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Q-SHIFT DIFFERENCE-DIFFERENTIAL POLYNOMIALS OF MEROMORPHIC FUNCTIONS SHARING A SMALL FUNCTION

HARINA P. WAGHAMORE, MANJUNATH B. E.

Abstract. In this paper, we deal with the uniqueness problem of q-shift difference-differential polynomials $F = (P(f)\prod_{j=1}^{d}(f(q_j z + c_j)^{v_j}))^k$ and $G = (P(g)\prod_{j=1}^{d}(g(q_j z + c_j)^{v_j}))^k$ where $P(z)$ is a polynomial with constant coefficients of degree n sharing small function. The results of this paper are an extension of the previous theorems given by N. V. Thin[7].

1. Introduction and main results

In what follows by a meromorphic function we mean that the function has poles as its singularities only in the complex plane $\mathbb{C}$, we assume that the reader is familier with standard notations such as $T(r,f)$, $m(r,f)$, $N(r,f)$([4], [10], [9]) and $S(r,f)$ denotes any quantity that satisfies the condition $S(r,f) = o(T(r,f))$ as $r \to \infty$ outside of a possible exceptional set of finite linear measure. A meromorphic function $a(z)(\not\equiv 0,\infty)$ de?ned in $\mathbb{C}$ is called a small function with respect to f if $T(r,a(z)) = S(r,f)$. Let f and g be two non-constant meromorphic functions in the complex plane $\mathbb{C}$. We say that $f,\ g$ share a counted multiplicities (CM) if $f-a,\ g-a$ have the same zeros with the same multiplicities and we say that $f,\ g$ share a ignoring multiplicities (IM) if we do not consider the multiplicities, where a is a small function of f and g. Let a be a finite complex number, and k a positive integer. We denote by $N_{(k}(r,a,f)$ the counting function for zeros of $f-a$ with multiplicities atleast k, and by $\overline{N}_{(k}(r,a,f)$ the one for which multiplicity is not counted. Similarly, we denote by $N_{k)}(r,a,f)$ the counting function for zeros of $f-a$ with multiplicities atmost k, and by $N_{k)}(r,a,f)$ the one for which multiplicity is not counted. Then

$$N_{(k}(r,a,f) = \overline{N}_{(1}(r,a,f) + \overline{N}_{(2}(r,a,f) + ... + \overline{N}_{(k}(r,a,f).$$

We denote and define order of $f(z)$ by

$$\rho(f) = lim_{r\to\infty} sup \frac{logT(r,f)}{logr}$$

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If a non-constant meromorphic function $f(z)$ is of zero order, then $\rho(f) = 0$.

In 2015, Zhao and Zhang[12] proved the following results.

Theorem A. [12] *Let $f(z)$ and $g(z)$ be transcendental entire functions of zero order and let $n,\ k$ be positive integers. If $n > 2k+5$, then $(f^n f(qz+c))^{(k)}$ and $(g^n g(qz+c))^{(k)}$ share z or 1 CM, then $f = tg$ for a constant t with $t^{n+1} = 1$.*

Theorem B. [12] *Let $f(z)$ and $g(z)$ be transcendental entire functions of zero order and let $n,\ k$ be positive integers. If $n > 5k+11$, then $(f^n f(qz+c))^{(k)}$ and $(g^n g(qz+c))^{(k)}$ share z or 1 IM, then $f = tg$ for a constant t with $t^{n+1} = 1$.*

In 2017, Thin[7] proved the following theorems for meromorphic functions.

