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EXISTENCE AND UNIQUENESS OF THE SOLUTION OF THE FRACTIONAL DIFFERENTIAL EQUATION VIA A NEW THREE STEPS ITERATION

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Abstract. In this paper, we study the existence, uniqueness, and other properties of solution of differential equation of fractional order involving the Caputo fractional derivative. The tool employed in the analysis is based on the application of a new three steps iteration process introduced by V. Karakaya, Y. Atalan, K. Dogan, and NH. Bouzara [26]. Furthermore, the study of various properties such as dependence on initial data, the closeness of solutions, and dependence on parameters and functions involved therein. The results obtained are illustrated through an example.

1. Introduction

We consider the following differential equation of fractional order involving the Caputo fractional derivative of the type:

$$\big(D^{\alpha}_{*a}\big)x(t) = \mathcal{F}\Big(t, x(t), x(a), x(b)\Big), \tag{1}$$

for $t \in I = [a,b],\ n-1 < \alpha \leq n\ (n \in \mathbb{N})$, with the given initial conditions

$$x^{(j)}(a) = c_j,\ j = 0, 1, 2, \cdots, n-1, \tag{2}$$

where $\mathcal{F} : I \times X \times X \times X \to X$ is continuous function and $c_j\ (j = 0, 1, 2, \ldots, n-1)$ are given elements in X.

The theory of iterative approximation of fixed points plays a significant role in the progress of differential and integral equations, and their applications. In this context, several researchers have introduced many iteration methods for certain classes of operators in the sense of their convergence, equivalence of convergence and rate of convergence etc. (see [2, 3, 7, 8, 12, 14, 15, 20, 21, 22, 23, 24, 25, 27, 29, 33, 34]). The most of iterations devoted for both analytical and numerical

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approaches. The new three steps iteration method, due to simplicity and fastness, has attracted the attention and hence, it is used in this paper.

The problems of existence, uniqueness and other properties of solutions of special forms of IVP (1)-(2) and its variants have been studied by several mathematicians under variety of hypotheses by using different techniques, [4, 5, 6, 9, 13, 16, 17, 18, 30, 31] and some of references cited therein.

The main objective of this paper is to use new three steps iteration method to establish the existence and uniqueness of solution of the initial value problem (1)-(2) and other qualitative properties of solutions.

2. Preliminaries

Before proceeding to the statement of our main results, we shall setforth some preliminaries and hypotheses that will be used in our subsequent discussion.

Let X be a Banach space with norm $\|\cdot\|$ and $I=[a,b]$ denotes an interval of the real line $\mathbb{R}$. We define $B=C^r(I,X)$ (where $r=n$ for $\alpha\in\mathbb{N}$ and $r=n-1$ for $\alpha\notin\mathbb{N}$.) as a Banach space of all r times continuously differentiable functions from I into X, endowed with the norm

$$\|x\|_B=\sup\{\|x(t)\|:x\in B\},\quad t\in I.$$

Definition 2.1. [32] *The Riemann Liouville fractional integral (left-sided) of a function $h\in C^1[a,b]$ of order $\alpha\in\mathbf{R}_+=(0,\infty)$ is defined by*

$$I_a^\alpha h(t)=\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}h(s)\,ds,$$

where Γ is the Euler gamma function.

Definition 2.2. [32] *Let $n-1<\alpha\le n,\ n\in\mathbb{N}$. Then the expression*

$$D_a^\alpha h(t)=\frac{d^n}{dt^n}\left[I_a^{n-\alpha}h(t)\right],\ t\in[a,b]$$

is called the (left-sided) Riemann Liouville derivative of h of order α whenever the expression on the right-hand side is defined.

Definition 2.3. [28] *Let $h\in C^n[a,b]$ and $n-1<\alpha\le n,\ n\in\mathbb{N}$. Then the expression*

$$\left(D_{*a}^\alpha\right)h(t)=I_a^{n-\alpha}h^{(n)}(t),\ t\in[a,b]$$

is called the (left-sided) Caputo derivative of h of order α.

Definition 2.4. [11] Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers that converge to a and b, respectively, and assume that there exists

$$l=\lim_{n\to\infty}\frac{|a_n-a|}{|b_n-b|}.$$

(a) If $l=0$, then it can be said that $\{a_n\}$ converges to a faster than $\{b_n\}$ converges to b.
(b) If $0<l<1$, then it can be said that $\{a_n\}$ and $\{b_n\}$ have the same rate of convergence.

Suppose that for two fixed point iteration procedures $\{u_n\}$ and $\{v_n\}$, both converging to the same fixed point p, the error estimates

$$\|u_n-p\|\le a_n,\ \forall\ n\in\mathbb{N},\tag{3}$$

$$\|v_n - p\| \leq b_n,\ \forall\ n \in \mathbb{N}, \tag{4}$$

are available, where $\{a_n\}$ and $\{b_n\}$ are two sequences of positive numbers (converging to zero). Then, in view of Definition 2.4, we will adopt the following concept.

Definition 2.5. [11] Let $\{u_n\}$ and $\{v_n\}$ be two fixed point iteration procedures that converge to the same fixed point p and satisfy (3) and (4), respectively. If $\{a_n\}$ converges faster than $\{b_n\}$, then it can be said that $\{u_n\}$ converges faster than $\{v_n\}$ to p.

Lemma 2.1. [19] *If the function $f = (f_1, \cdots, f_n) \in C^1[a, b]$, then the initial value problems*

$$\big(D^{\alpha_i}_{*a}\big)x_i(t) = f_i(t, x_1, \cdots, x_n),\ x_i^{(k)}(0) = c_k^i,\ i = 1, 2, \cdots, n,\ k = 1, 2, \cdots, m_i$$

where $m_i < \alpha_i \leq m_i + 1$ is equivalent to Volterra integral equations:

$$x_i(t) = \sum_{k=0}^{m_i} c_k^i \frac{t^k}{k!} + I_a^{\alpha_i} f_i(t, x_1, \cdots, x_n),\ 1 \leq i \leq n.$$

As a consequence of the Lemma 2.1, it is easy to observe that if $x \in B$ and $\mathcal{F} \in C^1[a, b]$, then $x(t)$ satisfies the following integral equation which is equivalent to (1)-(2) is

$$x(t) = \sum_{j=0}^{n-1} \frac{c_j}{j!}\ (t-a)^j + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\Big(s, x(s), x(a), x(b)\Big) ds. \tag{5}$$

Definition 2.6. *([26], p.626) The self-map $T : C \to C$ is called weak-contraction if there exist $\delta \in (0, 1)$ and $L \geq 0$ such that*

$$\|Tx - Ty\| \leq \delta\|x - y\| + L\|y - Ty\|.$$

Recently, V. Karakaya, Y. Atalan, K. Dogan, and NH. Bouzara [26] introduced the following new three steps iteration process:

$$\begin{cases} x_{k+1} = Ty_k, \\ y_k = (1-\xi_k)z_k + \xi_k T z_k, \\ z_k = Tx_k,\ k \in \mathbb{N} \cup \{0\}, \end{cases} \tag{6}$$

with the real control sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0, 1]$.

To prove existence and uniqueness, we require the following known results:

Theorem 2.1. *([26], p.626) Let (X, d) be a complete metric space and $T : X \to X$ be a weak contraction for which there exist $\delta \in (0, 1)$ and some $L_1 \geq 0$ such that*

$$\|Tx - Ty\| \leq \delta\|x - y\| + L_1\|x - Tx\|. \tag{7}$$

Then, T has a unique fixed point.

Theorem 2.2. *(([26], p.627)) Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a weak-contraction map satisfying condition (7). Let $\{x_k\}_{k=0}^{\infty}$ be an iterative sequence generated by the scheme (6) with a real control sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0, 1]$ satisfying $\sum_{k=0}^{\infty} \xi_k = \infty$. Then $\{x_k\}_{k=0}^{\infty}$ converges to a unique point x^* of T.*

Lemma 2.2. *(([35], p.4)) Let $\{\beta_k\}_{k=0}^{\infty}$ be a nonnegative sequence for which one assumes there exists $k_0 \in \mathbb{N}$, such that for all $k \geq k_0$ one has satisfied the inequality*

$$\beta_{k+1} \leq (1-\mu_k)\beta_k + \mu_k\gamma_k, \tag{8}$$

where $\mu_k \in (0,1)$, for all $k \in \mathbb{N} \cup \{0\}$, $\sum_{k=0}^{\infty} \mu_k = \infty$ and $\gamma_k \geq 0$, $\forall k \in \mathbb{N} \cup \{0\}$. Then the following inequality holds

$$0 \leq \lim_{k\to\infty} \sup \beta_k \leq \lim_{k\to\infty} \sup \gamma_k. \tag{9}$$

3. Existence and Uniqueness of Solutions via new three steps iteration

Now, we are able to state and prove the following main theorem which deals with the existence of solutions of the equations (1)-(2).

