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STOCHASTIC ITÔ-DIFFERENTIAL AND INTEGRAL OF FRACTIONAL-ORDERS

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ABSTRACT. The fractional calculus operators for the second order mean-square (continuous or Riemann integrable) stochastic processes have been discussed in some papers (see [5]-[11]). The combinations between this fractional calculus operators with the integer orders stochastic Itô-differential and stochastic Itô-integral can be found, recently, in [2]-[4]. In this paper, we introduce the definitions of the stochastic Itô-differential and the stochastic Itô-integral of fractional-orders. Some main properties will be proved. As applications some initial value problems of Itô-differential equations of fractional-orders will be studied in the classes $C([0,T],L_2(\Omega))$, $L_1([0,T],L_2(\Omega))$ and $L_2([0,T],L_2(\Omega))$.

1. Introduction

Let $I = [0,T]$. Let (Ω, F, P) be a fixed probability space, where Ω is a sample space, F is a σ-algebra and P is a probability measure. Let $X(t;\omega) = \{X(t), t \in I, \omega \in \Omega\}$ be a second order stochastic process, i.e., $E(X^2(t)) < \infty$, $t \in I$. Let $C(I, L_2(\Omega))$, $L_1(I, L_2(\Omega))$ and $L_2(I, L_2(\Omega))$ be the spaces of all mean square continuous, L_1 and L_2 mean square integrable second order stochastic processes on I. The norms of this Banach spaces are

$$||X||_C = \max_t \|X(t)\|_2, \quad \text{where} \quad \|X(t)\|_2 = (E(X^2(t)))^{\frac{1}{2}}$$

and

$$||X||_{L_1} = \int_0^T ||x(s)||_2 ds, \; ||X||_{L_2}^2 = \int_0^T ||x(s)||_2^2 ds$$

respectively.
Let $X(t)$ be a second order mean square continuous or Riemann integrable on $[0,T]$. The Riemann-Liouville fractional-order integral

$$I^\beta X(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s)\, ds, \; \beta \in (0,1] \tag{1}$$

and the fractional-order derivatives in the Riemann-Liouville and Caputo senses

$${}_*D^\alpha X(t) = {}^R D^\alpha X(t) = \frac{d}{dt} I^{1-\alpha} X(t) \text{ and } {}^C D^\alpha x(t) = I^{1-\alpha} \frac{d}{dt} X(t) \tag{2}$$

have been considered in [5]-[11].
Here, (in sec. 2) we give the definition and some of the main properties of the stochastic Itô-integral $F_\alpha f(t)$ and (in sec. 3) the stochastic Itô-differential ${}_*d^\alpha f(t)$ operators of fractional-order $\alpha \in (0,1)$ for the mean square continuous second order stochastic processes $\{f(t), t \in [0,T]\}$.
As an application, (in sec. 4) the existence of solutions of some problems of stochastic Itô-differential equations of fractional-orders will be studied.

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2. Fractional-order stochastic Itô-integral

For the the stochastic Itô-integral (see [9], [12] and [14]) we have

Definition 1. *Let* $f \in C(I, L_2(\Omega))$ *be a given second order mean square continuous processes. The stochastic Itô-integral of* f *with respect to the Brownian motion* w *is defined by*

$$F_1(f(t)) = \int_a^t f(s)dw(s) \tag{3}$$

Now, we can define the stochastic Itô-integral of fractional-order $\alpha \in (0,1)$ of the function $f \in C(I, L_2(\Omega))$ with respect to the Brownian motion w as follows

Definition 2. *Let* $f \in C(I, L_2(\Omega))$ *and* $\alpha \in (0,1)$. *We define the stochastic Itô-integral of fractional-order of the function* f *by*

$$F_\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)dw(s) \tag{4}$$

2.1. **Main properties.** For the main properties of the operator $F_\alpha f$, $f \in C(I, L_2(\Omega))$ we have the following lemmas.

Lemma 1. *A necessarily condition for the existence of the stochastic fractional-order Itô-integral* $F_\alpha f(t)$, $f \in C(I, L_2(\Omega))$ *is that* $\alpha > \frac{1}{2}$.