Theorem C. [7] *Let $f(z)$ and $g(z)$ be transcendental meromorphic (resp. entire) functions of zero order, q and c be a complex constants, $q \neq 0$, k be a positive integer, $a(z) \not\equiv 0$ be a meromorphic (resp. entire) small function and let $P(z) = a_n z^n + a_{n-1} z^{n-1} ... + a_1 z + a_0$ be a nonconstant polynomial with constant coefficients $a_0, a_1, .., a_{n-1}, a_n (\neq 0)$ and m be the number of the distinct zeros of $P(z)$. If $n \geq 2m(K+1) + 2k + 6$ (resp. $n \geq 2m(k+1)+4$) and $(P(f(z))f(qz+c))^{(k)}$ and $(P(g(z))g(qz+c))^{(k)}$ share $a(z)$, ∞ - CM, then one of the following two results holds:*

(1) $f(z) = tg(z)$ for a constant t such that $t^d = 1$, where $d = LCM(\lambda_j,\ j = 0, 1, 2, .., n)$ denotes the lowest common multiple of $\lambda_j (j = 0, 1, 2, .., n)$, and

$$\lambda_j = \begin{cases} j+1 & \text{if } a_j \neq 0, \\ n+1 & \text{if } a_j = 0, \end{cases}$$

(2) $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f(z), g(z)) = 0$, where

$$R(w_1, w_2) = P(w_1)w_1(qz+c) - P(w_2)w_2(qz+c).$$

Theorem D. [7] *Let $f(z)$ and $g(z)$ be transcendental meromorphic functions of zero order, q and c be a complex constants, $q \neq 0$, k be a positive integer, $a(z) \not\equiv 0$ be a meromorphic (resp. entire) small function and let $P(z) = a_n z^n + a_{n-1} z^{n-1} ... + a_1 z + a_0$ be a nonconstant polynomial with constant coefficients $a_0, a_1, .., a_{n-1}, a_n (\neq 0)$ and m be the distinct zeros of $P(z)$. If $n \geq 2m(K+2) + 3m(k+1) + 8k + 21$ and $(P(f(z))f(qz+c))^{(k)}$ and $(P(g(z))g(qz+c))^{(k)}$ share $a(z)$ - IM, then one of the following two results holds:*

(1) $(P(f)f(qz+c))^{(k)}(P(g)g(qz+c))^{(k)} \equiv a^2$,
(2) $f(z) = tg(z)$ for a constant t such that $t^d = 1$, where $d = LCM(\lambda_j,\ j = 0, 1, 2, .., n)$ denotes the lowest common multiple of $\lambda_j (j = 0, 1, 2, .., n)$, and

$$\lambda_j = \begin{cases} j+1 & \text{if } a_j \neq 0, \\ n+1 & \text{if } a_j = 0, \end{cases}$$

(3) $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f, g) = 0$, where

$$R(w_1, w_2) = P(w_1)w_1(qz+c) - P(w_2)w_2(qz+c).$$

In this paper, we replace the term $f(qz+c)$ and $g(qz+c)$ in Theorem C and Theorem D and obtained the following results.

Theorem 1.1. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic(resp. entire) functions of zero order, q_j and c_j are complex constants, $q_j \neq 0 (j = 1$ to $d)$, k, n, m are positive integers. Let $a(z)(\not\equiv 0)$ be a small function, let $P(z) = a_n z^n + a_{n-1}z^{n-1}...+a_1 z+a_0$ be a non-constant polynomial with constant coefficient $a_0, a_1, .., a_{n-1}, a_n(\neq 0)$ and m is the number of distinct zeros of $P(z)$. If $n > 2m(k+1)+2\lambda+(k+1)(d+1)+d$ (resp. $n \geq 2m(k+1)+4\lambda$) and $(P(f(z))\prod_{j=1}^{d}(f(q_j z+c_j)^{v_j}))^{(k)}$ and $(P(g(z))\prod_{j=1}^{d}(g(q_j z+c_j)^{v_j}))^{(k)}$ share $a(z), \infty$ CM then one of the following two cases holds:*

(1) $f(z) = tg(z)$ for a constant t such that $t^l = 1$, where $l = GCD\,(\lambda+\gamma_0, \lambda+\gamma_1, .., \lambda+\gamma_n)$, and

$$\gamma_j = \begin{cases} j+1 & \text{if } a_j \neq 0, \\ n+1 & \text{if } a_j = 0, \end{cases}$$

(2) $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f(z), g(z)) = 0$, where

$$R(w_1, w_2) = P(w_1)\prod_{j=1}^{d} w_1(q_j z+c_j)^{v_j} - P(w_2)\prod_{j=1}^{d} w_2(q_j z+c_j)^{v_j}.$$