Theorem 3.3. *Assume that there exists a function $p \in C(I, \mathbb{R}_+)$ and constants $\lambda,\ \beta,\ \gamma > 0$ such that for $t \in I$,*

$$\begin{aligned} &\|\mathcal{F}\Big(t, u_1, u_2, u_3\Big) - \mathcal{F}\Big(t, v_1, v_2, v_3\Big)\| \\ &\leq p(t)\Big[\lambda\|u_1 - v_1\| + \beta\|u_2 - v_2\| + \gamma\|u_3 - v_3\|\Big]. \end{aligned} \tag{10}$$

If $\Delta = {I_a}^{\alpha} p(t)\big(\lambda + \beta + \gamma\big) < 1$ $(t \in I)$, then the equations (1)-(2) has a unique solution $x \in B$, which is the required solution and is obtained by the three steps iterative method (6) starting with any element $x_0 \in B$. Moreover, if x_k is the $k-th$ successive approximation, then one has

$$\|x_{k+1} - x\|_B \leq \frac{\Delta^{2k+2}}{e^{\big(1-\Delta\big)\sum_{i=0}^{k}\xi_i}} \|x_0 - x\|_B. \tag{11}$$

Proof. Let $x(t) \in B$ and define the operator

$$\begin{aligned} (Tx)(t) = &\sum_{j=0}^{n-1} \frac{c_j}{j!}\ (t-a)^j \\ &+ \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\Big(s, x(s), x(a), x(b)\Big) ds,\ t \in I. \end{aligned} \tag{12}$$

Let $\{x_k\}_{k=0}^{\infty}$ be iterative sequence generated by new three steps iteration method (6) for the operator given in (12) with the real control sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0,1]$. We will show that $x_k \to x$ as $k \to \infty$. From (6), (12) and assumptions, we obtain

$$\begin{aligned} &\|z_k(t) - x(t)\| \\ &= \|(Tx_k)(t) - (Tx)(t)\| \\ &= \|\sum_{j=0}^{n-1} \frac{c_j}{j!}\ (t-a)^j + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\Big(s, x_k(s), x_k(a), x_k(b)\Big) ds \\ &\quad - \sum_{j=0}^{n-1} \frac{c_j}{j!}\ (t-a)^j - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\Big(s, x(s), x(a), x(b)\Big) ds\| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \|\mathcal{F}\Big(s, x_k(s), x_k(a), x_k(b)\Big) - \mathcal{F}\Big(s, x(s), x(a), x(b)\Big)\| ds \end{aligned}$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}p(s) \times \Big[\lambda\|x_k(s)-x(s)\| + \beta\|x_k(a)-x(a)\| + \gamma\|x_k(b)-x(b)\|\Big]ds. \tag{13}$$

Now, by taking supremum in the inequality (13), we obtain

$$\begin{aligned}\|z_k - x\|_B &\leq \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}p(s)\Big(\lambda+\beta+\gamma\Big)\|x_k - x\|_B ds\\ &\leq I_a{}^{\alpha}p(t)\Big(\lambda+\beta+\gamma\Big)\|x_k - x\|_B\\ &= \Delta\|x_k - x\|_B,\end{aligned} \tag{14}$$

and

$$\begin{aligned}\|y_k(t)-x(t)\| &= \|(1-\xi_k)z_k(t) + \xi_k(Tz_k)(t) - x(t)\|\\ &= \|(1-\xi_k)z_k(t) + \xi_k(Tz_k)(t) - (1-\xi_k)x(t) - \xi_k x(t)\|\\ &= \|(1-\xi_k)(z_k(t)-x(t)) + \xi_k\big((Tz_k)(t) - (Tx)(t)\big)\|\\ &\leq \Big[(1-\xi_k)\|z_k(t)-x(t)\| + \xi_k\|(Tz_k)(t)-(Tx)(t)\|\Big].\end{aligned} \tag{15}$$

Hence, by taking supremum in the inequality (15) and then use (14) to get

$$\begin{aligned}\|y_k - x\|_B &\leq (1-\xi_k)\|z_k - x\|_B + \xi_k\|Tz_k - Tx\|_B\\ &\leq (1-\xi_k)\|z_k - x\|_B + \xi_k\Delta\|z_k - x\|_B\\ &= \Big[1-\xi_k\Big(1-\Delta\Big)\Big]\|z_k - x\|_B\\ &\leq \Delta\Big[1-\xi_k\Big(1-\Delta\Big)\Big]\|x_k - x\|_B.\end{aligned} \tag{16}$$

Therefore, using (14) and (16), we obtain

$$\begin{aligned}\|x_{k+1} - x\|_B &= \|Ty_k - x\|_B\\ &= \|Ty_k - Tx\|_B\\ &\leq \Delta\|y_k - x\|_B\\ &\leq \Delta^2\Big[1-\xi_k\Big(1-\Delta\Big)\Big]\|x_k - x\|_B.\end{aligned} \tag{17}$$

Thus, by induction, we get

$$\|x_{k+1} - x\|_B \leq \Delta^{2k+2}\prod_{j=0}^{k}\Big[1-\xi_j\Big(1-\Delta\Big)\Big]\|x_0 - x\|_B. \tag{18}$$

Since $\xi_k \in [0,1]$ for all $k \in \mathbb{N}\cup\{0\}$, the definition of Δ yields,

$$\Rightarrow \xi_k\Delta < \xi_k$$

$$\Rightarrow \xi_k\Big(1-\Delta\Big) < 1,\ \forall\ k \in \mathbb{N}\cup\{0\}. \tag{19}$$

From the classical analysis, we know that $1 - x \leq e^{-x},\ \forall\ x \in [0,1]$. Hence by utilizing this fact with (19) in (18), we obtain

$$\begin{aligned}\|x_{k+1} - x\|_B &\leq \Delta^{2k+2}e^{-\big(1-\Delta\big)\sum_{j=0}^{k}\xi_j}\|x_0 - x\|_B\\ &= \frac{\Delta^{2k+2}}{e^{\big(1-\Delta\big)\sum_{i=0}^{k}\xi_i}}\|x_0 - x\|_B.\end{aligned} \tag{20}$$

Thus, we have proved (11). Since $\sum_{k=0}^{\infty} \xi_k = \infty$, then we have

$$e^{-\left(1-\Delta\right)\sum_{j=0}^{k} \xi_j} \to 0 \quad \text{as} \quad k \to \infty. \tag{21}$$

Hence using this, the inequality (20) implies $\lim_{k\to\infty} \|x_{k+1} - x\|_B = 0$ and therefore, we get $x_k \to x$ as $k \to \infty$. □

Remark: It is an interesting to note that the inequality (20) gives the bounds in terms of known functions, which majorizes the iterations for solutions of the equations (1)-(2) for $t \in I$.

4. Continuous dependence via new three steps iteration

In this section, we shall deal with continuous dependence of solution of the problem (1) on the initial data, functions involved therein and also on parameters.

4.1. **Dependence on initial data.** Suppose $x(t)$ and $\overline{x}(t)$ are solutions of (1) with initial data

$$x^{(j)}(a) = c_j, \; j = 0, 1, 2, \cdots, n-1, \tag{22}$$

and

$$\overline{x}^{(j)}(a) = d_j, \; j = 0, 1, 2, \cdots, n-1, \tag{23}$$

respectively, where c_j, d_j are elements of the space X.