Proof. From definition 2, we have

$$\begin{aligned} ||F_\alpha f(t)||_2^2 &= ||\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)dw(s)||^2 \leq \int_0^t \frac{(t-s)^{2\alpha-2}}{\Gamma^2(\alpha)} ||f(s)||_2^2 ds, \ \alpha > 1/2 \\ &\leq \frac{||f||_C^2}{\Gamma^2(\alpha)} \left(- \frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)} \Big|_0^t \right) = \frac{t^{2\alpha-1}}{\Gamma(2\alpha)\Gamma^2(\alpha)} ||f||_C^2 \\ &\leq \frac{T^{2\alpha-1}}{\Gamma(2\alpha)\Gamma^2(\alpha)} ||f||_C^2 \ , \end{aligned}$$

then

$$||F_\alpha f(t)||_2 \leq \frac{T^{\alpha-\frac{1}{2}}}{\sqrt{\Gamma(2\alpha)}\Gamma(\alpha)} ||f||_C. \tag{5}$$

□

Lemma 2. *Let* $\alpha \in (\frac{1}{2}, 1)$ *and* $f \in C(I, L_2(\Omega))$. *Then*

$$\lim_{\alpha \to 1} F_\alpha f(t) = F_1 f(t) = \int_a^t f(s)dw(s).$$

Proof.

$$\begin{aligned} F_1 f(t) - F_\alpha f(t) &= \int_0^t \left(1 - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\right) f(s)dw(s), \\ ||F_1 f(t) - F_\alpha f(t)||_2^2 &\leq \int_0^t \left(1 - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\right)^2 ||f(s)||_2^2 ds \\ &\leq ||f||_C^2 \int_0^t \left(\frac{(t-s)^{1-\alpha} - 1}{\Gamma(\alpha)(t-s)^{1-\alpha}}\right)^2 ds. \end{aligned}$$

Let $\alpha = 1 - \frac{1}{n}$, then we have [13]

$$(t-s)^{\frac{1}{n}} \to 1 \text{ as } n \to \infty.$$

Now

$$0 \leq ||F_1 f(t) - F_\alpha f(t)||_2^2 \leq ||f||_C^2 \int_0^t \left(\frac{(t-s)^{\frac{1}{n}} - 1}{\Gamma(\alpha)(t-s)^{\frac{1}{n}}}\right)^2 ds.$$

Then we deduce that

$$\lim_{\alpha \to 1} ||F_1 f(t) - F_\alpha f(t)||_2 = 0$$

and

$$\lim_{\alpha\to 1} F_\alpha f(t) = F_1 f(t) = \int_a^t f(s)dw(s).$$

□

Lemma 3. *Let* $\alpha \in (\frac{1}{2}, 1)$, $\beta > 0$ *and* $f \in C(I, L_2(\Omega))$. *Then*

$$I^\beta\big(F_\alpha f(t)\big) = F_{\alpha+\beta} f(t).$$

Proof. Firstly, as in [9] and [14], we can formally write $dw(s) = \frac{dw(s)}{ds}$, then

$$\begin{aligned} I^\beta\big(F_\alpha f(t)\big) &= I^\beta \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\frac{dw(s)}{ds}ds = I^\beta I^\alpha f(t)\frac{dw(t)}{dt} = I^{\beta+\alpha} f(t)\frac{dw(t)}{dt} \\ &= \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(s)dw(s) = F_{\alpha+\beta} f(t). \end{aligned}$$

Now, from the existence of $F_{\alpha+\beta} f(t)$, we obtain the result. Also we have

$$||F_{\alpha+\beta} f(t)||_2^2 \le \frac{T^{2(\alpha+\beta)-1}}{\Gamma(2(\alpha+\beta))\Gamma^2(\alpha+\beta)}||f||_C^2$$

and

$$||F_{\alpha+\beta} f(t)||_2 \le \frac{T^{\alpha+\beta-\frac{1}{2}}}{\sqrt{\Gamma(2(\alpha+\beta))}\Gamma(\alpha+\beta)}||f||_C.$$

□

Lemma 4. *Let* $\alpha \in (1/2, 1)$. *If* $f \in C(I, L_2(\Omega))$ *or* $f \in L_2(I, L_2(\Omega))$, $||f||_2^2 \le k$, *then*

$$F_\alpha f(t)\Big|_{t=0} = 0.$$

Proof. From definition 2, we obtain

$$\begin{aligned} ||F_\alpha f(t)||_2^2 &= ||\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)dw(s)||^2 \le \int_0^t \frac{(t-s)^{2\alpha-2}}{\Gamma^2(\alpha)}||f(s)||_2^2 ds \\ &\le \frac{||f||_C^2}{\Gamma^2(\alpha)}\left(-\frac{(t-s)^{2\alpha-1}}{\Gamma(2\alpha)}\Big|_0^t\right) = \frac{t^{2\alpha-1}}{\Gamma(2\alpha)\Gamma^2(\alpha)}||f||_C^2\ , \end{aligned}$$

