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ASYMPTOTIC CONVERGENCE CRITERIA FOR NONHOMOGENEOUS LINEAR FRACTIONAL ORDER SYSTEMS

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Abstract. This paper investigates the asymptotic convergence analysis of the nonhomogeneous systems that involve Caputo derivative operator of real order. By making the use of comparison methodology, some new asymptotic convergence results are proposed and it is shown that the responses of such time-varying systems converge to the point 0. Finally, an incommensurate order electrical circuit system that involves time-varying coefficients is considered to demonstrate the applicability of some developed results.

1. Introduction

Fractional calculus is an important branch of mathematics that deals with the descriptions of the possibility of computation of unknown functions via suitable derivative and integral operators of real order and studies the relationships between them [1, 2, 3, 4]. A fractional order system is a system that is described by a set of equations where the unknown variables are associated with some derivative operators of real order in these equations. Such a system is called a commensurate system if all variables are associated with the same real order otherwise it is called an incommensurate order system.

The issue of asymptotic convergence of response analysis of linear systems that involve real order (fractional order) derivatives, e.g., Caputo derivative, Hadamard derivative, Caputo-Hadamard derivative, etc. plays an important role and are useful in various areas of system design, modeling, and engineering applications [5, 6, 7, 8, 9, 10, 11]. It allows to predict the local behaviours of simple dynamics of nonlinear fractional order systems as well as provides a platform for controlling the complicated dynamics of various systems. In the literature, the asymptotic stability and control problems of linear systems receive significant research importance of various researchers (see, e.g., [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]). In these works, they showed that, with the introduction of various control methods,

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the unstable responses to systems can be effectively stabilized under some reasonable analytic conditions. The introduction of control inputs to such systems in many situations automatically gives rise to the representation of a nonhomogeneous fractional order system.

Most of all these existing works in the literature only focused on the stability and stabilization problem of the commensurate fractional linear time-invariant (autonomous) system. Recently, in the work in [24], an asymptotic convergence result has been established for a scalar nonhomogeneous commensurate linear fractional differential equation with constant coefficient. In [25], an asymptotic boundedness result has been derived for a nonhomogeneous commensurate linear fractional order time-varying system using quadratic Lyapunov function. In [26], an asymptotic convergence criterion has been proposed for the delayed nonhomogeneous incommensurate linear fractional order system with constant coefficients.

However, we observe that the response behaviour of the nonhomogeneous incommensurate linear fractional system with time-varying coefficient has not been investigated yet. This may be due to the fact that incommensurate order systems are not easier than that of their commensurate counterpart, and hence the system analysis is difficult. On the other hand, finding the closed form solutions to such systems seems very difficult in contrast to its time-invariant ones. For the analytic solutions to autonomous nonhomogeneous/homogeneous Caputo fractional order systems with constant coefficients, we refer the readers to the works in [27, 28, 29, 30]. Consequently, we are often interested in reasonable mathematical conditions, as well as a new theory, to draw some conclusions about the responses of such time-varying systems without having the knowledge of its exact solutions. It should be pointed out that most of the existing research so far dealt with the case of fractional order systems that involved zero initial time instant, i.e, the lower limit associated with the derivative operator is set at 0. This could be perhaps due to the lack of advanced mathematical tools available in the current literature to tackle issues of non-zero initial time instant. Recently, in [11], a new modified Laplace integral transform and a new modified Mellin transform have been introduced that allows one to specify the initial conditions at a non-zero time instant for Caputo derivative case. But for its applications to incommensurate order systems that involve nonzero time instant, some advanced properties of such transforms are often required for obtaining reasonable asymptotic stability criteria.

Recently, in [31], the authors have developed a few asymptotic stability results for homogeneous incommensurate fractional nonautonomous (time-varying) systems using the methodology of fractional comparison method [32]. In that work, many different analytic criteria for such systems have been established for the asymptotic analysis of the responses to such systems. But the results of the existing works cannot be applied directly to the incommensurate nonhomogeneous time-varying systems. The purpose of this work is to develop some asymptotic convergence criteria for nonhomogeneous incommensurate linear time-varying fractional order system

$$ {}^{C}D_{0,t}^{\widehat{\alpha}}x(t) = A(t)x(t) + f(t), \tag{1.1}$$

subject to the initial condition $x(0) = x_0$, where $x(t) = (x_1(t), \cdots, x_n(t))^T \in \mathbb{R}^n$, ${}^{C}D_{0,t}^{\widehat{\alpha}}x(t) = ({}^{C}D_{0,t}^{\alpha_1}x_1(t), \cdots, {}^{C}D_{0,t}^{\alpha_n}x_n(t))^T$, fractional orders $\alpha_1, \alpha_2, \cdots, \alpha_n \in (0, 1]$, the continuous matrix $A(t) = (a_{ij}(t)) \in \mathbb{R}^{n\times n}$ and some suitable function $f(t) = (f_1(t), \cdots, f_n(t))^T \in \mathbb{R}^n$.

First, we develop a new lemma that deals with the nonnegativity of fractional differential inequality. Then by making the use of Laplace transform, we develop a new convergence theorem for autonomous incommensurate fractional systems along with a forcing term. Using these results along with the comparison methodology [32], new asymptotic convergence theorems are established for nonhomogeneous incommensurate order linear time-varying systems whenever the coefficient matrix is in both standard form and block matrix form. Based on these theorems, various simplified analytic conditions are proposed and it is shown that the responses of such time-varying systems converge to the point 0. Finally, we consider an electrical time-varying circuit system to demonstrate the potential application of some proposed theoretical results.