Then looking at the steps as in the proof of Theorem 3.3, we define the operator for the equation (1) with initial conditions (23)

$$\begin{aligned}(\overline{T}\overline{x})(t) = {} & \sum_{j=0}^{n-1} \frac{d_j}{j!} \; (t-a)^j \\ & + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\Big(s, \overline{x}(s), \overline{x}(a), \overline{x}(b)\Big) ds, \; t \in I.\end{aligned} \tag{24}$$

We shall deal with the continuous dependence of solutions of equation (1) on initial data.

Theorem 4.4. *Suppose the function $\mathcal{F}$ in equation* (1) *satisfies the condition* (10)*. Consider the sequences $\{x_k\}_{k=0}^{\infty}$ and $\{\overline{x}_k\}_{k=0}^{\infty}$ generated by new three steps iteration method associated with operators T in* (12) *and $\overline{T}$ in* (24)*, respectively with the real sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0, 1]$ satisfying $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$. If the sequence $\{\overline{x}_k\}_{k=0}^{\infty}$ converges to $\overline{x}$, then we have*

$$\|x - \overline{x}\|_B \leq \frac{5M}{\Big(1 - \Delta\Big)}, \tag{25}$$

where

$$M = \sum_{j=0}^{n-1} \frac{\|c_j - d_j\|}{j!} \; (b-a)^j.$$

Proof. Suppose the sequences $\{x_k\}_{k=0}^{\infty}$ and $\{\overline{x}_k\}_{k=0}^{\infty}$ generated by new three steps iteration method associated with operators T in (12) and $\overline{T}$ in (24), respectively with the real control sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0,1]$ satisfying $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$. From iteration (6) and equations (12); (24) and assumptions, we obtain

$$
\begin{aligned}
&\|z_k(t) - \overline{z}_k(t)\| \\
&= \|(Tx_k)(t) - (\overline{T}\overline{x}_k)(t)\| \\
&= \| \sum_{j=0}^{n-1} \frac{c_j}{j!}\ (t-a)^j + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\Big(s, x_k(s), x_k(a), x_k(b)\Big) ds \\
&\quad - \sum_{j=0}^{n-1} \frac{d_j}{j!}\ (t-a)^j - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\Big(s, \overline{x}_k(s), \overline{x}_k(a), \overline{x}_k(b)\Big) ds \| \\
&\leq \sum_{j=0}^{n-1} \frac{\|c_j - d_j\|}{j!}\ (b-a)^j + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \\
&\qquad \times \|\mathcal{F}\Big(s, x_k(s), x_k(a), x_k(b)\Big) - \mathcal{F}\Big(s, \overline{x}_k(s), \overline{x}_k(a), \overline{x}_k(b)\Big)\| ds \\
&\leq M + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} p(s) \\
&\qquad \times \Big[\lambda \|x_k(s) - \overline{x}_k(s)\| + \beta \|x_k(a) - \overline{x}_k(a)\| + \gamma \|x_k(b) - \overline{x}_k(b)\|\Big] ds. \qquad (26)
\end{aligned}
$$

Now, by taking supremum in the inequality (26), we obtain

$$
\begin{aligned}
\|z_k - \overline{z}_k\|_B &\leq M + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} p(s) \Big(\lambda + \beta + \gamma\Big) \|x_k - \overline{x}_k\|_B ds \\
&\leq M + {I_a}^{\alpha} p(t) \Big(\lambda + \beta + \gamma\Big) \|x_k - \overline{x}_k\|_B \\
&= M + \Delta \|x_k - \overline{x}_k\|_B, \qquad (27)
\end{aligned}
$$

and

$$
\begin{aligned}
\|y_k(t) - \overline{y}_k(t)\| &= \|(1-\xi_k)\big(z_k(t) - \overline{z}_k(t)\big) + \xi_k\big((Tz_k)(t) - (\overline{T}\overline{z}_k)(t)\big)\| \\
&\leq \Big[(1-\xi_k)\|z_k(t) - \overline{z}_k(t)\| + \xi_k \|(Tz_k)(t) - (\overline{T}\overline{z}_k)(t)\|\Big]. \qquad (28)
\end{aligned}
$$

Hence, by taking supremum in the inequality (28) and then use the idea from (26) to get

$$
\begin{aligned}
\|y_k - \overline{y}_k\|_B &\leq (1-\xi_k)\|z_k - \overline{z}_k\|_B + \xi_k \|Tz_k - \overline{T}\overline{z}_k\|_B \\
&\leq (1-\xi_k)\|z_k - \overline{z}_k\|_B + \xi_k \Big[M + \Delta \|z_k - \overline{z}_k\|_B\Big] \\
&= \xi_k M + \Big[1 - \xi_k \Big(1 - \Delta\Big)\Big] \|z_k - \overline{z}_k\|_B \\
&\leq \xi_k M + \Big[1 - \xi_k \Big(1 - \Delta\Big)\Big] \Big[M + \Delta \|x_k - \overline{x}_k\|_B\Big] \\
&\leq \Delta \xi_k M + M + \Delta \Big[1 - \xi_k \Big(1 - \Delta\Big)\Big] \|x_k - \overline{x}_k\|_B. \qquad (29)
\end{aligned}
$$

Therefore, using the idea from (26) and (29) along with hypotheses $\Delta < 1$, and $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$, the resulting inequality becomes

$$
\|x_{k+1} - \overline{x}_{k+1}\|_B = \|Ty_k - \overline{T}\overline{y}_k\|_B
$$

$$
\begin{aligned}
&\le M + \Delta\|y_k - \overline{y}_k\|_B \\
&\le M + \|y_k - \overline{y}_k\|_B \\
&\le M + \Delta\xi_k M + M + \Delta\Big[1 - \xi_k\Big(1-\Delta\Big)\Big]\|x_k - \overline{x}_k\|_B \\
&\le 2M + \xi_k M + \Big[1 - \xi_k\Big(1-\Delta\Big)\Big]\|x_k - \overline{x}_k\|_B \\
&\le 2\xi_k(2M) + \xi_k M + \Big[1 - \xi_k\Big(1-\Delta\Big)\Big]\|x_k - \overline{x}_k\|_B \\
&\le \Big[1 - \xi_k\Big(1-\Delta\Big)\Big]\|x_k - \overline{x}_k\|_B + \xi_k\Big(1-\Delta\Big)\frac{5M}{\Big(1-\Delta\Big)}. \qquad (30)
\end{aligned}
$$

We denote

$$
\begin{aligned}
\beta_k &= \|x_k - \overline{x}_k\|_B \ge 0, \\
\mu_k &= \xi_k\Big(1-\Delta\Big) \in (0,1), \\
\gamma_k &= \frac{5M}{\Big(1-\Delta\Big)} \ge 0.
\end{aligned}
$$

The assumption $\frac{1}{2} \le \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$ implies $\sum_{k=0}^{\infty} \xi_k = \infty$. Now, it can be easily seen that (30) satisfies all the conditions of Lemma 2.2 and hence we have

$$
\begin{aligned}
&0 \le \limsup_{k\to\infty} \beta_k \le \limsup_{k\to\infty} \gamma_k \\
\Rightarrow\ &0 \le \limsup_{k\to\infty} \|x_k - \overline{x}_k\|_B \le \limsup_{k\to\infty} \frac{5M}{\Big(1-\Delta\Big)} \\
\Rightarrow\ &0 \le \limsup_{k\to\infty} \|x_k - \overline{x}_k\|_B \le \frac{5M}{\Big(1-\Delta\Big)}. \qquad (31)
\end{aligned}
$$

Using the assumptions $\lim_{k\to\infty} x_k = x$, $\lim_{k\to\infty} \overline{x}_k = \overline{x}$, we get from (31) that

$$
\|x - \overline{x}\|_B \le \frac{5M}{\Big(1-\Delta\Big)}, \qquad (32)
$$

which shows that the dependency of solutions of IVPs (1)-(2) and (1) with the conditions (23) on given initial data. □

4.2. **Closeness of solutions.** Consider the problem (1)-(2) and the corresponding problem

$$
(D^{\alpha}_{*a})\overline{x}(t) = \overline{\mathcal{F}}\Big(t, \overline{x}(t), \overline{x}(a), \overline{x}(b)\Big), \qquad (33)
$$

for $t \in I = [a,b],\ n-1 < \alpha \le n\ (n \in \mathbb{N})$, with the given initial conditions

$$
\overline{x}^{(j)}(a) = d_j,\ j = 0,1,2,\cdots,n-1, \qquad (34)
$$

where $\overline{\mathcal{F}}$ is defined as $\mathcal{F}$ and d_j $(j = 0,1,2,\ldots,n-1)$ are given elements in X.

