gamma,\gamma',\delta,\delta',\kappa\in\mathbb{C}$, $x>0$; $Re(\kappa)>0$, $\Lambda=1+[Re(\kappa)]$; $Re(\varrho)-\Lambda>max\{0,Re(-\gamma+\delta),Re(-\gamma-\gamma'-\delta'+\kappa)\}$, then

$$\left({}^{C}D_{-}^{\gamma,\gamma',\delta,\delta',\kappa}t^{\varrho-1}\right)(x)$$
$$=x^{\varrho+\gamma+\gamma'-\kappa-1}\frac{\Gamma(\varrho)\Gamma(\varrho+\gamma-\delta-\Lambda)\Gamma(\varrho+\delta+\gamma+\gamma'-\kappa-\Lambda)}{\Gamma(\varrho-\delta-\Lambda)\Gamma(\varrho-\kappa+\gamma+\gamma')\Gamma(\varrho-\kappa+\gamma+\delta'-\Lambda)}. \tag{49}$$

Lemma 25: If $\gamma,\gamma',\delta,\delta',\kappa\in\mathbb{C}$, $x>0$; $Re(\kappa)>0$, $\Lambda=1+[Re(\kappa)]$; $Re(\varrho)+\Lambda>1+min\{Re(-\delta),Re(\gamma'+\delta-\kappa),Re(\gamma+\gamma'-\kappa)+1+[Re(\kappa)]\}$, then

$$\left({}^{C}D_{-}^{\gamma,\gamma',\delta,\delta',\kappa}t^{-\varrho}\right)(x)$$
$$=x^{\gamma+\gamma'-\varrho-\kappa}\frac{\Gamma(\varrho+\delta'+\Lambda)\Gamma(\varrho+\kappa-\gamma-\gamma')\Gamma(\varrho-\gamma'-\delta+\gamma+\Lambda)}{\Gamma(\varrho)\Gamma(\varrho+\Lambda-\gamma'+\delta')\Gamma(\varrho-\gamma-\gamma'-\delta+\kappa+\Lambda)}. \tag{50}$$

Theorem 26: The following results holds

$$\left({}^{C}D_{0+}^{\gamma,\gamma',\delta,\delta',\kappa}\left[t^{\varrho-1}J_{\omega;p,q}^{\sigma;\varsigma,\lambda}(t\tau;\vartheta)\right]\right)(x)$$
$$=\frac{\sqrt{\pi}}{\Gamma(\omega+1)}x^{\varrho+\gamma+\gamma'-\kappa-1}R_{p,q}^{\sigma;\varsigma,\lambda}\left(1,\frac{1}{2};\omega+1;-x\tau;\vartheta\right)$$
$$* {}_3\Psi_4\left[\begin{array}{c|c}(\varrho,1),(\varrho+\gamma-\delta-\Lambda,1),(\varrho+\gamma+\gamma'+\delta-\kappa-\Lambda,1) & \\ & -\tau x \\ \left(\frac{1}{2},\sigma\right),(\varrho-\delta-\Lambda,1),(\varrho-\kappa+\gamma+\gamma',1),(\varrho-\kappa+\gamma+\delta'-\Lambda,1) & \end{array}\right], \tag{51}$$

where $\Lambda=1+[Re(\kappa)]$, $Re(-\gamma-\gamma'-\delta'+\kappa)\}$, $min\{Re(p),Re(q)\}>0$, $Re(\varrho)-\Lambda>max\{0,Re(-\gamma+\delta)$, $min\{Re(\varsigma),\ Re(\lambda)\}>0$, $\vartheta\in(0,\infty)\backslash\{1\}$, $Re(\omega)>-1$, when $p=q=1$, $Re(\kappa)>0$, $Re(\sigma)>0$.

Proof: Proof is easy, with the help of $(p,q;\vartheta)$-extended σ-Gauss hypergeometric function (20), generalized Wright function (21), Hadamard product (convolution) (22) along with (19) and (49), after little algebra, we obtain desired results (51).