Remark 1.1. *In Theorem 1.1, if we take $\lambda = d = 1$ then $\prod_{j=1}^{d} f(q_j z+c_j)^{v_j} = f(qz+c)$ and we get $n > 2m(k+1)+2k+5$ (resp. $n \geq 2m(k+1)+4$) and hence Theorem 1.1 reduces to Theorem C .*

Example 1.1. *Let $P(z) = (z-1)^6(z+1)^6 z^{11}$, $f(z) = sin(z)$, $g(z) = cos(z)$. Take $d = 1 = q$, $c = 2\pi$, $k = 0$ then it is easy to verify that, $(P(f(z))\prod_{j=1}^{d}(f(q_j z+c_j)^{v_j}))^{(k)}$ and $(P(g(z))\prod_{j=1}^{d}(g(q_j z+c_j)^{v_j}))^{(k)}$ share $a(z), \infty$ CM. Here f and g satisfy the algebraic equation $R(f, g) = 0$,*

$$\text{i.e., } P(f)\prod_{j=1}^{d} f(q_j z+c_j)^{v_j} - P(g)\prod_{j=1}^{d} g(q_j z+c_j)^{v_j} = 0$$

Theorem 1.2. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of zero order, q_j and c_j are complex constants, $q_j \neq 0$ for all $j = 1$ to d, k, n, m are positive integers. Let $a(z)(\not\equiv 0)$ be a small function, let $P(z) = a_n z^n + a_{n-1}z^{n-1}...+a_1 z+a_0$ be a non-constant polynomial with constant coefficient $a_0, a_1, .., a_{n-1}, a_n(\neq 0)$ and m is the number of distinct zeros of $P(z)$.*
If $n > 2m(k+2)+3m(k+1)+4k(d+1)+8d+5\lambda+7$ and $(P(f(z))\prod_{j=1}^{d}(f(q_j z+c_j)^{v_j}))^{(k)}$ and $(P(g(z))\prod_{j=1}^{d}(g(q_j z+c_j)^{v_j}))^{(k)}$ share $a(z)$ IM then one of the following two cases holds:

(1) $(P(f(z))\prod_{j=1}^{d}(f(q_j z+c_j)^{v_j}))^{(k)}.(P(g(z))\prod_{j=1}^{d}(g(q_j z+c_j)^{v_j}))^{(k)} \equiv a(z)^2$,

(2) $f(z) = tg(z)$ for a constant t such that $t^l = 1$, where $l = GCD(\lambda+\gamma_0, \lambda+\gamma_1, .., \lambda+\gamma_n)$, and

$$\gamma_j = \begin{cases} j+1 & \text{if } a_j \neq 0, \\ n+1 & \text{if } a_j = 0, \end{cases}$$

(3) $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f(z), g(z)) = 0$, where

$$R(w_1, w_2) = P(w_1)\prod_{j=1}^{d} w_1(q_j z + c_j)^{v_j} - P(w_2)\prod_{j=1}^{d} w_2(q_j z + c_j)^{v_j}.$$

Remark 1.2. *In Theorem 1.2, if we take $\lambda = d = 1$ then $\prod_{j=1}^{d} f(q_j z + c_j)^{v_j} = f(qz + c)$, and we get $n > 2m(k + 2) + 3m(k + 1) + 8k + 20$, our results coincides with Theorem D.*

As a particular case of the above theorems, we deduce the following corollaries.