Then looking at the steps as in the proof of Theorem 3.3, we define the operator for the equations (33)- (34)

$$(\overline{T}\overline{x})(t) = \sum_{j=0}^{n-1} \frac{d_j}{j!}\ (t-a)^j + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \overline{\mathcal{F}}\Big(s, \overline{x}(s), \overline{x}(a), \overline{x}(b)\Big) ds,\ t \in I. \tag{35}$$

The next theorem deals with the closeness of solutions of the problems (1)-(2) and (33)-(34).

Theorem 4.5. *Consider the sequences* $\{x_k\}_{k=0}^{\infty}$ *and* $\{\overline{x}_k\}_{k=0}^{\infty}$ *generated by new three steps iteration method associated with operators* T *in* (12) *and* $\overline{T}$ *in* (35), *respectively with the real sequence* $\{\xi_k\}_{k=0}^{\infty}$ *in* $[0,1]$ *satisfying* $\frac{1}{2} \leq \xi_k$ *for all* $k \in \mathbb{N} \cup \{0\}$. *Assume that*

(i) *all conditions of Theorem 3.3 hold, and* $x(t)$ *and* $\overline{x}(t)$ *are solutions of* (1)-(2) *and* (33)-(34) *respectively.*

(ii) *there exist non negative constant* ϵ *such that*

$$\|\mathcal{F}\Big(t, u_1, u_2, u_3\Big) - \overline{\mathcal{F}}\Big(t, u_1, u_2, u_3\Big)\| \leq \epsilon,\ \forall\ t \in I. \tag{36}$$

If the sequence $\{\overline{x}_k\}_{k=0}^{\infty}$ *converges to* $\overline{x}$, *then we have*

$$\|x - \overline{x}\|_B \leq \frac{5\Big[M + \frac{\epsilon(b-a)^{\alpha}}{\Gamma(\alpha+1)}\Big]}{\Big(1-\Delta\Big)}. \tag{37}$$

Proof. Suppose the sequences $\{x_k\}_{k=0}^{\infty}$ and $\{\overline{x}_k\}_{k=0}^{\infty}$ generated by new three steps iteration method associated with operators T in (12) and $\overline{T}$ in (35), respectively with the real control sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0,1]$ satisfying $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$. From iteration (6) and equations (12); (35) and hypotheses, we obtain

$$\begin{aligned}
&\|z_k(t) - \overline{z}_k(t)\| \\
&= \|(Tx_k)(t) - (\overline{T}\overline{x}_k)(t)\| \\
&= \|\sum_{j=0}^{n-1} \frac{c_j}{j!}\ (t-a)^j + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\Big(s, x_k(s), x_k(a), x_k(b)\Big) ds \\
&\quad - \sum_{j=0}^{n-1} \frac{d_j}{j!}\ (t-a)^j - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \overline{\mathcal{F}}\Big(s, \overline{x}_k(s), \overline{x}_k(a), \overline{x}_k(b)\Big) ds\| \\
&\leq \sum_{j=0}^{n-1} \frac{\|c_j - d_j\|}{j!}\ (b-a)^j + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \\
&\qquad \times \|\mathcal{F}\Big(s, x_k(s), x_k(a), x_k(b)\Big) - \overline{\mathcal{F}}\Big(s, \overline{x}_k(s), \overline{x}_k(a), \overline{x}_k(b)\Big)\| ds \\
&\leq M + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \\
&\qquad \times \|\mathcal{F}\Big(s, x_k(s), x_k(a), x_k(b)\Big) - \mathcal{F}\Big(s, \overline{x}_k(s), \overline{x}_k(a), \overline{x}_k(b)\Big)\| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \|\mathcal{F}\Big(s, \overline{x}_k(s), \overline{x}_k(a), \overline{x}_k(b)\Big) - \overline{\mathcal{F}}\Big(s, \overline{x}_k(s), \overline{x}_k(a), \overline{x}_k(b)\Big)\| ds
\end{aligned}$$

$$
\begin{aligned}
&\leq M+\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\epsilon ds+\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}p(s)\\
&\quad\times\Big[\lambda\|x_k(s)-\overline{x}_k(s)\|+\beta\|x_k(a)-\overline{x}_k(a)\|+\gamma\|x_k(b)-\overline{x}_k(b)\|\Big]ds\\
&\leq M+\frac{\epsilon(t-a)^\alpha}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}p(s)\\
&\quad\times\Big[\lambda\|x_k(s)-\overline{x}_k(s)\|+\beta\|x_k(a)-\overline{x}_k(a)\|+\gamma\|x_k(b)-\overline{x}_k(b)\|\Big]ds\\
&\leq M+\frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}p(s)\\
&\quad\times\Big[\lambda\|x_k(s)-\overline{x}_k(s)\|+\beta\|x_k(a)-\overline{x}_k(a)\|+\gamma\|x_k(b)-\overline{x}_k(b)\|\Big]ds. \qquad (38)
\end{aligned}
$$

Recalling the derivations obtained in equations (27) and (29), the above inequality becomes

$$
\|z_k-\overline{z}_k\|_B\leq M+\frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)}+\Delta\|x_k-\overline{x}_k\|_B, \qquad (39)
$$

and similarly, it is seen that

$$
\begin{aligned}
\|y_k-\overline{y}_k\|_B\leq{}& \Delta\xi_k\Big[M+\frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)}\Big]+\Big[M+\frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)}\Big]\\
&+\Delta\Big[1-\xi_k\Big(1-\Delta\Big)\Big]\|x_k-\overline{x}_k\|_B. \qquad (40)
\end{aligned}
$$

Therefore, using the idea from (38) and (40) along with hypotheses $\Delta<1$, and $\frac{1}{2}\leq\xi_k$ for all $k\in\mathbb{N}\cup\{0\}$, the resulting inequality becomes

$$
\begin{aligned}
&\|x_{k+1}-\overline{x}_{k+1}\|_B\\
&=\|Ty_k-\overline{T}\overline{y}_k\|_B\\
&\leq M+\frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)}+\Delta\|y_k-\overline{y}_k\|_B\\
&\leq\Big[M+\frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)}\Big]+\|y_k-\overline{y}_k\|_B\\
&\leq\Big[M+\frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)}\Big]+\Delta\xi_k\Big[M+\frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)}\Big]+\Big[M+\frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)}\Big]\\
&\quad+\Delta\Big[1-\xi_k\Big(1-\Delta\Big)\Big]\|x_k-\overline{x}_k\|_B\\
&\leq 4\xi_k\Big[M+\frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)}\Big]+\xi_k\Big[M+\frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)}\Big]+\Big[1-\xi_k\Big(1-\Delta\Big)\Big]\|x_k-\overline{x}_k\|_B\\
&\leq\Big[1-\xi_k\Big(1-\Delta\Big)\Big]\|x_k-\overline{x}_k\|_B+\xi_k\Big(1-\Delta\Big)\frac{5\Big[M+\frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)}\Big]}{\Big(1-\Delta\Big)}. \qquad (41)
\end{aligned}
$$

We denote

$$
\begin{aligned}
\beta_k&=\|x_k-\overline{x}_k\|_B\geq 0,\\
\mu_k&=\xi_k\Big(1-\Delta\Big)\in(0,1),
\end{aligned}
$$

$$\gamma_k = \frac{5\Big[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)}\Big]}{\Big(1-\Delta\Big)} \geq 0.$$

The assumption $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$ implies $\sum_{k=0}^{\infty} \xi_k = \infty$. Now, it can be easily seen that (41) satisfies all the conditions of Lemma 2.2 and hence we have

$$\begin{aligned} & 0 \leq \limsup_{k\to\infty} \beta_k \leq \limsup_{k\to\infty} \gamma_k \\ & \Rightarrow 0 \leq \limsup_{k\to\infty} \|x_k - \overline{x}_k\|_B \leq \limsup_{k\to\infty} \frac{5\Big[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)}\Big]}{\Big(1-\Delta\Big)} \\ & \Rightarrow 0 \leq \limsup_{k\to\infty} \|x_k - \overline{x}_k\|_B \leq \frac{5\Big[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)}\Big]}{\Big(1-\Delta\Big)}. \end{aligned} \tag{42}$$

Using the assumptions $\lim_{k\to\infty} x_k = x$, $\lim_{k\to\infty} \overline{x}_k = \overline{x}$, we get from (42) that

$$\|x - \overline{x}\|_B \leq \frac{5\Big[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)}\Big]}{\Big(1-\Delta\Big)}, \tag{43}$$

which shows that the dependency of solutions of IVP (1)-(2) on both the function involved from the right hand side of the given equation and initial data. □

Remark: The inequality (43) relates the solutions of the problems (1)-(2) and (33)-(34) in the sense that, if $\mathcal{F}$ and $\overline{\mathcal{F}}$ are close as $\epsilon \to 0$, then not only the solutions of the problems (1)-(2) and (33)-(34) are close to each other (i.e. $\|x - \overline{x}\|_B \to 0$), but also depends continuously on the functions involved therein and initial data.