Theorem 27: The following results holds

$$\left({}^{C}D_{-}^{\gamma,\gamma',\delta,\delta',\kappa}\left[t^{\varrho-1}J_{\omega;p,q}^{\sigma;\varsigma,\lambda}(\tau t^{-1};\vartheta)\right]\right)(x)$$
$$=\frac{\sqrt{\pi}}{\Gamma(\omega+1)}x^{-\varrho+\gamma+\gamma'-\kappa}R_{p,q}^{\sigma;\varsigma,\lambda}\left(1,\frac{1}{2};\omega+1;-x\tau;\vartheta\right)$$
$$*\,{}_3\Psi_4\left[\begin{array}{c}(\varrho+\delta'+\Lambda,1),(\varrho+\kappa-\gamma-\gamma',1),(\varrho-\gamma'-\delta+\gamma+\Lambda,1)\\ \\ \left(\frac{1}{2},\sigma\right),(\varrho,1),(\varrho-\gamma'+\delta'+\Lambda,1),(\varrho-\gamma-\gamma'-\delta+\kappa+\Lambda,1)\end{array}\middle|-\tau x^{-1}\right], \tag{52}$$

where $\Lambda=1+[Re(\kappa)]$, $Re(\varrho)+\Lambda>1+min\{Re(-\delta'),Re(\gamma'+\delta-\kappa),Re(\gamma+\gamma'+\kappa)+1+[Re(\kappa)]\}$, $min\{Re(p),Re(q)\}>0$, $min\{Re(\varsigma),Re(\lambda)\}>0$, $\vartheta\in(0,\infty)\backslash\{1\}$, $Re(\omega)>-1$, when $p=q=1$, $Re(\kappa)>0$, $Re(\sigma)>0$.

Proof: Proof is easy, with the help of $(p,q;\vartheta)$-extended σ-Gauss hypergeometric function (20), generalized Wright function (21),Hadamard product (convolution) (22) along with (19) and (50), after little algebra, we obtain desired result (52).

5. Solutions of fractional kinetic equations

Various special functions such as Mittag-Leffler-type, K-type, H-type, I-type Bessel-type, Aleph-type, S-type, hypergeometric-type, and plenty of others, see Kiryakova [23, 26, 28] ; and integral transform such as Laplace, Sumudu were used by different researchers to study fractional kinetic equations. In this section, we are using $(p,q;\vartheta)$-extended Bessel-Wright function $J_{\omega;p,q}^{\sigma;\varsigma;\lambda}(z;\vartheta)$ and Laplace transform, for possible solutions of fractional kinetic equations.

Definition 28: The two-parameters Mittag-Leffler (Wiman) function is defined (see, Wiman [49], Gorenflo et al. [13] and Paneva-Konovska [17]), as follows:

$$E_{\phi,\varphi}(z)=\sum_{\eta=0}^{\infty}\frac{z^{\eta}}{\Gamma(\phi\eta+\varphi)}. \tag{53}$$

where $\phi,\varphi,z\in\mathbb{C}, min\{Re(\phi),Re(\varphi)\}>0$.

Theorem 29: Suppose $d,\alpha>0$ with $\eta\in(0,\infty)\backslash\{1\}$, $Re(\omega)>-1$, $min\{Re(p),Re(q)\}>0$, $min\{Re(\varsigma),Re(\lambda)\}>0$, so the extended fractional kinetic equation

$$K(t)-K_0J_{\omega;p,q}^{\sigma;\varsigma,\lambda}(t;\vartheta)=-d^{\alpha}{}_0D_t^{-\alpha} \tag{54}$$

has a solution

$$K(t)=K_0\frac{\sqrt{\pi}}{\Gamma(\omega+1)}\sum_{\eta=0}^{\infty}\frac{B_{p,q}\left(\sigma\eta+\frac{1}{2},\omega+\frac{1}{2}\right)}{B\left(\frac{1}{2},\omega+\frac{1}{2}\right)\Gamma\left(\sigma\eta+\frac{1}{2}\right)}(-t)^{\eta}E_{\alpha,\eta+1}(-d^{\alpha}t^{\alpha}). \tag{55}$$