Corollary 1.1. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of zero order such that $q_j \neq 0$ for all $j = 1$ to d, where q_j and c_j are distinct nonzero complex constants. Let $\lambda = \sum_{j=1}^{d} v_j$, k, n are positive integers, $a(z)(\not\equiv 0)$ be a small function of $f(z)$ and $g(z)$, and α a complex constant. If $n > 3k + d(k+2) + 2\lambda + 3$ and $((f - \alpha)^n \prod_{j=1}^{d} (f(q_j z + c_j)^{v_j}))^{(k)}$ and $((g - \alpha)^n \prod_{j=1}^{d} (g(q_j z + c_j)^{v_j}))^{(k)}$ share $a(z), \infty$ CM then one of the following two cases holds:*
(1) $f(z) = tg(z)$ for a constant t with $t^{n+\lambda} = 1$,
(2) $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f(z), g(z)) = 0$, where

$$R(w_1, w_2) = (w_1 - \alpha)^n \prod_{j=1}^{d} w_1(q_j z + c_j)^{v_j} - (w_2 - \alpha)^n \prod_{j=1}^{d} w_2(q_j z + c_j)^{v_j}.$$

Corollary 1.2. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of zero order such that $q_j \neq 0$ for all $j = 1$ to d, where q_j and c_j are distinct non zero complex constants. Let $\lambda = \sum_{j=1}^{d} v_j$, k, n are positive integers, $a(z)(\not\equiv 0)$ be a small function of $f(z)$ and $g(z)$, and α a complex constant. If $n > 9k+4d(k+2)+5\lambda+11$ and $((f - \alpha)^n \prod_{j=1}^{d} (f(q_j z + c_j)^{v_j}))^{(k)}$ and $((g - \alpha)^n \prod_{j=1}^{d} (g(q_j z + c_j)^{v_j}))^{(k)}$ share $a(z)$ IM then one of the following two cases holds:*
(1) $((f - \alpha)^n \prod_{j=1}^{d} (f(q_j z + c_j)^{v_j}))^{(k)}.((g - \alpha)^n \prod_{j=1}^{d} (g(q_j z + c_j)^{v_j}))^{(k)} \equiv a^2$,
(2) $f(z) = tg(z)$ for a constant t with $t^{n+\lambda} = 1$,
(3) $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f(z), g(z)) = 0$, where

$$R(w_1, w_2) = (w_1 - \alpha)^n \prod_{j=1}^{d} w_1(q_j z + c_j)^{v_j} - (w_2 - \alpha)^n \prod_{j=1}^{d} w_2(q_j z + c_j)^{v_j}.$$

2. Some Preliminary Results

To prove our theorems we require the following lemmas.

Lemma 2.1. [5]. *Let $f(z)$ be a nonconstant zero order meromorphic function and let q, c be a nonzero complex number. Then on a set of logarithmic density 1, we have*

$$m\left(r, \frac{f(qz + c)}{f(z)}\right) = S(r, f).$$

Lemma 2.2. [8]. *Let $f(z)$ be a nonconstant meromorphic function of zero order and let q, c be two nonzero complex constants. Then on a set of logarithmic density 1, we have*

$$N\left(r, f(qz+c)\right) = N(r,f) + S(r,f),$$
$$N\left(r, \frac{1}{f(qz+c)}\right) = N(r, \frac{1}{f}) + S(r,f).$$

Lemma 2.3. [8]. *Let $f(z)$ be a nonconstant meromorphic function of zero order and let q, c be two nonzero complex constants. Then on a set of logarithmic density 1, we have*

$$T\left(r, f(qz+c)\right) = T(r,f) + S(r,f).$$

Lemma 2.4. [10]. *Let $f(z)$ be a nonconstant meromorphic function, then*

$$T\left(r, P_n(f)\right) = T(r,f) + S(r,f).$$

Lemma 2.5. [6]. *Let $f(z)$ be a nonconstant meromorphic function, and let p, k be a positive integers. Then*

$$N_p\left(r, \frac{1}{f^{(k)}}\right) \leq T\left(r, f^{(k)}\right) - T(r,f) + N_{p+k}\left(r, \frac{1}{f}\right) + S(r,f),$$
$$N_p\left(r, \frac{1}{f^{(k)}}\right) \leq N_{p+k}\left(r, \frac{1}{f}\right) + k\overline{N}\left(r,f\right) + S(r,f).$$

Lemma 2.6. [11]. *Let $f(z)$ and $g(z)$ be a nonconstant meromorphic functions and let $a(z)(\not\equiv 0,\infty)$ be a small function of $f(z)$ and $g(z)$. If $f(z)$ and $g(z)$ share $a(z)$ IM, then one of the following three cases holds:*