4.3. **Dependence on Parameters.** We next consider the following problems

$$\big(D^\alpha_{*a}\big)x(t) = \mathcal{F}\Big(t, x(t), x(a), x(b), \mu_1\Big), \tag{44}$$

for $t \in I = [a, b]$, $n-1 < \alpha \leq n$ $(n \in \mathbb{N})$, with the given initial conditions

$$x^{(j)}(a) = c_j, \ j = 0, 1, 2, \cdots, n-1, \tag{45}$$

and

$$\big(D^\alpha_{*a}\big)\overline{x}(t) = \mathcal{F}\Big(t, \overline{x}(t), \overline{x}(a), \overline{x}(b), \mu_2\Big), \tag{46}$$

for $t \in I = [a, b]$, $n-1 < \alpha \leq n$ $(n \in \mathbb{N})$, with the given initial conditions

$$\overline{x}^{(j)}(a) = d_j, \ j = 0, 1, 2, \cdots, n-1, \tag{47}$$

where $\mathcal{F} : I \times X \times X \times X \times \mathbb{R} \to X$ is continuous function, c_j, d_j $(j = 0, 1, 2, \ldots, n-1)$ are given elements in X and constants μ_1, μ_2 are real parameters.

Let $x(t),\ \overline{x}(t) \in B$ and following steps from the proof of Theorem 3.3, define the operators for the equations (44) and (46), respectively

$$(Tx)(t) = \sum_{j=0}^{n-1} \frac{c_j}{j!}\ (t-a)^j + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\Big(s, x(s), x(a), x(b), \mu_1\Big) ds,\ t \in I. \tag{48}$$

and

$$(\overline{T}\overline{x})(t) = \sum_{j=0}^{n-1} \frac{d_j}{j!}\ (t-a)^j + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\Big(s, \overline{x}(s), \overline{x}(a), \overline{x}(b), \mu_2\Big) ds,\ t \in I. \tag{49}$$

The following theorem proves the continuous dependency of solutions on parameters.

Theorem 4.6. *Consider the sequences* $\{x_k\}_{k=0}^{\infty}$ *and* $\{\overline{x}_k\}_{k=0}^{\infty}$ *generated by new three steps iteration method associated with operators* T *in* (48) *and* $\overline{T}$ *in* (49), *respectively with the real sequence* $\{\xi_k\}_{k=0}^{\infty}$ *in* $[0,1]$ *satisfying* $\frac{1}{2} \leq \xi_k$ *for all* $k \in \mathbb{N} \cup \{0\}$. *Assume that*

(i) $x(t)$ *and* $\overline{x}(t)$ *are solutions of* (44)-(45) *and* (46)-(47) *respectively.*

(ii) *there exist constants* $\overline{\lambda},\ \overline{\beta},\ \overline{\gamma} > 0$ *such that the function* $\mathcal{F}$ *satisfy the conditions:*

$$\|\mathcal{F}\Big(t, u_1, u_2, u_3, \mu_1\Big) - \mathcal{F}\Big(t, v_1, v_2, v_3, \mu_1\Big)\| \leq \overline{p}(t)\Big[\overline{\lambda}\|u_1 - v_1\| + \overline{\beta}\|u_2 - v_2\| + \overline{\gamma}\|u_3 - v_3\|\Big].$$

and

$$\|\mathcal{F}\Big(t, u_1, u_2, u_3, \mu_1\Big) - \mathcal{F}\Big(t, u_1, u_2, u_3, \mu_2\Big)\| \leq r(t)\Big|\mu_1 - \mu_2\Big|,$$

where $\overline{p},\ r \in C(I, \mathbb{R}_+)$.

If the sequence $\{\overline{x}_k\}_{k=0}^{\infty}$ *converges to* $\overline{x}$, *then we have*

$$\|x - \overline{x}\|_B \leq \frac{5\Big[M + |\mu_1 - \mu_2| {I_a}^{\alpha} r(t)\Big]}{\Big(1 - \overline{\Delta}\Big)}, \tag{50}$$

where $\overline{\Delta} = {I_a}^{\alpha}\overline{p}(t)\big(\overline{\lambda} + \overline{\beta} + \overline{\gamma}\big) < 1$ $(t \in I)$.

Proof. Suppose the sequences $\{x_k\}_{k=0}^{\infty}$ and $\{\overline{x}_k\}_{k=0}^{\infty}$ generated by new three steps iteration method associated with operators T in (48) and $\overline{T}$ in (49), respectively with the real control sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0,1]$ satisfying $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$. From iteration (6) and equations (48); (49) and hypotheses, we obtain

$$\begin{aligned} &\|z_k(t) - \overline{z}_k(t)\| \\ &= \|(Tx_k)(t) - (\overline{T}\overline{x}_k)(t)\| \\ &= \|\sum_{j=0}^{n-1} \frac{c_j}{j!}\ (t-a)^j + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\Big(s, x_k(s), x_k(a), x_k(b), \mu_1\Big) ds \end{aligned}$$

$$
\begin{aligned}
&- \sum_{j=0}^{n-1} \frac{d_j}{j!}\ (t-a)^j - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\Big(s, \overline{x}_k(s), \overline{x}_k(a), \overline{x}_k(b), \mu_2\Big) ds\| \\
&\le \sum_{j=0}^{n-1} \frac{\|c_j - d_j\|}{j!}\ (b-a)^j + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \\
&\quad \times \|\mathcal{F}\Big(s, x_k(s), x_k(a), x_k(b), \mu_1\Big) - \mathcal{F}\Big(s, \overline{x}_k(s), \overline{x}_k(a), \overline{x}_k(b), \mu_2\Big)\| ds \\
&\le M + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \\
&\quad \times \|\mathcal{F}\Big(s, x_k(s), x_k(a), x_k(b), \mu_1\Big) - \mathcal{F}\Big(s, \overline{x}_k(s), \overline{x}_k(a), \overline{x}_k(b), \mu_1\Big)\| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \\
&\quad \times \|\mathcal{F}\Big(s, \overline{x}_k(s), \overline{x}_k(a), \overline{x}_k(b), \mu_1\Big) - \mathcal{F}\Big(s, \overline{x}_k(s), \overline{x}_k(a), \overline{x}_k(b), \mu_2\Big)\| ds \\
&\le M + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} r(s) |\mu_1 - \mu_2| ds + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \overline{p}(s) \\
&\quad \times \Big[\overline{\lambda}\|x_k(s) - \overline{x}_k(s)\| + \overline{\beta}\|x_k(a) - \overline{x}_k(a)\| + \overline{\gamma}\|x_k(b) - \overline{x}_k(b)\|\Big] ds \\
&\le M + |\mu_1 - \mu_2| {I_a}^{\alpha} r(t) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \overline{p}(s) \\
&\quad \times \Big[\overline{\lambda}\|x_k(s) - \overline{x}_k(s)\| + \overline{\beta}\|x_k(a) - \overline{x}_k(a)\| + \overline{\gamma}\|x_k(b) - \overline{x}_k(b)\|\Big] ds \\
&\le M + |\mu_1 - \mu_2| {I_a}^{\alpha} r(t) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \overline{p}(s) \\
&\quad \times \Big[\overline{\lambda}\|x_k(s) - \overline{x}_k(s)\| + \overline{\beta}\|x_k(a) - \overline{x}_k(a)\| + \overline{\gamma}\|x_k(b) - \overline{x}_k(b)\|\Big] ds. \qquad (51)
\end{aligned}
$$