Proof: Applying the Laplace transform (see, Mousa [33]) to both sides of equation (54), and using (19), we obtain

$$L\{K(t);s\}=K_0L\left\{\frac{\sqrt{\pi}}{\Gamma(\omega+1)}\sum_{\eta=0}^{\infty}\frac{B_{p,q}^{\varsigma,\lambda}\left(\sigma\eta+\frac{1}{2},\omega+\frac{1}{2};\vartheta\right)}{B\left(\frac{1}{2},\omega+\frac{1}{2}\right)\Gamma\left(\sigma\eta+\frac{1}{2}\right)}\frac{(-t)^{\eta}}{\eta!};s\right\}-d^{\alpha}L\{{}_0D_t^{-\alpha};s\}$$

by changing the position of summation and the Laplace operator, gives

$$L\{K(t);s\}=K_0\frac{\sqrt{\pi}}{\Gamma(\omega+1)}\sum_{\eta=0}^{\infty}\frac{B_{p,q}^{\varsigma,\lambda}\left(\sigma\eta+\frac{1}{2},\omega+\frac{1}{2};\vartheta\right)}{B\left(\frac{1}{2},\omega+\frac{1}{2}\right)\Gamma\left(\sigma\eta+\frac{1}{2}\right)}\frac{(-1)^{\eta}}{\eta!}L\left\{t^{\eta};s\right\}-d^{\alpha}L\{{}_0D_t^{-\alpha};s\}$$

Applying the results obtain by Srivastava et al. [47]

$$L\{{}_0D_t^{-\alpha};s\}=s^{-\alpha}K(s) \tag{56}$$

and

$$L\left\{t^{\eta};s\right\}=\frac{\Gamma(\eta+1)}{s^{\eta+1}},\ Re(\eta)>-1$$

resulting in

$$K(s)\left\{1+d^{\alpha}s^{-\alpha}\right\}=K_0\frac{\sqrt{\pi}}{\Gamma(\omega+1)}\sum_{\eta=0}^{\infty}\frac{B_{p,q}^{\varsigma,\lambda}\left(\sigma\eta+\frac{1}{2},\omega+\frac{1}{2};\vartheta\right)}{B\left(\frac{1}{2},\omega+\frac{1}{2}\right)\Gamma\left(\sigma\eta+\frac{1}{2}\right)}\frac{(-1)^{\eta}}{\eta!}\frac{\Gamma(\eta+1)}{s^{\eta+1}}.$$

Further, considering the finding of Kachhia et al. [**?**], we have

$$\left\{1+d^{\alpha}s^{-\alpha}\right\}^{-1}=\sum_{\varpi=0}^{\infty}\left(-d^{\alpha}s^{-\alpha}\right)^{\varpi},\ \left|d^{\alpha}s^{-\alpha}\right|<1$$

yields

$$\begin{aligned}K(s)&=K_0\frac{\sqrt{\pi}}{\Gamma(\omega+1)}\sum_{\eta=0}^{\infty}\frac{B_{p,q}^{\varsigma,\lambda}\left(\sigma\eta+\frac{1}{2},\omega+\frac{1}{2};\vartheta\right)}{B\left(\frac{1}{2},\omega+\frac{1}{2}\right)\Gamma\left(\sigma\eta+\frac{1}{2}\right)}(-1)^{\eta}\frac{1}{s^{\eta+1}}\sum_{\varpi=0}^{\infty}(-1)^{\varpi}d^{\alpha\varpi}s^{-\alpha\varpi}\\&=K_0\frac{\sqrt{\pi}}{\Gamma(\omega+1)}\sum_{\eta=0}^{\infty}\frac{B_{p,q}^{\varsigma,\lambda}\left(\sigma\eta+\frac{1}{2},\omega+\frac{1}{2};\vartheta\right)}{B\left(\frac{1}{2},\omega+\frac{1}{2}\right)\Gamma\left(\sigma\eta+\frac{1}{2}\right)}(-1)^{\eta}\sum_{\varpi=0}^{\infty}(-1)^{\varpi}d^{\alpha\varpi}s^{-(\alpha\varpi+\eta+1)}\end{aligned}$$