This paper is organized as follows. In Section 2, some known definitions and a few new results are introduced. In Section 3, the main asymptotic convergence results are proposed. In Section 4, an electrical circuit system is considered and a few results are demonstrated. In Section 5, the conclusions are drawn.

2. Preliminary definitions and new results

The following standard notations are used in this work.
$\mathbb{R}_+$ is the set of positive real numbers, $\mathbb{Q}_+$ the set of positive rational numbers, $\mathbb{Z}_+$ the set of positive integers, $\arg(z)$ the argument of a complex number $z \in \mathbb{C}$ and Y^T is the transpose of the vector or matrix Y. The symbol $i = 1(1)n$ means $i = 1, 2, \cdots, n$. Given a vector $x \in \mathbb{R}^n$, $x \geq 0$ means its components are nonnegative. Given two vectors $x, y \in \mathbb{R}^n$, $x \leq y$ means the inequality $x_i \leq y_i$ holds for $i = 1(1)n$. gcd and lcm, respectively, mean the greatest common divisor and least common multiple. For a vector $x \in \mathbb{R}^n$, $\|x\|$ denotes the Euclidean norm. Given a real symmetric matrix $M \in \mathbb{R}^{n\times n}$, $\lambda_{max}(M)$ and $\lambda_{min}(M)$, respectively, mean its maximal and minimal eigenvalues.

Definition 2.1. [1, 2, 3] *The α-order Riemann-Liouville integral of $\xi : [0, \infty) \to \mathbb{R}$ is defined by*

$$ {}^{RL}D_{0,t}^{-\alpha}\xi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \xi(\tau) d\tau, \ t > 0, \tag{2.1}$$

where $\alpha \in \mathbb{R}_+$ and $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$ is the Gamma function.

Definition 2.2. [1, 2, 3] *The α-order, with $\alpha \in \mathbb{R}_+$, Caputo derivative of $\xi : [0, \infty) \to \mathbb{R}$ is defined by*

$$ {}^{C}D_{0,t}^{\alpha}\xi(t) = \begin{cases} {}^{RL}D_{0,t}^{-(n-\alpha)} \left(\frac{d^n\xi(t)}{dt^n}\right), & if\ \alpha \in (n-1, n), \\ \frac{d^n\xi(t)}{dt^n}, & if\ \alpha = n, \end{cases} \tag{2.2}$$

where $n \in \mathbb{Z}_+$.

Definition 2.3. [1] *The Laplace transform of a function $\xi : [0, \infty) \to \mathbb{R}$ is defined as*

$$ \mathcal{L}(\xi(t)) = \int_0^\infty e^{-st} \xi(t) dt. \tag{2.3}$$

Here, we introduce the following new definition.

Definition 2.4. *A function $g = (g_1, \cdots, g_n)^T : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is of class W^* if, for every fixed $t \in \mathbb{R}$, there exist some $h = (h_1, \cdots, h_n)^T : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ with $g_i(t, u) \leq h_i(t, \widehat{u})$, $i = 1(1)n$, for all u, $\widehat{u} \in \mathbb{R}^n$ such that $u_j \leq \widehat{u_j}$, $u_i = \widehat{u_i}$, $j = 1(1)n$, $i \neq j$, where u_i denotes the i-th component of u.*

When we consider $g = h : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, Definition 2.4 reduces to the *class W* function (see [33]).

Definition 2.5. [34] *A matrix $A \in \mathbb{R}^{n\times n}$ is called a Metzler matrix if its off diagonal elements are non-negative.*

We introduce here a new comparison lemma.

Lemma 2.1. *Let $\xi(t)$ be the solution to the inequality*

$$ {}^C D^{\widehat{\alpha}}_{0,t}\xi(t) \leq A\xi(t) + g_1(t), \;\; \xi(0) = \xi_0 \geq 0, \tag{2.4}$$

where ${}^C D^{\widehat{\alpha}}_{0,t}\xi(t) = ({}^C D^{\alpha_1}_{0,t}\xi_1(t), \cdots, {}^C D^{\alpha_n}_{0,t}\xi_n(t))^T$, $\alpha_i \in (0, 1]$ for $i = 1(1)n$, $A = (k_{ij}) \in \mathbb{R}^{n\times n}$ is a Metzler matrix with negative diagonal elements and $g_1 : [0, \infty) \to \mathbb{R}^n$ is a nonnegative continuous function. Let $\eta(t)$ be the solution to the system

$$ {}^C D^{\widehat{\alpha}}_{0,t}\eta(t) = A\eta(t) + g_2(t), \;\; \eta(0) \geq \xi(0), \tag{2.5}$$

where $g_2 : [0, \infty) \to \mathbb{R}^n$ is a nonnegative continuous function. If the inequality $g_1(t) \leq g_2(t)$ holds for all $t \geq 0$, then the inequality $0 \leq \xi(t) \leq \eta(t)$ holds for all $t \geq 0$.