Recalling the derivations obtained in equations (27) and (29), the above inequality becomes

$$
\|z_k - \overline{z}_k\|_B \le M + |\mu_1 - \mu_2| {I_a}^{\alpha} r(t) + \overline{\Delta}\|x_k - \overline{x}_k\|_B, \qquad (52)
$$

and similarly, it is seen that

$$
\begin{aligned}
\|y_k - \overline{y}_k\|_B \le \overline{\Delta}\xi_k \Big[M + |\mu_1 - \mu_2| {I_a}^{\alpha} r(t)\Big] + \Big[M + |\mu_1 - \mu_2| {I_a}^{\alpha} r(t)\Big] \\
+ \overline{\Delta}\Big[1 - \xi_k\Big(1 - \overline{\Delta}\Big)\Big] \|x_k - \overline{x}_k\|_B. \qquad (53)
\end{aligned}
$$

Therefore, using the idea from (51) and (53) along with hypotheses $\Delta < 1$, and $\frac{1}{2} \le \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$, the resulting inequality becomes

$$
\begin{aligned}
&\|x_{k+1} - \overline{x}_{k+1}\|_B \\
&= \|Ty_k - \overline{T}\overline{y}_k\|_B \\
&\le \Big[M + |\mu_1 - \mu_2| {I_a}^{\alpha} r(t)\Big] + \overline{\Delta}\|y_k - \overline{y}_k\|_B \\
&\le \Big[M + |\mu_1 - \mu_2| {I_a}^{\alpha} r(t)\Big] + \|y_k - \overline{y}_k\|_B
\end{aligned}
$$

$$\leq \Big[1-\xi_k\Big(1-\overline{\Delta}\Big)\Big]\|x_k-\overline{x}_k\|_B+\xi_k\Big(1-\overline{\Delta}\Big)\frac{5\Big[M+|\mu_1-\mu_2|{I_a}^{\alpha}r(t)\Big]}{\Big(1-\overline{\Delta}\Big)}. \tag{54}$$

We denote

$$\begin{aligned}\beta_k &= \|x_k-\overline{x}_k\|_B \geq 0,\\ \mu_k &= \xi_k\Big(1-\overline{\Delta}\Big)\in(0,1),\\ \gamma_k &= \frac{5\Big[M+|\mu_1-\mu_2|{I_a}^{\alpha}r(t)\Big]}{\Big(1-\overline{\Delta}\Big)}\geq 0.\end{aligned}$$

The assumption $\frac{1}{2}\leq\xi_k$ for all $k\in\mathbb{N}\cup\{0\}$ implies $\sum_{k=0}^{\infty}\xi_k=\infty$. Now, it can be easily seen that (54) satisfies all the conditions of Lemma 2.2 and hence we have

$$\begin{aligned}&0\leq\limsup_{k\to\infty}\beta_k\leq\limsup_{k\to\infty}\gamma_k\\ &\Rightarrow 0\leq\limsup_{k\to\infty}\|x_k-\overline{x}_k\|_B\leq\limsup_{k\to\infty}\frac{5\Big[M+|\mu_1-\mu_2|{I_a}^{\alpha}r(t)\Big]}{\Big(1-\overline{\Delta}\Big)}\\ &\Rightarrow 0\leq\limsup_{k\to\infty}\|x_k-\overline{x}_k\|_B\leq\frac{5\Big[M+|\mu_1-\mu_2|{I_a}^{\alpha}r(t)\Big]}{\Big(1-\overline{\Delta}\Big)}.\end{aligned} \tag{55}$$

Using the assumptions, $\lim_{k\to\infty}x_k=x$, $\lim_{k\to\infty}\overline{x}_k=\overline{x}$, we get from (55) that

$$\|x-\overline{x}\|_B\leq\frac{5\Big[M+|\mu_1-\mu_2|{I_a}^{\alpha}r(t)\Big]}{\Big(1-\overline{\Delta}\Big)}, \tag{56}$$

which shows the dependence of solutions of the problem (1)-(2) on parameters μ_1 and μ_2. □

Remark: The result deals with the property of a solution called "dependence of solutions on parameters". Here the parameters are scalars and also note that the initial conditions do not involve parameters. The dependence on parameters is an important aspect in various physical problems.

5. Example

We consider the following problem:

$$\big(D_*^{\alpha}\big)x(t)=\frac{3t^2}{5}\Big[\frac{t^2-\sin^2(x(t))}{2}+\frac{\cos(x(0))+\cos(x(1))}{3}\Big], \tag{57}$$

for $t\in[0,1],\ 2<\alpha=\frac{5}{2}\leq 3$, with the given initial conditions

$$x(0)=0,\ x'(0)=0,x''(0)=1, \tag{58}$$

Comparing this equation with the equation (1), we get $\mathcal{F} \in C(I \times \mathbb{R}^3, \mathbb{R})$ with $c_0 = 0,\ c_1 = 0,\ c_2 = 1$ and

$$\mathcal{F}(t, x(t), x(0), x(1)) = \frac{3t^2}{5}\Big[\frac{t^2 - \sin^2(x(t))}{2} + \frac{\cos(x(0)) + \cos(x(1))}{3}\Big].$$

Now, we have

$$\begin{aligned}
&\Big|\mathcal{F}(t, x(t), x(0), x(1)) - \mathcal{F}(t, \overline{x}(t), \overline{x}(0), \overline{x}(1))\Big| \\
&\leq \Big|\frac{3t^2}{5}\Big|\Big[\Big|\frac{t^2 - \sin^2(x(t))}{2} - \frac{t^2 - \sin^2(\overline{x}(t))}{2}\Big| \\
&\quad + \Big|\frac{\cos(x(0)) + \cos(x(1))}{3} - \frac{\cos(\overline{x}(0)) + \cos(\overline{x}(1))}{3}\Big|\Big] \\
&\leq \frac{3t^2}{5}\Big[\frac{2}{2}\Big|\sin(x(t)) - \sin(\overline{x}(t))\Big| + \frac{1}{3}\Big|\cos(x(0)) - \cos(\overline{x}(0))\Big| \\
&\quad + \frac{1}{3}\Big|\cos(x(1)) - \cos(\overline{x}(1))\Big|\Big] \\
&\leq \frac{3t^2}{5}\Big[\Big|(x(t) - \overline{x}(t)\Big| + \frac{1}{3}\Big|x(0) - \overline{x}(0)\Big| + \frac{1}{3}\Big|x(1) - \overline{x}(1)\Big|\Big]. \qquad (59)
\end{aligned}$$

Taking sup norm, we obtain

$$|\mathcal{F}(t, x(t), x(0), x(1)) - \mathcal{F}(t, \overline{x}(t), \overline{x}(0), \overline{x}(1))| \leq \frac{3t^2}{5}\Big(1 + \frac{1}{3} + \frac{1}{3}\Big)|x - \overline{x}|, \qquad (60)$$

where $p(t) = \frac{3t^2}{5}, \quad \lambda = 1, \quad \beta = \frac{1}{3}, \quad \gamma = \frac{1}{3}$ and hence the condition (10) holds.

5.1. **Existence and Uniqueness.** Therefore, we the estimate Δ for the given value of $\alpha = \frac{5}{2}$:

$$\begin{aligned}
\Delta &= {I_a}^{\alpha} p(t)\big(\lambda + \beta + \gamma\big) \\
&= {I_0}^{\alpha}\frac{3t^2}{5}\Big(1 + \frac{1}{3} + \frac{1}{3}\Big) \\
&= \frac{3}{5}\Big(1 + \frac{1}{3} + \frac{1}{3}\Big)({I_0}^{\alpha})(t^2) \\
&= ({I_0}^{\alpha})(t^2) \\
&= \frac{2t^{\alpha+2}}{\Gamma(\alpha + 3)} \\
&= \frac{2t^{\frac{9}{2}}}{\Gamma(\frac{11}{2})} \\
&\leq \frac{2}{\Gamma(\frac{11}{2})},\ (t \leq 1) \\
&\leq \frac{64}{945\sqrt{\pi}} \\
&\simeq 0.03821 \\
&< 1. \qquad (61)
\end{aligned}$$

We define the operator $T : B \to B$ by

$$(Tx)(t) = \frac{t^2}{2} + \frac{1}{\Gamma(\frac{5}{2})}\int_0^t (t - s)^{\frac{3}{2}}\frac{3s^2}{5}$$

$$\times\Big[\frac{s^2-\sin^2(x(s))}{2}+\frac{\cos(x(0))+\cos(x(1))}{3}\Big]ds,\ t\in I. \tag{62}$$

Since all the conditions of Theorem 3.3 are satisfied and so by its conclusion, the sequence $\{x_k\}$ associated with the iterative method (6) for the operator T in (62) converges to a unique solution $x \in B$.