(1) $T\left(r,f\right) \leq N_2\left(r,\frac{1}{f}\right) + N_2(r,f) + N_2\left(r,\frac{1}{g}\right) + N_2(r,g) + 2\left(\overline{N}\left(r,\frac{1}{f}\right) + \overline{N}(r,f)\right) + \left(\overline{N}\left(r,\frac{1}{g}\right) + \overline{N}(r,g)\right) + S(r,f) + S(r,g)$, *and similar inequality holds for $T(r,g)$,*

(2) $fg \equiv 1$,

(3) $f \equiv g$.

Lemma 2.7. *Let $f(z)$ be a transcendental meromorphic function of zero order and $F = P(z)\prod_{j=1}^{d} f(q_j z + c_j)^{v_j}$, $q_j(\not\equiv 0)$, c_j $(j = 1$ to $d)$ are complex constants, n, d be a positive integers. Then*

$$(n-d)T\left(r,f\right) + S(r,f) \leq T\left(r,F\right).$$

Proof. From first fundamental theorem, lemma 2.4 and lemma 2.1, we obtain

$$\begin{aligned}
(n+1)T\left(r,f\right) = T\left(r, f(z)P(f)\right) + S(r,f) &\leq T\left(r, \frac{f(z)F}{\prod_{j=1}^{d} f(q_j z + c_j)^{v_j}}\right) + S(r,f),\\
&\leq T\left(r,F\right) + T\left(r, \frac{\prod_{j=1}^{d} f(q_j z + c_j)^{v_j}}{f(z)}\right) + S(r,f),\\
&\leq T\left(r,F\right) + m\left(r, \frac{\prod_{j=1}^{d} f(q_j z + c_j)^{v_j}}{f(z)}\right) + N\left(r, \frac{\prod_{j=1}^{d} f(q_j z + c_j)^{v_j}}{f(z)}\right) + S(r,f),\\
&\leq T\left(r,F\right) + (d+1)\,T\left(r,f\right) + S(r,f),
\end{aligned}$$

$\therefore (n-d)T(r,f) + s(r,f) \leq T\left(r,F\right)$ on a set of logarithmic density 1. □

Lemma 2.8. *Let $f(z)$ be a transcendental entire function of zero order and $F(z) = P(z)\prod_{j=1}^{d} f(q_jz+c_j)^{v_j}$, where $P(z)$ is polynomial of degree n and $q_j(\not\equiv 0)$, c_j ($j=1$ to d) are complex constants, n, d be a positive integers. Then*

$$nT(r,f)+S(r,f)\le T(r,F).$$

Proof. From first fundamental theorem, lemma 2.4 and lemma2.1, we obtain

$$\begin{aligned}(n+1)T(r,f) = T(r,f(z)P(f))+S(r,f) &\le T\left(r,\frac{f(z)F}{\prod_{j=1}^{d} f(q_jz+c_j)^{v_j}}\right)+S(r,f),\\ &\le T(r,F)+T\left(r,\frac{\prod_{j=1}^{d} f(q_jz+c_j)^{v_j}}{f(z)}\right)+S(r,f),\\ &\le T(r,F)+m\left(r,\frac{\prod_{j=1}^{d} f(q_jz+c_j)^{v_j}}{f(z)}\right)+S(r,f),\\ &\le T(r,F)+T(r,f)+S(r,f),\end{aligned}$$