Proof. 1) Set $W(t) = {}^C D^{\widehat{\alpha}}_{0,t}\xi(t) - A\xi(t) - g_1(t)$. Define $Z(t) = \eta(t) - \xi(t)$. Then, by using (2.5), one gets a new system

$$ {}^C D^{\widehat{\alpha}}_{0,t}Z(t) = AZ(t) + g_2(t) - g_1(t) - W(t), \tag{2.6}$$

with initial condition $Z(0) = \eta(0) - \xi(0) \geq 0$. If the inequality $g_1(t) \leq g_2(t)$ holds, then the solution $Z(t)$ to the system (2.6) is non-negative [26, 34], since the matrix A is Metzler and $W(t) \leq 0$. As a result, the solution $y(t)$ to system (2.5) is nonnegative. Hence, the inequality $0 \leq \xi(t) \leq \eta(t)$ holds for all $t \geq 0$. □

Proof. 2) Consider the inequality (2.4) with the equality (2.5). Since the matrix A is Metzler with negative diagonal elements and if the inequality $g_1(t) \leq g_2(t)$ holds for all $t \geq 0$, the function $g(u) = Au + g_1(t)$ is clearly of *class* W^*. Thus, it follows from fractional comparison principle [32] that $0 \leq \xi(t) \leq \eta(t)$, $\forall t \geq 0$. This completes the proof. □

Next, we introduce the following definition.

Definition 2.6. *The system* (1.1) *is said to globally asymptotically converge (GAC) to* 0 *if $\|x(t)\| \to 0$ as t tends to ∞ for any value of $x(0)$.*

Then, we introduce the following new result.

Theorem 2.1. *Consider the nonhomogeneous fractional order system*

$$ {}^C D^{\widehat{\alpha}}_{0,t}y(t) = Ay(t) + g_3(t), \;\; y(0) = y_0, \tag{2.7}$$

where $y(t) = (y_1(t), \cdots, y_n(t))^T \in \mathbb{R}^n$, ${}^C D^{\widehat{\alpha}}_{0,t}y(t) = ({}^C D^{\alpha_1}_{0,t}y_1(t), \cdots, {}^C D^{\alpha_n}_{0,t}y_n(t))^T$, fractional orders $\alpha_1, \alpha_2, \cdots, \alpha_n \in (0, 1]$, the matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is constant and the known function $g_3 : [0, \infty) \to \mathbb{R}^n$ is continuous, differentiable and bounded. If the conditions

$C_1)$ *all the roots of characteristic equation*

$$\det\left[diag\left(s^{\alpha_1}, s^{\alpha_2}, \cdots, s^{\alpha_n}\right) - A\right] = 0 \tag{2.8}$$

satisfy $|\arg(s)| > \frac{\pi}{2}$,

$C_2)$ $\lim_{t\to\infty} g_3(t) = 0$,

are satisfied, then the system (2.7) *is GAC to* 0.

Proof. Taking Laplace transform on system (2.7), one gets

$$\left[diag\left(s^{\alpha_1}, s^{\alpha_2}, \cdots, s^{\alpha_n}\right) - A\right] Y(s) = G_3(s) - \left[diag\left(s^{\alpha_1-1}, s^{\alpha_2-1}, \cdots, s^{\alpha_n-1}\right)\right] y(0) \tag{2.9}$$

where $\mathcal{L}(y(t)) = Y(s)$ and $\mathcal{L}(g_3(t)) = G_3(s)$. Then, multiplying both sides of (2.9) by s gives

$$\left[diag\left(s^{\alpha_1}, s^{\alpha_2}, \cdots, s^{\alpha_n}\right) - A\right] \cdot sY(s) = sG_3(s) - \left[diag\left(s^{\alpha_1}, s^{\alpha_2}, \cdots, s^{\alpha_n}\right)\right] y(0). \tag{2.10}$$

Now if condition $C_1)$ holds, then one can consider the system (2.10) in $\mathrm{Re}(s) \geq 0$ and get

$$sY(s) = \left[diag\left(s^{\alpha_1}, s^{\alpha_2}, \cdots, s^{\alpha_n}\right) - A\right]^{-1} \left(sG_3(s) - \left[diag\left(s^{\alpha_1}, s^{\alpha_2}, \cdots, s^{\alpha_n}\right)\right] y(0)\right). \tag{2.11}$$

Further, if condition $C_2)$ holds, then by using the Laplace final value theorem [28, 35] in (2.11), one obtains

$$\lim_{t\to\infty} y(t) = 0. \tag{2.12}$$

□

Corollary 2.1. *Consider the nonohomogeneous system* (2.7) *with* $\alpha_k = \ell_k\alpha \in (0, 1]$, $k = 1(1)n$, *where* $\ell_k = \frac{u_k}{v_k} \in \mathbb{Q}_+$ *with* $gcd(u_k, v_k) = 1$, $k = 1(1)n$, *and* $\alpha \in \mathbb{R}_+$. *Let* $M = lcm(v_1, v_2, \cdots, v_n)$. *Under the assumptions of Theorem 2.1, if the conditions*

$C_1)$ *all the roots of*

$$\det\left[diag\left(\lambda^{M\ell_1}, \lambda^{M\ell_2}, \cdots, \lambda^{M\ell_n}\right) - A\right] = 0 \tag{2.13}$$

satisfy $|\arg(\lambda)| > \frac{\pi}{2M}\alpha$,

$C_2)$ $\lim_{t\to\infty} g_3(t) = 0$,

are satisfied, then the system (2.7) *is GAC to* 0.

Proof. Take $s^{\frac{\alpha}{M}} = \lambda$. Then, condition (2.8) gets changed to (2.13). Consequently, the result follows from Theorem 2.1. □

The following results are consequences of Corollary 2.1.