This convergence under a new three steps iteration process is faster than the S-iteration, Picard, Mann and Ishikawa iteration processes.

Now, we will discuss the simplicity and fastness of the new three steps iteration method. By refereing [10, 24, 26, 33], the definitions of a_k, b_k, c_k, d_k and e_k under S-iteration, Picard iteration, Mann iteration, Ishikawa iteration and a new three steps iteration are given, respectively:

(a) $a_k = \nu^k\Big[1-(1-\nu)\alpha\beta\Big]^k\|u_1 - x^*\|$,
(b) $b_k = \nu^k\|u_1 - x^*\|$,
(c) $c_k = \Big[1-(1-\nu)\beta\Big]^k\|u_1 - x^*\|$,
(d) $d_k = \Big[1-(1-\nu)^2\beta\Big]^k\|u_1 - x^*\|$,
(e) $e_k = \nu^{2k}\Big[1-(1-\nu)\alpha\beta\Big]^k\|u_1 - x^*\|$,

where $\nu \in [0,1)$ is contracting factor. For given $u_1 \in \mathbb{R}$, the convergence of sequences $\{a_k\}$, $\{b_k\}$, $\{c_k\}$, $\{d_k\}$ and $\{e_k\}$ depend only on the factors $\Delta_1 = \nu^k\Big[1-(1-\nu)\alpha\beta\Big]^k$, $\Delta_2 = \nu^k$, $\Delta_3 = \Big[1-(1-\nu)\beta\Big]^k$, $\Delta_4 = \Big[1-(1-\nu)^2\beta\Big]^k$ and $\Delta_5 = \nu^{2k}\Big[1-(1-\nu)\alpha\beta\Big]^k$ respectively. Therefore, the following comparison table shows the values of the factors Δ_1, Δ_2, Δ_3, Δ_4 and Δ_5 under respective iteration processes for the numerical example discussed in this paper with $\nu = \Delta = 0.0382096649$ and $\xi_k = \alpha_k = \beta_k = \frac{1}{2}$:

Table 1. Comparison Table

Iteration (k)	S-iteration (Δ_1)	P-iteration (Δ_2)	M-iteration (Δ_3)	I-iteration (Δ_4)	3 steps-iteration (Δ_5)
1	0.029022243	0.038209665	0.519104832	0.537479676	0.00110893
2	0.000842291	0.001459978	0.269469827	0.288884402	0.00000123
3	0.000024445	0.000055785	0.139883089	0.155269495	0.000000001
4	0.000000709	0.000002132	0.072613988	0.083454198	0
5	0.000000021	0.000000081	0.037694272	0.044854935	0
6	0.000000001	0.000000003	0.019567279	0.024108616	0
7	0	0	0.010157469	0.012957891	0
8	0	0	0.005272791	0.006964603	0
9	0	0	0.002737131	0.003743333	0
10	0	0	0.001420858	0.002011965	0
⋮	⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮	⋮
28	0	0	0.000000011	0.000000028	0
29	0	0	0.000000006	0.000000015	0
30	0	0	0.000000003	0.000000008	0
31	0	0	0.000000001	0.000000004	0
32	0	0	0.000000001	0.000000002	0
33	0	0	0	0.000000001	0
34	0	0	0	0.000000001	0
35	0	0	0	0	0

Hence, observing the above table and Definitions 2.4, 2.5, it is easy to see that $\lim_{k\to\infty} \frac{e_k}{a_k} = 0$, $\lim_{k\to\infty} \frac{e_k}{b_k} = 0$, $\lim_{k\to\infty} \frac{e_k}{c_k} = 0$ and $\lim_{k\to\infty} \frac{e_k}{d_k} = 0$. Therefore, we conclude that the new three steps iteration process is faster than the S-iteration, Picard, Mann and Ishikawa iteration processes. The following is the graphical presentations of the above table:

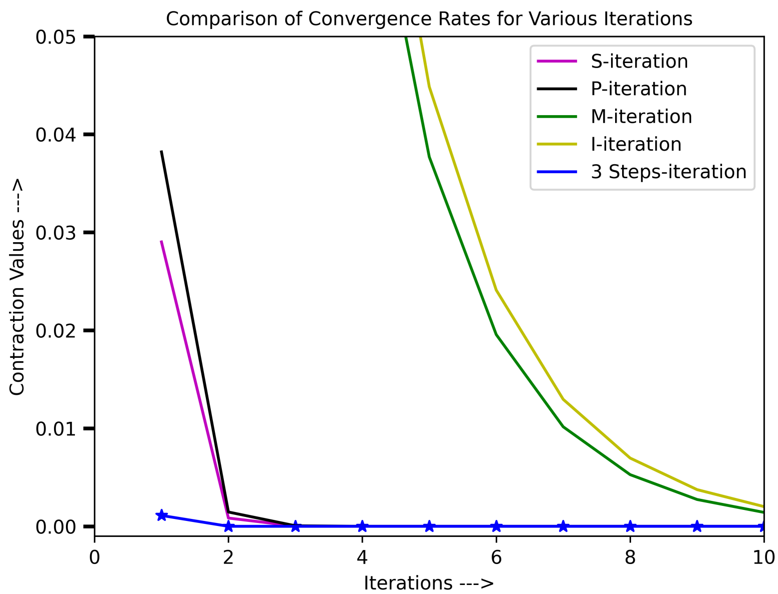

FIGURE 1. Comparison of rate of convergence

FIGURE 1 shows that the three-steps iteration scheme reaches a fixed point at the 4^{th} step, whereas the S, Picard, Mann, and Ishikawa iterations do so at the 7^{th}, 7^{th}, 33^{rd} and 35^{th} steps, respectively.

5.2. **Error Estimate.** Further, we also have for any $x_0 \in B$

$$\begin{aligned} \|x_{k+1} - x\|_B &\leq \frac{\Delta^{2k+2}}{e^{\left(1-\Delta\right)\sum_{i=0}^{k}\xi_i}} \|x_0 - x\|_B \\ &\leq \frac{\left[\frac{2}{\Gamma(\alpha+3)}\right]^{2k+2}}{e^{\left[1-\frac{2}{\Gamma(\alpha+3)}\right]\sum_{i=0}^{k}\xi_i}} \|x_0 - x\| \\ &\leq \frac{\left(\frac{2}{\Gamma(\alpha+3)}\right)^{2k+2}}{e^{\left(1-\frac{2}{\Gamma(\alpha+3)}\right)\sum_{i=0}^{k}\frac{1}{1+i}}} \|x_0 - x\|, \end{aligned} \tag{63}$$

where we have chosen $\xi_i = \frac{1}{1+i} \in [0,1]$. The estimate obtained in (63) is called a bound for the error (due to truncation of computation at the $k-$th iteration).