then

$$0 \le ||F_\alpha f(t)||_2 \le \frac{t^{\alpha-\frac{1}{2}}}{\sqrt{\Gamma(2\alpha)}\Gamma(\alpha)}||f||_C$$

and as $t \to 0$, then we get $F_\alpha f(t)\Big|_{t=0} = 0.$ □

The following corollary can be also proved

Corollary 1. *The results of Lemmas 1, 2, and 4 can be also obtain if* $f \in L_2(I, L_2(\Omega))$ *and* $||f||_2^2 \le k$.

Lemma 5. *Let* $\alpha + \beta > 1$, $f \in C(I, L_2(\Omega))$. *Then we have*

(i)

$$\frac{d}{dt} F_{\alpha+\beta} f(t) = F_{\alpha+\beta-1} f(t)$$

(ii)

$$I^{1-\beta}\frac{d}{dt} F_{\alpha+\beta} f(t) = F_\alpha f(t)$$

(iii)

$$I^{1-\alpha} F_\alpha f(t) = F_1 f(t)$$

Proof. (i) $F_{\alpha+\beta} f(t) = I^1 F_{\alpha+\beta-1} f(t)$, then

$$\frac{d}{dt} F_{\alpha+\beta} f(t) = \frac{d}{dt} I^1 F_{\alpha+\beta-1} f(t) = F_{\alpha+\beta-1} f(t).$$

(ii) $I^{1-\beta}\frac{d}{dt}F_{\alpha+\beta}f(t) = I^{1-\beta}F_{\alpha+\beta-1}f(t) = F_{\alpha+\beta-1+1-\beta}f(t) = F_{\alpha}f(t)$.
Then

$$I^{1-\beta}\frac{d}{dt}F_{\alpha+\beta}f(t) = F_{\alpha}f(t).$$

(iii) $I^{1-\alpha}F_{\alpha}f(t) = F_{\alpha+1-\alpha}f(t) = F_1 f(t)$,

□

Lemma 6.

$$F_\alpha : C(I, L_2(\Omega)) \to C(I, L_2(\Omega)).$$

Proof. Let $f \in C(I, L_2(\Omega))$, then

$$\begin{aligned} F_\alpha f(t_2) - F_\alpha f(t_1) &= \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)dw(s) - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)dw(s) \\ &= \int_0^{t_1} \frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)dw(s) + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)dw(s). \end{aligned}$$

But

$$\begin{aligned} \|\int_0^{t_1} \frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)dw(s)\|_2^2 &\le \|f\|_C^2 \Big(\int_0^{t_1} \Big(\frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)} \Big)^2 ds \\ &\le \|f\|_C^2 \Big(\int_0^{t_1} \Big(\frac{(t_1-s)^{1-\alpha} - (t_2-s)^{1-\alpha}}{\Gamma(\alpha)(t_1-s)^{1-\alpha}(t_2-s)^{1-\alpha}} \Big)^2 ds \end{aligned}$$

and

$$\|\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)dw(s)\|_2^2 \le \|f\|_C^2 \int_{t_1}^{t_2} \frac{(t_2-s)^{2\alpha-2}}{\Gamma^2(\alpha)} ds.$$

Then by (Lebesgue Theorem [1]) $\forall \epsilon > 0,\ \exists \delta > 0$ such that $|t_2 - t_1| < \delta$ implies that

$$\|F_\alpha f(t_2) - F_\alpha f(t_1)\|_2 < \epsilon.$$

□

Lemma 7.

$$F_\alpha : L_2(I, L_2(\Omega)) \to L_2(I, L_2(\Omega)).$$

Proof. For $X \in L_2([0,T], L_2(\Omega))$, we have

$$\begin{aligned} \|F_\alpha X(t)\|_2^2 &= \|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} X(s)dw(s)\|_2^2 \\ &\le \int_0^t \frac{(t-s)^{2\alpha-2}}{\Gamma^2(\alpha)} \|X(s)\|_2^2 ds \end{aligned}$$

and

$$\|F_\alpha g_1(t, X(t))\|_{L_2}^2 \le \int_0^T \|X(s)\|_2^2 \int_s^T \frac{(t-s)^{2\alpha-2}}{\Gamma^2(\alpha)} dt\ ds \le \|X\|_{L_2}^2\ \frac{T^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}.$$