Corollary 2.2. *Consider the system* (2.7) *with* $\alpha_1 = \frac{1}{n}\alpha, \alpha_2 = \frac{2}{n}\alpha, \cdots, \alpha_n = \frac{n}{n}\alpha$, *and* $\alpha \in (0, 1]$. *Under the assumptions of Theorem 2.1, if the conditions*

$C_1)$ *all the roots of*

$$\det\left[diag\left(\lambda, \lambda^2, \cdots, \lambda^n\right) - A\right] = 0 \tag{2.14}$$

satisfy $|\arg(\lambda)| > \frac{\pi}{2n}\alpha$,

$C_2)$ $\lim_{t\to\infty} g_3(t) = 0$,

are satisfied, then the system (2.7) *is GAC to* 0.

Corollary 2.3. *Consider the system* (2.7) *with* $\alpha_1 = \frac{n}{n}\alpha, \alpha_2 = \frac{n-1}{n}\alpha, \cdots, \alpha_n = \frac{1}{n}\alpha$, *and* $\alpha \in (0, 1]$. *Under the assumptions of Theorem 2.1, if the conditions*

C_1) *all the roots of*

$$\det\left[diag\left(\lambda^n, \lambda^{n-1}, \cdots, \lambda\right) - A\right] = 0 \tag{2.15}$$

satisfy $|\arg(\lambda)| > \frac{\pi}{2n}\alpha$,

C_2) $\lim_{t\to\infty} g_3(t) = 0$,

are satisfied, then the system (2.7) *is GAC to* 0.

3. Main asymptotic theory

In this section, we develop the main theory of nonhomogeneous time-varying system (1.1) by considering two of its generic forms. Here we consider the system (1.1) when the coefficient matrix takes

(i) the standard form, i.e., the entries of the coefficient matrix can be viewed as scalar functions,

(ii) the block matrix form, i.e., the entries of the coefficient matrix can be viewed as block matrices.

Next, the asymptotic convergence conditions to such forms are presented in different subsections.

3.1. **Standard form.**

Let the system (1.1) be of the form

$$\begin{pmatrix} {}^{C}D_{0,t}^{\alpha_1} x_1(t) \\ \vdots \\ {}^{C}D_{0,t}^{\alpha_n} x_n(t) \end{pmatrix} = \begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & \ddots & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{pmatrix} \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} + \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix} \tag{3.1}$$

with respect to the initial values

$$x_i(0) = x_{i0},\ i = 1(1)n. \tag{3.2}$$

Assumption 3.1. *Assume the following for the system* (3.1)*:*

A_1) $a_{ii}(t) \le -\delta_{ii}$, $\delta_{ii} > 0$, $\forall t \ge 0$, $i = 1(1)n$,

A_2) $|a_{ij}(t)| \le \delta_{ij}$, $\delta_{ij} \ge 0$, $\forall t \ge 0$, $i \ne j \in \{1, 2, \cdots, n\}$.

Assumption 3.2. *The vector function* f *of system* (3.1) *satisfies the following:*

A_1) f *is continuous, differentiable and bounded.*

A_2) $\lim_{t\to\infty} f(t) = 0$.

Now we define the following:

(i) the constants:

$$c_{ii} = 2\delta_{ii} - \sum_{j=1,\ j\ne i}^{n} \delta_{ij} + 1 \text{ for } i = 1, 2, \cdots, n, \tag{3.3}$$

(ii) the matrix:

$$\Delta = \begin{pmatrix} -c_{11} & \delta_{12} & \cdots & \delta_{1n} \\ \delta_{21} & -c_{22} & \cdots & \delta_{2n} \\ \vdots & \cdots & \ddots & \vdots \\ \delta_{n1} & \delta_{n2} & \cdots & -c_{nn} \end{pmatrix}. \tag{3.4}$$

We introduce the main asymptotic convergence theorem for (3.1).

Theorem 3.1. *Consider the nonhomogeneous system* (3.1) *with* $\alpha_1, \alpha_2, \cdots, \alpha_n \in (0,1]$. *Let Assumptions 3.1 and 3.2 hold. Consider the constants and the matrix given by* (3.3) *and* (3.4)*, respectively. If the conditions*

$C_1)$ $c_{ii} > 0$ *for* $i = 1(1)n$,

$C_2)$ *all the roots of*

$$\det\left[diag\left(s^{\alpha_1}, s^{\alpha_2}, \cdots, s^{\alpha_n}\right) - \Delta\right] = 0 \tag{3.5}$$

lie in the sector $|\arg(s)| > \frac{\pi}{2}$,

are true, then the system (3.1)-(3.2) *is* GAC *to* 0.

Proof. Take $V(t) = \sum_{i=1}^{n} V_i(t)$, where $V_i(t) = x_i^2(t)$ for $i = 1, 2, \cdots, n$. Then, by using Lemma 1 of [36] for the system (3.1), one gets

$$\begin{aligned}
{}^{C}D_{0,t}^{\alpha_i} V_i(t) &\le 2\left(a_{i1}(t)x_i(t)x_1(t) + \cdots + a_{in}(t)x_i(t)x_n(t)\right) + 2x_i(t)f_i(t) \\
&\le 2a_{ii}(t)x_i^2(t) + 2\sum_{j=1,\ j\neq i}^{n} |a_{ij}(t)||x_i(t)||x_j(t)| + 2|x_i(t)||f_i(t)| \\
&\le 2a_{ii}(t)x_i^2(t) + \sum_{j=1,\ j\neq i}^{n} |a_{ij}(t)|\left(x_i^2(t) + x_j^2(t)\right) + x_i^2(t) + f_i^2(t) \\
&\le -\left(2\delta_{ii} - \sum_{j=1,\ j\neq i}^{n} \delta_{ij} + 1\right) x_i^2(t) + \sum_{j=1,\ j\neq i}^{n} \delta_{ij} x_j^2(t) + f_i^2(t) \qquad (3.6) \\
&= -c_{ii}x_i^2(t) + \sum_{j=1,\ j\neq i}^{n} \delta_{ij} x_j^2(t) + f_i^2(t), \quad i = 1(1)n, \qquad (3.7)
\end{aligned}$$