5.3. **Continuous dependence.** One can check easily the continuous dependence of solutions of equation (1) on initial data. Indeed, for $c_0 = c_1 = d_0 = d_1 = 0,\ c_2 = 1,\ d_2 = \frac{1}{2}$, we have

$$\begin{aligned}\|x-\overline{x}\|_B &\le \frac{5M}{\Big(1-\Delta\Big)}\\ &\le \frac{5\sum_{j=0}^{2}\frac{\|c_j-d_j\|}{j!}\ (b-a)^j}{\Big(1-\Delta\Big)}\\ &\le \frac{5\frac{1-\frac{1}{2}}{2!}}{\Big(1-\frac{2}{\Gamma(\frac{11}{2})}\Big)}\\ &\le \frac{\frac{5}{4}}{\Big(1-0.0382096649\Big)}\\ &\le \frac{5}{3.84716134}\\ &\simeq 1.2997. \end{aligned}\tag{64}$$

5.4. **Closeness of Solutions.** Next, we consider the perturbed equation:

$${}^cD^{\alpha}\overline{x}(t) = \frac{3t^2}{5}\Big[\frac{t^2-\sin^2(\overline{x}(t))}{2} + \frac{\cos(\overline{x}(0))+\cos(\overline{x}(1))}{3} - t^2 + \frac{1}{7}\Big], \tag{65}$$

$t \in [0,1],\ 2 < \alpha = \frac{5}{2} \le 3$, with the given initial conditions

$$\overline{x}(0) = 0,\ \overline{x}'(0) = 0,\ \overline{x}''(0) = \frac{1}{2}. \tag{66}$$

Similarly, comparing it with the equation (33), we have

$$\overline{\mathcal{F}}(t,\overline{x}(t),\overline{x}(0),\overline{x}(1)) = \frac{3t^2}{5}\Big[\frac{t^2-\sin^2(\overline{x}(t))}{2} + \frac{\cos(\overline{x}(0))+\cos(\overline{x}(1))}{3} - t^2 + \frac{1}{7}\Big].$$

One can easily define the mapping $\overline{T} : B \to B$ by

$$\begin{aligned}(\overline{T}\overline{x})(t) = \frac{t^2}{4} + \frac{1}{\Gamma(\frac{5}{2})}\int_0^t (t-s)^{\frac{3}{2}}\frac{3s^2}{5}\\ \times\Big[\frac{s^2-\sin^2(\overline{x}(s))}{2} + \frac{\cos(\overline{x}(0))+\cos(\overline{x}(1))}{3} - s^2 + \frac{1}{7}\Big]ds,\ t \in I.\end{aligned}\tag{67}$$

In perturbed equation, all conditions of Theorem 3.3 are also satisfied and so by its conclusion, the sequence $\{\overline{x}_k\}$ associated with the new three steps iterative method (6) for the operator $\overline{T}$ in (67) converges to a unique solution $\overline{x} \in B$.
Now, we have the following estimate:

$$\begin{aligned}&|\mathcal{F}(t,x(t),x(0),x(1)) - \overline{\mathcal{F}}(t,x(t),x(0),x(1))|\\ &= |\frac{3t^2}{5}\Big[\frac{t^2-\sin^2(x(t))}{2} + \frac{\cos(x(0))+\cos(x(1))}{3}\Big]\\ &\quad - \frac{3t^2}{5}\Big[\frac{t^2-\sin^2(x(t))}{2} + \frac{\cos(x(0))+\cos(x(1))}{3} - t^2 + \frac{1}{7}\Big]|\\ &= |\frac{3t^2}{5}||t^2 - \frac{1}{7}|\end{aligned}$$

$$\leq \frac{3}{5}\big(1+\frac{1}{7}\big) \qquad (t \leq 1)$$
$$= \frac{24}{35} = \epsilon. \tag{68}$$

Consider the sequences $\{x_k\}_{k=0}^{\infty}$ with $x_k \to x$ as $k \to \infty$ and $\{\overline{x}_k\}_{k=0}^{\infty}$ with $\overline{x}_k \to \overline{x}$ as $k \to \infty$ generated by new three steps iteration method associated with operators T in (62) and $\overline{T}$ in (67), respectively with the real sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0,1]$ satisfying $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$. Then we have from Theorem 4.4 that

$$\begin{aligned}
\|x-\overline{x}\|_B &\leq \frac{5\Big[M + \frac{\epsilon(b-a)^{\alpha}}{\Gamma(\alpha+2)}\Big]}{\Big(1-\Delta\Big)} \\
&\leq \frac{\frac{5}{4} + 5 \times \frac{24}{35}\frac{1}{\Gamma(\frac{9}{2})}}{\Big(1-\frac{2}{\Gamma(\frac{11}{2})}\Big)} \\
&\leq \frac{\frac{5}{4} + 5 \times \frac{24}{35}\frac{32}{105\sqrt{\pi}}}{3.84716134} \\
&\leq \frac{\frac{5}{4} + \frac{768}{735\sqrt{\pi}}}{3.84716134} \\
&\leq \frac{\frac{5}{4} + 0.589520545}{3.84716134} \\
&\leq \frac{1.83952055}{3.84716134} \\
&\simeq 0.4782.
\end{aligned} \tag{69}$$

This shows that the closeness of solutions and dependency of solutions on functions involved therein.

5.5. **Dependence on Parameters.** Finally, we shall prove the dependency of solutions on real parameters.
We consider the following integral equations involving real parameters $\mu_1,\ \mu_2$:

$${}^cD^{\alpha}x(t) = \frac{3t^2}{5}\Big[\frac{t^2-\sin^2(x(t))}{2} + \frac{\cos(x(0))+\cos(x(1))}{3} + \mu_1\Big], \tag{70}$$

and

$${}^cD^{\alpha}\overline{x}(t) = \frac{3t^2}{5}\Big[\frac{t^2-\sin^2(\overline{x}(t))}{2} + \frac{\cos(\overline{x}(0))+\cos(\overline{x}(1))}{3} + \mu_2\Big], \tag{71}$$

$t \in [0,1],\ 2 < \alpha = \frac{5}{2} \leq 3.$

Based on the above discussion, one can observe that $p(t) = \overline{p}(t) = r(t) = \frac{3t^2}{5}$ and therefore, we have $\Delta = \overline{\Delta}$. Hence by making similar arguments and from Theorem 4.6, one can have

$$\begin{aligned}
\|x-\overline{x}\|_B &\leq \frac{5\Big[M + |\mu_1-\mu_2| {I_a}^{\alpha} r(t)\Big]}{\Big(1-\overline{\Delta}\Big)} \\
&\leq \frac{5\Big[\frac{1}{4} + |\mu_1-\mu_2| {I_0}^{\alpha} r(t)\Big]}{\Big(1-\frac{2}{\Gamma(\alpha+3)}\Big)}
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{5\left[\frac{1}{4}+|\mu_1-\mu_2| I_0{}^{\frac{5}{2}}\left(\frac{3t^2}{5}\right)\right]}{\left(1-\frac{2}{\Gamma(\frac{11}{2})}\right)} \\
&\le \frac{\left[\frac{5}{4}+|\mu_1-\mu_2| 6\frac{t^{\frac{9}{2}}}{\Gamma(\frac{11}{2})}\right]}{\left(1-\frac{2}{\Gamma(\frac{11}{2})}\right)} \\
&\le \frac{\left[\frac{5}{4}+|\mu_1-\mu_2| \frac{6}{\Gamma(\frac{11}{2})}\right]}{3.84716134} \\
&\le \frac{\left[\frac{5}{4}+|\mu_1-\mu_2| \frac{96}{945\sqrt{\pi}}\right]}{3.84716134} \\
&\le \frac{\left[\frac{5}{4}+|\mu_1-\mu_2| \frac{96}{945\sqrt{\pi}}\right]}{3.84716134}. \qquad (72)
\end{aligned}
$$

In particular, if we choose $\mu_1 = \frac{9}{4}$ and $\mu_2 = \frac{9}{6}$, then the above inequality (72) takes the form

$$
\begin{aligned}
\|x-\overline{x}\|_B &\le \frac{\left[\frac{5}{4}+\frac{18}{24}\frac{96}{945\sqrt{\pi}}\right]}{3.84716134} \\
&\le \frac{\left[\frac{5}{4}+\frac{8}{105\sqrt{\pi}}\right]}{3.84716134} \\
&\le \frac{\left[1.25+0.042985873\right]}{3.84716134} \\
&\le \frac{1.29298587}{3.84716134} \\
&\simeq 0.3361 \qquad (73)
\end{aligned}
$$

6. Conclusions

Firstly, we proved the existence and uniqueness of the solution to the IVP (1)-(2) by a new three steps iterative approach as the main result. Further, we discussed various properties of solutions like continuous dependence on the initial data, closeness of solutions, and dependence on parameters and functions involved therein. Finally, we provided an appropriate example to support all of the findings along with the comparison table and graphical representation showing that a new three steps iteration method is faster than S-iteration, Picard, Mann and Ishikawa iteration processes.

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