Then

$$\|F_\alpha X(t)\|_{L_2} \le \|X\|_{L_2}\ \frac{T^{\alpha-1/2}}{\sqrt{(2\alpha-1)}\Gamma(\alpha)}.$$

□

3. Fractional-order stochastic Itô-differential

Let $\alpha \in (0,1)$ and $f \in C(I, L_2(\Omega))$. Then we can give the following definition

Definition 3. *The fractional-order stochastic Itô-differential of the function $f \in C(I, L_2(\Omega))$ can be defined by*

$$_{*}d^{\alpha} f(t) = dI^{1-\alpha} f(t)$$

where d is the Itô-differential

Lemma 8. *The left inverse of the stochastic Itô-integral of fractional order $\alpha \in (0,1)$ is the stochastic Itô-differential of fractional-order $\alpha \in (0,1)$*

$$_{*}d^{\alpha} F_{\alpha} f(t) = f(t)dw(t), \; f \in C(I, L_2(\Omega)).$$

Proof. By direct application of the stochastic Itô-differential of fractional-order and the stochastic Itô-integral of fractional-order, we obtain

$$_{*}d^{\alpha} F_{\alpha} f(t) = dI^{1-\alpha} F_{\alpha} f(t) = dF_1 f(t) = f(t)dw(t)$$

□

Lemma 9. *Let $f, \; g \in C(I, L_2(\Omega))$ and $k_1, \; k_2$ are two constants. Then*

$$_{*}d^{\alpha}\Big(k_1 f(t) + k_2 g(t)\Big) = k_1 \; _{*}d^{\alpha} f(t) + k_2 \; _{*}d^{\alpha} g(t)$$

4. Stochastic differential equations

4.1. **Continuous solution.** Let $f \in C(I, L_2(\Omega))$. Consider the three initial value problems of the stochastic Itô-differential equation of fractional-order

$$_{*}d^{\alpha} X(t) \; = \; dI^{1-\alpha} X(t) = f(t)dw(t) \tag{6}$$

with each one of the following initial condition

(i) The initial condition $X(0) = 0$.
(ii) The nonlocal condition $I^{1-\alpha} X(t)\big|_{t=0} = 0$.
(iii) The weighted condition $t^{1-\alpha} X(t)\big|_{t=0} = 0$.

Theorem 1. *The solution of each of the initial value problems (6)-(i), (6) with (ii) and (6) with (iii) is given by*

$$X(t) = F_{\alpha} f(t) \in C(I, L_2(\Omega)) \tag{7}$$

Proof. Consider the initial value problem (6)-(i).

$$dI^{1-\alpha} X(t) = f(t)dw(t), \; X(0) = x_0.$$

Integrating both sides, we get

$$I^{1-\alpha} X(t) = c + F_1 f(t).$$

Operating both sides by I^{α}, we have

$$I^1 X(t) = I^{\alpha}(c + F_1 f(t)) = \frac{ct^{\alpha}}{\Gamma(\alpha+1)} + F_{1+\alpha} f(t).$$

Next, differentiating and putting $t = 0$ we get

$$X(t) = \frac{ct^{\alpha-1}}{\Gamma(\alpha)} + F_{\alpha} f(t).$$

$$X(0) = \frac{ct^{\alpha-1}}{\Gamma(\alpha)}\bigg|_{t=0} + F_{\alpha} f(t)\big|_{t=0} \to \infty.$$

Then we must choose $c = 0$, so we have $X(0) = 0$. Using Lemma 6, we deduce that the solution of the problem (6)-(i) is given by

$$X(t) = F_{\alpha} f(t) \in C(I, L_2(\Omega)).$$

Conversely, we have

$$_{*}d^{\alpha} F_{\alpha} f(t) \; = \; dI^{1-\alpha} F_{\alpha} f(t) = \; dF_1 f(t) \; = \; f(t)dw(t).$$

Also $F_\alpha f(t)|_0 = 0$, then $X(0) = 0$. This proves that $X(t) = F_\alpha f(t) \in C(I, L_2(\Omega))$ is the solution of the initial value problem (6)-(i). □

Consider now the problem (6) and (ii)

$$dI^{1-\alpha}X(t) = f(t)dw(t),\ I^{1-\alpha}X(t)\big|_{t=0} = 0.$$

Integrating both sides and using condition (ii), we get

$$I^{1-\alpha}X(t) - I^{1-\alpha}X(t)\Big|_{t=0} = F_1 f(t) = I^{1-\alpha}X(t).$$