where the Assumption 3.1 and values of c_{ii}'s are, respectively, used in (3.6) and (3.7). Set $\bar{V}(t) = (V_1(t), \cdots, V_n(t))^T$ and $h(t) = \left(f_1^2(t), \cdots, f_n^2(t)\right)^T$. Consequently, one gets the vector-matrix fractional differential inequality

$${}^{C}D_{0,t}^{\widehat{\alpha}} \bar{V}(t) \le \Delta \bar{V}(t) + h(t) \tag{3.8}$$

with $\bar{V}(0) = \left(x_1^2(0), \cdots, x_n^2(0)\right)^T$. Consider a new vector-matrix fractional comparison system

$${}^{C}D_{0,t}^{\widehat{\alpha}} \bar{U}(t) = \Delta \bar{U}(t) + h(t) \tag{3.9}$$

with $\bar{U}(0) = \bar{V}(0)$. In view of condition $C_1)$, by Lemma 2.1, it follows that

$$0 \le \bar{V}(t) \le \bar{U}(t). \tag{3.10}$$

Under Assumption 3.2, if condition $C_2)$ is satisfied, then by Theorem 2.1, one can conclude from (3.9) that $\lim_{t\to\infty} \bar{U}(t) = 0$. As a result, one can get from (3.10) that $\lim_{t\to\infty} \bar{V}(t) = 0$. Thus, it is established that $\lim_{t\to\infty} \|x(t)\| = 0$. Hence the result is proved. □

Corollary 3.1. *Consider the nonhomogeneous system* (3.1) *with* $\alpha_k = \ell_k \alpha \in (0,1]$ *for* $k = 1(1)n$, *where* $\ell_k = \frac{u_k}{v_k} \in \mathbb{Q}_+$. *Set* $M = lcm(v_1, v_2, \cdots, v_n)$. *Under Assumptions 3.1 and 3.2 along with the constants and the matrix given by* (3.3) *and* (3.4)*, respectively, if the conditions*

C_1) *$c_{ii} > 0$ for $i = 1(1)n$,*

C_2) *all the roots of*

$$\det\left[diag\left(\lambda^{M\ell_1}, \lambda^{M\ell_2}, \cdots, \lambda^{M\ell_n}\right) - \Delta\right] = 0 \tag{3.11}$$

lie in the sector $|\arg(\lambda)| > \frac{\pi}{2M}\alpha$,

hold, then the system (3.1)-(3.2) *is GAC to* 0.

Corollary 3.2. *Consider* (3.1) *with $\alpha_1 = \frac{1}{n}\alpha, \alpha_2 = \frac{2}{n}\alpha, \cdots, \alpha_n = \frac{n}{n}\alpha$, where $\alpha \in (0, 1]$. Under Assumptions 3.1 and 3.2 along with the constants and the matrix given by* (3.3) *and* (3.4), *respectively, if the following conditions hold:*

C_1) *$c_{ii} > 0$, $i = 1(1)n$,*

C_2) *all the characteristic roots of polynomial*

$$\det\left[diag\left(\lambda, \lambda^2, \cdots, \lambda^n\right) - \Delta\right] = 0 \tag{3.12}$$

lie in the sector $|\arg(\lambda)| > \frac{\pi}{2n}\alpha$,

then the system (3.1)-(3.2) *is GAC to* 0.

Corollary 3.3. *Consider the system* (3.1) *with $\alpha_1 = \frac{n}{n}\alpha, \alpha_2 = \frac{n-1}{n}\alpha, \cdots, \alpha_n = \frac{1}{n}\alpha$, where $\alpha \in (0, 1]$. Under Assumptions 3.1–3.2 along with the constants in* (3.3) *and matrix* (3.4), *if the conditions*

C_1) *$c_{ii} > 0$, $i = 1(1)n$,*

C_2) *all the characteristic roots of polynomial*

$$\det\left[diag\left(\lambda^n, \lambda^{n-1}, \cdots, \lambda\right) - \Delta\right] = 0 \tag{3.13}$$

lie in the sector $|\arg(\lambda)| > \frac{\pi}{2n}\alpha$,

hold, then the system (3.1)-(3.2) *is GAC to* 0.