$\therefore nT(r,f)+S(r,f)\le T(r,F)$ on a set of logarithmic density 1. □

3. Proof Of The Theorems

3.1. Proof of Theorem 1.1.

Proof. Let $F(z) = P(f)\prod_{j=1}^{d} f(q_jz+c_j)^{v_j}$ and $F(z)^{(k)} = (P(f)\prod_{j=1}^{d} f(q_jz+c_j)^{v_j})^{(k)}$ and$G(z) = P(g)\prod_{j=1}^{d} g(q_jz+c_j)^{v_j}$ and $G(z)^{(k)} = (P(g)\prod_{j=1}^{d} g(q_jz+c_j)^{v_j})^{(k)}$. Since $F^{k}(z)$ and $G^{(k)}(z)$ share $a(z),\infty$ CM, there exist a nonzero constant A such that

$$\frac{(P(f)\prod_{j=1}^{d} f(q_jz+c_j)^{v_j})^{(k)}/a(z)-1}{(P(g)\prod_{j=1}^{d} g(q_jz+c_j)^{v_j})^{(k)}/a(z)-1}=A, \tag{1}$$

and we get

$$(P(f)\prod_{j=1}^{d} f(q_jz+c_j)^{v_j})^{(k)}-a(z)(1-A)=A(P(g)\prod_{j=1}^{d} g(q_jz+c_j)^{v_j})^{(k)}.$$

Now, we prove that $A=1$, let on contrary $A=1$.
Using the Second fundamental theorem and by Lemma 2.5, we get

$$\begin{aligned}T\left(r,F^{(k)}\right) &\le \overline{N}\left(r,F^{(k)}\right)+\overline{N}\left(r,\frac{1}{F^{(k)}}\right)+\overline{N}\left(r,\frac{1}{F^{(k)}-\frac{a}{1-A}}\right)+S(r,f),\\ &\le \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F^{(k)}}\right)+\overline{N}\left(r,\frac{1}{G^{(k)}}\right)+S(r,f),\\ &\le \overline{N}(r,F)+T\left(r,F^{(k)}\right)-T(r,F)+N_{k+1}\left(r,\frac{1}{F}\right)+k\overline{N}(r,G)+N_{k+1}\left(r,\frac{1}{G}\right)\\ &\quad+S(r,f)+S(r,g),\end{aligned}$$

which implies

$$\begin{aligned}T(r,F) &\le \overline{N}(r,F)+N_{k+1}\left(r,\frac{1}{F}\right)+k\overline{N}(r,G)+N_{k+1}\left(r,\frac{1}{G}\right)+S(r,f)+S(r,g),\\ &\le [m(k+1)+\lambda+d+1]\,T(r,f)+[m(k+1)+k(1+d)+\lambda]\,T(r,g)+S(r,f)+S(r,g),\end{aligned}$$

$$(n-d)T\left(r,f\right)\leq\left[m(k+1)+\lambda+d+1\right]T\left(r,f\right)+\left[k(d+1)+m(k+1)+\lambda\right]T\left(r,g\right)\\+S(r,f)+S(r,g). \tag{2}$$

Similarly, we get

$$(n-d)T\left(r,g\right)\leq\left[m(k+1)+\lambda+d+1\right]T\left(r,g\right)+\left[k(d+1)+m(k+1)+\lambda\right]T\left(r,f\right)\\+S(r,f)+S(r,g). \tag{3}$$

From 2 and 3, we get

$$(n-d)\left[T(r,f)+T(r,g)\right]\leq\left[2m(k+1)+2\lambda+(d+1)(k+1)\right]\left(T(r,f)+T\left(r,g\right)\right)\\+S(r,f)+S(r,g),$$

i.e., $[n-2m(k+1)+2\lambda+(d+1)(k+1)+d]\left(T(r,f)+T(r,g)\right)\leq S(r,f)+S(r,g)$,

this is contradiction to $n>2m(k+1)+2\lambda+(d+1)(k+1)+d$. Thus, we get $A=1$. Hence from 1, we have

$$(P(f)\prod_{j=1}^{d}f(q_jz+c_j)^{v_j})^{(k)}=(P(g)\prod_{j=1}^{d}g(q_jz+c_j)^{v_j})^{(k)},$$

and we get

$$P(f)\prod_{j=1}^{d}f(q_jz+c_j)^{v_j}=P(g)\prod_{j=1}^{d}g(q_jz+c_j)^{v_j}+\beta(z), \tag{4}$$

where $\beta(z)$ is a polynomial of degree atmost $k-1$. Suppose $\beta(z)\not\equiv 0$, then we get

$$\frac{P(f)\prod_{j=1}^{d}f(q_jz+c_j)^{v_j}}{\beta(z)}=\frac{P(g)\prod_{j=1}^{d}g(q_jz+c_j)^{v_j}}{\beta(z)}+1$$