Now by using (56) and taking inverse Laplace transform and result obtained by Kachhia et al. [**?**], $L^{-1}\left\{s^{-\eta}\right\}=\frac{t^{\eta-1}}{\Gamma(\Gamma(\eta))},\ Re(\eta)>0$, we obtain

$$\begin{aligned}K(t)&=K_0\frac{\sqrt{\pi}}{\Gamma(\omega+1)}\sum_{\eta=0}^{\infty}\frac{B_{p,q}^{\varsigma,\lambda}\left(\sigma\eta+\frac{1}{2},\omega+\frac{1}{2};\vartheta\right)}{B\left(\frac{1}{2},\omega+\frac{1}{2}\right)\Gamma\left(\sigma\eta+\frac{1}{2}\right)}(-1)^{\eta}\sum_{\varpi=0}^{\infty}(-1)^{\varpi}d^{\alpha\varpi}\frac{t^{\alpha\varpi+\eta}}{\Gamma(\alpha\varpi+\eta+1)}\\&=K_0\frac{\sqrt{\pi}}{\Gamma(\omega+1)}\sum_{\eta=0}^{\infty}\frac{B_{p,q}^{\varsigma,\lambda}\left(\sigma\eta+\frac{1}{2},\omega+\frac{1}{2};\vartheta\right)}{B\left(\frac{1}{2},\omega+\frac{1}{2}\right)\Gamma\left(\sigma\eta+\frac{1}{2}\right)}(-t)^{\eta}\sum_{\varpi=0}^{\infty}\frac{(-d^{\alpha}t^{\alpha})^{\varpi}}{\Gamma(\alpha\varpi+\eta+1)}\\&=K_0\frac{\sqrt{\pi}}{\Gamma(\omega+1)}\sum_{\eta=0}^{\infty}\frac{B_{p,q}^{\varsigma,\lambda}\left(\sigma\eta+\frac{1}{2},\omega+\frac{1}{2};\vartheta\right)}{B\left(\frac{1}{2},\omega+\frac{1}{2}\right)\Gamma\left(\sigma\eta+\frac{1}{2}\right)}(-t)^{\eta}E_{\alpha,\eta+1}(-d^{\alpha}t^{\alpha}).\end{aligned}$$

Thus we proved our Theorem 29.

Theorem 30: Suppose $d,\alpha>0$ with $min\{Re(p),Re(q)\}>0$, $min\{Re(\varsigma),Re(\lambda)\}>0$, $\eta\in(0,\infty)\backslash\{1\}$, $Re(\omega)>-1$, so the extended fractional kinetic equation

$$K(t)-K_0J_{\omega;p,q}^{\sigma;\varsigma,\lambda}(d^{\alpha}t^{\alpha};\vartheta)=-d^{\alpha}{}_0D_t^{-\alpha} \tag{57}$$

has a solution

$$K(t)=K_0\frac{\sqrt{\pi}}{\Gamma(\omega+1)}\sum_{\eta=0}^{\infty}\frac{B_{p,q}\left(\sigma\eta+\frac{1}{2},\omega+\frac{1}{2}\right)}{B\left(\frac{1}{2},\omega+\frac{1}{2}\right)\Gamma\left(\sigma\eta+\frac{1}{2}\right)}\frac{(-d^{\alpha}t^{\alpha})^{\eta}}{\eta!}\Gamma(\alpha\eta+1)E_{\alpha,\alpha\eta+1}(-d^{\alpha}t^{\alpha}). \tag{58}$$

Proof: First of all obtain Laplace transform of both sides of (57), then use the definition of $(p,q;\vartheta)$-extended Bessel-Wright function (19) and also (56); further simplify and take inverse of the Laplace transform and by using (53), we obtained the required results (58).