3.2. **Block matrix form.** Suppose the system (1.1) is represented in the form

$$\begin{pmatrix} {}^{C}D_{0,t}^{\alpha_1}x_1(t) \\ \vdots \\ {}^{C}D_{0,t}^{\alpha_d}x_d(t) \end{pmatrix} = \begin{pmatrix} A_{11}(t) & \cdots & A_{1d}(t) \\ \vdots & \ddots & \vdots \\ A_{d1}(t) & \cdots & A_{dd}(t) \end{pmatrix} \begin{pmatrix} x_1(t) \\ \vdots \\ x_d(t) \end{pmatrix} + \begin{pmatrix} f_1(t) \\ \vdots \\ f_d(t) \end{pmatrix}, \tag{3.14}$$

$$x_1(0) = x_1^0, \cdots, x_d(0) = x_d^0, \tag{3.15}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \cdots, x_{in_i}(t))^T \in \mathbb{R}^{n_i}$, the matrices $A_{ij}(t) \in \mathbb{R}^{n_i \times n_j}$ are continuous on $[0, \infty)$ for $i,\ j \in \{1, 2, \cdots, d\}$, $\alpha_1, \alpha_2, \cdots, \alpha_d \in (0, 1]$, $f_i \in \mathbb{R}^{n_i}$ and $\sum_{i=1}^{d} n_i = n$.

Assumption 3.3. *Assume that there exist symmetric and positive definite matrices $P_i \in \mathbb{R}^{n_i \times n_i}$ such that*

A_1) *the diagonal blocks of* (3.14) *satisfy*

$$P_i A_{ii}(t) + A_{ii}^T(t) P_i = Q_{ii}(t),\ \lambda_{max}\left(Q_{ii}(t)\right) \le -\delta_{ii},\ \delta_{ii} > 0,\ \ \forall t \ge 0,\ i = j, \tag{3.16}$$

A_2) *the following relations hold for* (3.14)*:*

$$A_{ij}^T(t) A_{ij}(t) = Q_{ij}(t),\ \lambda_{max}\left(Q_{ij}(t)\right) \le \delta_{ij},\ \ \delta_{ij} \ge 0,\ \forall t \ge 0,\ i \ne j, \tag{3.17}$$

with $i, j \in \{1, 2, \cdots, d\}$.

Assumption 3.4. *The components of vector function f of system* (3.14) *satisfy the following:*

$A_1)$ *$f_1, \cdots, f_d$ are continuous, differentiable and bounded.*
$A_2)$ *$\lim_{t\to\infty} f_i(t) = 0$, where $i = 1(1)d$.*

Next, we set up the following:
(i) Define

$$c_{ij} = \begin{cases} \frac{\delta_{ii} - d\lambda_{max}(P_i^T P_i)}{\lambda_{max}(P_i)}, & \text{for } i = j, \\ \frac{\delta_{ij}}{\lambda_{min}(P_j)}, & \text{for } i \neq j, \end{cases} \tag{3.18}$$

where $i = 1(1)d$ and $j = 1(1)d$.
(ii) Define

$$\Delta_C = \begin{bmatrix} -c_{11} & c_{12} & \cdots & c_{1d} \\ c_{21} & -c_{22} & \cdots & c_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ c_{d1} & c_{d2} & \cdots & -c_{dd} \end{bmatrix}. \tag{3.19}$$

Here, we introduce the main asymptotic convergence theorem for the system (3.14).

Theorem 3.2. *Consider the nonhomogeneous system* (3.14) *with $\alpha_1, \cdots, \alpha_d \in (0, 1]$. Under Assumptions 3.3–3.4 along with the constants* (3.18) *and the matrix* (3.19)*, if the conditions*
$C_1)$ *$c_{ii} > 0$ for $i = 1(1)d$,*
$C_2)$ *all the characteristic roots of*

$$\det\left[diag\left(s^{\alpha_1}, \cdots, s^{\alpha_d}\right) - \Delta_C\right] = 0 \tag{3.20}$$

lie in the sector $|\arg(s)| > \frac{\pi}{2}$,

hold, then the system (3.14)-(3.15) *is GAC to* 0.

Proof. Let $V(t) = \sum_{i=1}^{d} V_i(t) = \sum_{i=1}^{d} x_i^T(t) P_i x_i(t)$, where $P_i \in \mathbb{R}^{n_i \times n_i}$'s are symmetric positive definite matrices for $i = 1(1)d$. Using Lemma 4 of [37] for the system (3.14), one can get

$$\begin{aligned} {}^{C}D_{0,t}^{\alpha_i} V_i(t) &\leq x_i^T(t) P_i {}^{C}D_{0,t}^{\alpha_i} x_i(t) + \left({}^{C}D_{0,t}^{\alpha_i} x_i(t)\right)^T P_i x_i(t) \\ &= x_i^T(t) P_i \left[A_{i1}(t)x_1(t) + \cdots + A_{id}(t)x_d(t)\right] \\ &\quad + \left[A_{i1}(t)x_1(t) + \cdots + A_{id}(t)x_d(t)\right]^T P_i x_i(t) + 2x_i^T(t) P_i f_i(t) \\ &= x_i^T(t)\left[P_i A_{ii}(t) + A_{ii}^T(t) P_i\right] x_i(t) + 2x_i^T(t) P_i \left[\sum_{j=1, j\neq i}^{d} A_{ij}(t) x_j(t)\right] \\ &\quad + 2x_i^T(t) P_i f_i(t) \\ &\leq x_i^T(t) Q_{ii}(t) x_i(t) + d x_i^T(t) P_i^T P_i x_i(t) + \left[\sum_{j=1, j\neq i}^{d} x_j(t)^T Q_{ij}(t) x_j(t)\right] \\ &\quad + f_i^T(t) f_i(t) \\ &\leq -\left[\delta_{ii} - d\lambda_{max}(P_i^T P_i)\right] x_i^T(t) x_i(t) + \left[\sum_{j=1, j\neq i}^{d} \delta_{ij} x_j(t)^T x_j(t)\right] \end{aligned}$$