Operating both sides by I^α, we have

$$I^1 X(t) = I^\alpha(F_1 f(t)) = F_{1+\alpha}f(t).$$

Next, differentiating both sides, then the solution of the problem (6)-(ii) is given by

$$X(t) = F_\alpha f(t) \in C(I, L_2(\Omega)).$$

For the problem (6) and (iii)

$$dI^{1-\alpha}X(t) = f(t)dw(t),\ t^{1-\alpha}X(t)\big|_{t=0} = 0.$$

Integrating both sides and using condition (iii), we get

$$I^{1-\alpha}X(t) - c = F_1 f(t).$$

Operating both sides by I^α, we have

$$I^1 X(t) = I^\alpha(F_1 f(t)) + I^\alpha c = F_{1+\alpha}f(t) + \frac{ct^\alpha}{\Gamma(\alpha+1)}$$

and differentiating both sides, we obtain

$$X(t) = F_\alpha f(t) + \frac{ct^{\alpha-1}}{\Gamma(\alpha)}$$

and putting $t = 0$, we get

$$t^{1-\alpha}X(t)\Big|_{t=0} = t^{1-\alpha}(F_1 f(t))\Big|_{t=0} + c,$$

then $c = 0$ and the solution of the problem (6)-(iii) is given by

$$X(t) = F_\alpha f(t) \in C(I, L_2(\Omega)).$$

4.2. **Integrable solution.** In this subsection, we establish the existence of a unique solution $X \in L_1(I, L_2(\Omega))$ for the following nonlinear stochastic Itô-differential equation of fractional-order

$$_*d^\alpha X(t) \ = \ dI^{1-\alpha}X(t) = f(t)dw(t),\ \alpha \in (\frac{1}{2}, 1), \tag{8}$$

with the nonlocal or weighted condition (respectively)

$$I^{1-\alpha}X(t)\big|_{t=0} = b \ or\ t^{1-\alpha}X(t)\big|_{t=0} = \ \frac{b}{\Gamma(\alpha)} \tag{9}$$

where b is a second order random variable.

Theorem 2. *Let $f \in C(I, L_2(\Omega)$, then there exists a unique solution $X \in L_1(I, L_2(\Omega))$ of (8) with each one of the initial conditions (9). Moreover, this solution is given by*

$$X(t) = \frac{bt^{\alpha-1}}{\Gamma(\alpha)} + F_\alpha f(t). \tag{10}$$

Proof. Integrating (8) and using the two condition (9), we obtain

$$X(t) = \frac{bt^{\alpha-1}}{\Gamma(\alpha)} + F_\alpha f(t)$$

and

$$X(t) = \frac{bt^{\alpha-1}}{\Gamma(\alpha)} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)dw(s),$$

then

$$\begin{aligned} \|X(t)\|_2 &= \|\frac{bt^{\alpha-1}}{\Gamma(\alpha)} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)dw(s)\|_2 \\ &\leq \frac{||b||_2 t^{\alpha-1}}{\Gamma(\alpha)} + ||\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)dw(s)\|_2. \end{aligned}$$

But

$$||\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)dw(s)\|_2^2 \leq ||f||_C^2 \frac{t^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)},$$

then

$$\|X(t)\|_2 \leq \frac{||b||_2 t^{\alpha-1}}{\Gamma(\alpha)} + ||f||_C \frac{t^{\alpha-1/2}}{\sqrt{2\alpha-1}\,\Gamma(\alpha)}.$$

Hence

$$\begin{aligned} \|X(t)\|_{L_1} &= \int_0^T \|X(t)\|_2 dt \\ &\leq \frac{||b||_2 T^\alpha}{\alpha\Gamma(\alpha)} + ||f||_C \frac{2\ T^{\alpha+1/2}}{(2\alpha+1)\sqrt{2\alpha-1}\,\Gamma(\alpha)}. \end{aligned}$$

Consequently, the solution of the initial value problem (8) with the initial condition $I^{1-\alpha}X(t)\big|_{t=0} = b$ is given by

$$X(t) = \frac{bt^{\alpha-1}}{\Gamma(\alpha)} + F_\alpha f(t) \in L_1(I, L_2(\Omega)).$$

Similarly, the solution of the initial value problem (8) with the initial condition $t^{1-\alpha}X(t)\big|_{t=0} = \frac{b}{\Gamma(\alpha)}$ is given by

$$X(t) = \frac{bt^{\alpha-1}}{\Gamma(\alpha)} + F_\alpha f(t) \in L_1(I, L_2(\Omega)).$$

□

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