Theorem 31: Suppose $d, \alpha > 0$ $a \neq d$ with $\eta \in (0,\infty)\backslash\{1\}$, $Re(\omega) > -1$, $min\{Re(p), Re(q)\} > 0$, $min\{Re(\varsigma), Re(\lambda)\} > 0$, so the extended fractional kinetic equation

$$K(t) - K_0 J^{\sigma;\varsigma,\lambda}_{\omega;p,q}(d^\alpha t^\alpha; \vartheta) = -a^\alpha {}_0D_t^{-\alpha} \tag{59}$$

has a solution

$$K(t) = K_0 \frac{\sqrt{\pi}}{\Gamma(\omega+1)} \sum_{\eta=0}^{\infty} \frac{B_{p,q}\left(\sigma\eta + \frac{1}{2}, \omega + \frac{1}{2}\right)}{B\left(\frac{1}{2}, \omega + \frac{1}{2}\right)\Gamma\left(\sigma\eta + \frac{1}{2}\right)} \frac{(-d^\alpha t^\alpha)^\eta}{\eta!} \Gamma(\alpha\eta+1) E_{\alpha,\alpha\eta+1}(-d^\alpha t^\alpha). \tag{60}$$

Proof: First of all find the Laplace transform of both sides of (59), then use the definition of $(p,q;\vartheta)$-extended Bessel-Wright function (19) and also (56); further simplify and take inverse of the Laplace transform and by using (53), we obtain the required result (60).

6. Conclusion

If we substitute $\vartheta = e$ and $\varsigma = \lambda = 1$, then the new $(p,q;\vartheta)$-extended Bessel-Wright function $J^{\sigma;\varsigma,\lambda}_{\omega;p,q}(z;\vartheta)$ in (19) reduces to the (p,q)-extended Bessel-Wright function $J^{\sigma}_{\omega;p,q}(z)$ in (2), i.e.

$$J^{\sigma;1,1}_{\omega;p,q}(z;e) = J^{\sigma}_{\omega;p,q}(z).$$

If we set $\vartheta = e$, $\varsigma = \lambda = 1$ and $p = q = 0$ then the new $(p,q;\vartheta)$-extended Bessel-Wright function $J^{\sigma;\varsigma,\lambda}_{\omega;p,q}(z;\vartheta)$ in (19) reduces to the extended Bessel-Wright function $J^{\sigma}_{\omega}(z)$ in (1), i.e.

$$J^{\sigma;1,1}_{\omega;0,0}(z;e) = J^{\sigma}_{\omega}(z).$$

If we consider $\vartheta = e$, $\sigma = 1$, $\varsigma = \lambda = 1$ and $p = q = 0$ then the new $(p,q;\vartheta)$-extended Bessel-Wright function $J^{\sigma;\varsigma,\lambda}_{\omega;p,q}(z;\vartheta)$ in (19) reduces to the Bessel-Clifford function $C_\omega(z)$ (refer to [37]), i.e.

$$J^{1;1,1}_{\omega;0,0}(z;e) = C_\omega(z).$$

Hence on replacing the parameters and variable appropriately some existing fractional integration and differentiation formulas exist in literatures, can be obtained (see for example [48]). Therefore, the new $(p,q;\vartheta)$-extended Bessel-Wright function $J^{\sigma;\varsigma,\lambda}_{\omega;p,q}(z;\vartheta)$ is expected to have vast application in science and technology.

Also, fractional calculus is a rapidly growing area of mathematics concerned with the study of fractional derivatives and integrals of the fractional order. Many applications of fractional calculus can be found in nuclear interactions, image processing, earthquake prediction, biological systems, signal processing, electro-chemistry, fluid dynamics, stochastic dynamic systems, optics, control theory, plasma physics, electronics, controlled thermonuclear fusion, quantum mechanics and many other real-life application problems, see details, Hilfer [15], Aziz and Kumawat [5], Mishra et al. [32], Ray et al. [41], Eze and Oyesanya [12] and Abdo et al. [6].

Acknowledgements. The research work of the M.P. Chaudhary was sponsored by the Major Research Project of the National Board of Higher Mathematics (NBHM) of the Department of Atomic Energy (DAE) of the Government of India by its sanction letter (Ref. No. 02011/12/2020 NBHM (R.P.)/R D II/7867) dated 19 October 2020. Some results of this article was presented in the 1^{st} International Alumni's Mathematics UET Conference, Lahore, Pakistan, February 26, 2022.

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