$$+ f_i^T(t) f_i(t) \tag{3.21}$$

$$\leq -\frac{\left[\delta_{ii} - d\lambda_{max}(P_i^T P_i)\right]}{\lambda_{max}(P_i)} V_i(t) + \sum_{j=1, j\neq i}^{d} \frac{\delta_{ij}}{\lambda_{min}(P_j)} V_j(t) + f_i^T(t) f_i(t)$$

$$= -c_{ii} V_i(t) + \sum_{j=1, j\neq i}^{d} c_{ij} V_j(t) + f_i^T(t) f_i(t), \tag{3.22}$$

for $i = 1(1)d$, where Assumption 3.3 and c_{ij} from (3.18) are used in (3.21) and (3.22). Set $\bar{V}(t) = (V_1(t), \cdots, V_d(t))^T$ and $\bar{h}(t) = \left(f_1^T(t) f_1(t), \cdots, f_d^T(t) f_d(t)\right)^T$. Then, one gets the vector-matrix fractional differential inequality

$${}^C D_{0,t}^{\widehat{\alpha}} \bar{V}(t) \leq \Delta_C \bar{V}(t) + \bar{h}(t) \tag{3.23}$$

with initial conditions

$$\bar{V}(0) = \left(x_1^T(0) P_1 x_1(0), \cdots, x_d^T(0) P_d x_d(0)\right)^T. \tag{3.24}$$

Now, consider its vector-matrix comparison system

$${}^C D_{0,t}^{\widehat{\alpha}} \bar{U}(t) = \Delta_C \bar{U}(t) + \bar{h}(t),\ \bar{U}(0) = \bar{V}(0). \tag{3.25}$$

If condition C_1) is satisfied, then Lemma 2.1 gives

$$0 \leq \bar{V}(t) \leq \bar{U}(t). \tag{3.26}$$

Now under Assumption 3.4, if condition C_2) holds, then by Theorem 2.1, one can conclude from (3.25) that $\lim_{t\to\infty} \bar{U}(t) = 0$. Consequently, it follows from (3.26) that $\lim_{t\to\infty} \bar{V}(t) = 0$. It implies that $\lim_{t\to\infty} \|x(t)\| = 0$. This completes the proof. □

Corollary 3.4. *Consider the nonhomogeneous system* (3.14) *with* $\alpha_k = \ell_k \alpha \in (0, 1]$ *for* $k = 1(1)d$, *where* $\ell_k = \frac{u_k}{v_k} \in \mathbb{Q}_+$ *with* $gcd(u_k, v_k) = 1$, *and* $\alpha \in \mathbb{R}_+$. *Set* $M = lcm(v_1, v_2, \cdots, v_d)$. *Let Assumptions 3.3 and 3.4 hold. Moreover, consider the constants defined by* (3.18) *and the matrix given by* (3.19). *If the conditions*

C_1) $c_{ii} > 0$ *for* $i = 1(1)d$,

C_2) *all the characteristic roots of*

$$\det\left[diag\left(\lambda^{M\ell_1}, \lambda^{M\ell_2}, \cdots, \lambda^{M\ell_d}\right) - \Delta_C\right] = 0 \tag{3.27}$$

lie in the sector $|\arg(\lambda)| > \frac{\pi}{2M}\alpha$,

are satisfied, then the system (3.14)-(3.15) *is GAC to* 0.

Corollary 3.5. *Consider* (3.14) *with* $\alpha_1 = \frac{1}{d}\alpha, \alpha_2 = \frac{2}{d}\alpha, \cdots, \alpha_d = \frac{d}{d}\alpha$ *where* $\alpha \in (0, 1]$. *Under Assumptions 3.3 and 3.4 along with the constants* (3.18) *and the matrix* (3.19), *if the conditions*

C_1) $c_{ii} > 0$ *for* $i = 1(1)d$,

C_2) *all the roots of polynomial equation*

$$\det\left[diag\left(\lambda, \lambda^2, \cdots, \lambda^d\right) - \Delta_C\right] = 0 \tag{3.28}$$

lie in the sector $|\arg(\lambda)| > \frac{\pi}{2d}\alpha$,

are satisfied, then the system (3.14)-(3.15) *is GAC to* 0.

Corollary 3.6. *Consider the system* (3.14) *with* $\alpha_1 = \frac{d}{d}\alpha, \alpha_2 = \frac{d-1}{d}\alpha, \cdots, \alpha_d = \frac{1}{d}\alpha$, *where* $\alpha \in (0, 1]$. *Under Assumptions 3.3 and 3.4 along with the constants* (3.18) *and the matrix* (3.19), *if the conditions*

$C_1)$ *c_{ii} for $i = 1(1)d$,*

$C_2)$ *all the characteristic roots of polynomial*

$$\det\left[diag\left(\lambda^d, \lambda^{d-1}, \cdots, \lambda\right) - \Delta_C\right] = 0 \tag{3.29}$$

lie in the sector $|\arg(\lambda)| > \frac{\pi}{2d}\alpha$,

are true, then the system (3.14)-(3.15) *is GAC to* 0.

4. An illustrative example

This section demonstrates some proposed theoretical results on an electrical time-varying circuit system. The standard time-varying circuit system that involves integer derivatives was addressed in the work in [38]. An extension to the standard commensurate fractional order time-varying circuit system was proposed in [39]. Now we consider the modified standard incommensurate version of electrical circuit system

$$\begin{pmatrix} {}^C D_{0,t}^{\alpha_1} i_1(t) \\ {}^C D_{0,t}^{\alpha_2} i_2(t) \end{pmatrix} = A(t)\begin{pmatrix} i_1(t) \\ i_2(t) \end{pmatrix} + B(t)\begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix} \tag{4.1}$$

with respect to the initial conditions $i_k(0) = i_{k0}$ for $k = 1, 2$, where fractional orders $\alpha_1, \alpha_2 \in (0, 1]$, currents $i_k(t)$ for $k = 1, 2$, source voltages $e_k(t) \geq 0$ for $k = 1, 2$, the coefficient matrices
$A(t) = \begin{pmatrix} -\frac{R_1(t)+R_3(t)}{L_1(t)} & \frac{R_3(t)}{L_1(t)} \\ \frac{R_3(t)}{L_2(t)} & -\frac{R_2(t)+R_3(t)}{L_2(t)} \end{pmatrix}$ and $B(t) = \begin{pmatrix} \frac{1}{L_1(t)} & 0 \\ 0 & \frac{1}{L_2(t)} \end{pmatrix}$ with resistance $R_k(t) \geq 0$ for $k = 1, 2, 3$ and inductance $L_k(t) \geq 0$ for $k = 1, 2$.

Here we set up the following circuit system parameters:

$$R_1(t) = 5 + e^{-t} + 2e^t,\ R_2(t) = 3 + \sin(t) + 2e^{2t},\ R_3(t) = 1 + e^{-t},$$
$$L_1(t) = e^t,\ L_2(t) = e^{2t},$$
$$e_1(t) = \sin(t) + 2,\ \ e_2(t) = \sin(2t) + 3.$$

Clearly, the system (4.1) is of the form (3.1). We now demonstrate the following two fractional order cases for the system (4.1).
Case (i): Let $\alpha_1 = \frac{\sqrt{7}}{5}$ and $\alpha_2 = \frac{\sqrt{7}}{3}$. Here the application of Corollary 3.1 is applied to the system (4.1). Take $\delta_{11} = 2$, $\delta_{12} = 2$, $\delta_{21} = 2$ and $\delta_{22} = 2$. Observe that Assumptions 3.1 and 3.2 are satisfied. Consider the constants (see (3.3)) as

$$c_{11} = 2\delta_{11} - \delta_{12} + 1 = 3,$$
$$c_{22} = 2\delta_{22} - \delta_{21} + 1 = 3.$$

It shows that condition $C_1)$ holds. Set $\ell_1 = \frac{1}{5}$, $\ell_2 = \frac{1}{3}$, $\alpha = \sqrt{7}$ and $M = 15$. Then, equation (3.11) becomes

$$\lambda^8 + 3\lambda^5 + 3\lambda^3 + 5 = 0. \tag{4.2}$$

Consequently, one gets

$$\min_i\{|\arg(\lambda_i)|\} \approx 0.7146, \tag{4.3}$$

where λ_i's are the roots of (4.2) for $i = 1(1)8$. Observe that the estimate (4.3) is greater than $\frac{\pi}{2M}\alpha = \frac{\pi}{30}\sqrt{7} \approx 0.2771$. Thus, condition $C_2)$ of Corollary 3.1 also holds. As a result, by Corollary 3.1, one ensures that the system (4.1) should be

GAC to 0. Consequently, the responses $i_1(t)$ and $i_2(t)$ to the system (4.1) must approach to 0 as time increases to ∞.

Case (ii): Let $\alpha_1 = \frac{9}{20}$ and $\alpha_2 = \frac{9}{10}$. Here we apply Corollary 3.2 to system (4.1) to examine the state response. In this case, the following parameters are chosen: $\delta_{11} = 2$, $\delta_{12} = 2$, $\delta_{21} = 2$, $\delta_{22} = 2$. Clearly, Assumptions 3.1 and 3.2 are satisfied. Observe that condition C_1) of Corollary 3.2 is satisfied, since $c_{11} = 3 > 0$ and $c_{22} = 3 > 0$ (see (3.3)). Set $\alpha = \frac{9}{10}$ and $n = 2$. Then, equation (3.12) becomes

$$\lambda^3 + 3\lambda^2 + 3\lambda + 5 = 0. \tag{4.4}$$

Solving equation (4.4), one gets

$$\min_i \{|\arg(\lambda_i)|\} \approx 1.7198, \tag{4.5}$$

where λ_i's are the roots of (4.4) for $i = 1, 2, 3$. Clearly, the obtained estimate in (4.5) is greater than $\frac{\pi}{2n}\alpha = \frac{\pi}{4} \times \frac{9}{10} \approx 0.7069$. Thus, condition C_2) of Corollary 3.2 is also satisfied. As a result, by Corollary 3.2, the system (4.1) should be GAC to 0. Hence, the responses $i_1(t)$ and $i_2(t)$ to the system (4.1) must approach 0 as time approaches ∞.

5. Conclusions

This work proposes some theoretical asymptotic convergence criteria for the analysis of the nonhomogeneous incommensurate fractional order linear time-varying system. It is shown that, whenever the coefficient matrix of such a system and the forcing function satisfy some prior assumptions, then the conditions of the proposed results guarantee the convergence of the responses to such systems to 0. The assumption of coefficient matrix enables the construction of a suitable constant Metzler matrix and while the conditions of proposed results guarantee the limiting behavior of state responses to such systems. Some asymptotic convergence results of the proposed theory are successively demonstrated on an electrical time-varying circuit system to show the effectiveness.

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