Therefore from Lemma 2.7, and the second fundamental theorem, we have

$$\begin{aligned}(n-d)T(r,f)&\leq T\left(r,\frac{P(f)\prod_{j=1}^{d}f(q_jz+c_j)^{v_j}}{\beta(z)}\right)+S(r,f),\\&\leq\overline{N}\left(r,\frac{P(f)\prod_{j=1}^{d}f(q_jz+c_j)^{v_j}}{\beta(z)}\right)+\overline{N}\left(r,\frac{\beta(z)}{P(f)\prod_{j=1}^{d}f(q_jz+c_j)^{v_j}}\right)\\&+\overline{N}\left(r,\frac{\beta(z)}{P(g)\prod_{j=1}^{d}g(q_jz+c_j)^{v_j}}\right)+S(r,f),\\&\leq\overline{N}\left(r,f\right)+dT(r,f)+mT(r,f)+\lambda T(r,f)+mT(r,g)+\lambda T(r,g)+S(r,f),\end{aligned}$$

$$(n-d)T\left(r,f\right)\leq\left[m+\lambda+d+1\right]T(r,f)+\left[m+\lambda\right]T(r,g)+S(r,f) \tag{5}$$

Similarly,

$$(n-d)T\left(r,g\right)\leq\left[m+\lambda+d+1\right]T(r,g)+\left[m+\lambda\right]T(r,f)+S(r,f) \tag{6}$$

From 5 and 6, we obtain

$$\left[n-2(m+\lambda)-2d-1\right]\left(T(r,f)+T(r,g)\right)\leq S(r,f)+S(r,g.)$$

This is a contradiction to $n > 2m(k+1)+2\lambda+(k+1)(1+d)+d$. Therefore $\beta(z) \equiv 0$.

Hence 4 becomes

$$P(f)\prod_{j=1}^{d} f(q_jz+c_j)^{v_j} = P(g)\prod_{j=1}^{d} g(q_jz+c_j)^{v_j} \tag{7}$$

That is
$(a_nf^n + a_{n-1}f^{n-1} + .. + a_1f + a_0)(\prod_{j=1}^{d} f(q_jz+c_j)^{v_j}) = (a_ng^n + a_{n-1}g^{n-1} + .. + a_1g + a_0)(\prod_{j=1}^{d} g(q_jz+c_j)^{v_j})$,
let $h = \frac{f}{g}$, we consider the following cases

Case 1. If $h(z)$ is a constant then substituting $f(z) = h(z)g(z)$ in 7, we have $(a_n(gh)^n + a_{n-1}(gh)^{n-1} + .. + a_1(gh) + a_0)(\prod_{j=1}^{d} g(q_jz+c_j)^{v_j} g(q_jz+c_j)^{v_j}) = (a_ng^n + a_{n-1}g^{n-1} + .. + a_1g + a_0)(\prod_{j=1}^{d} g(q_jz+c_j)^{v_j})$,

$$\prod_{j=1}^{d} g(q_jz+c_j)^{v_j}\left[a_ng^n\left(h^{n+\lambda}-1\right) + a_{n-1}g^{n-1}\left(h^{n+\lambda-1}-1\right) + ... + a_0\left(h^{\lambda}-1\right)\right] = 0 \tag{8}$$

Where a_n is a non-zero complex constant and $\prod_{j=1}^{d} g(q_jz+c_j)^{v_j} \not\equiv 0$,
Since $g(z)$ is non-constant meromorphic function, then from 8

$$a_ng^n\left(h^{n+\lambda}-1\right) + a_{n-1}g^{n-1}\left(h^{n+\lambda-1}-1\right) + ... + a_0\left(h^{\lambda}-1\right) = 0 \tag{9}$$

If $a_n(\not\equiv 0)$ and $a_{n-1} = a_{n-2} = ... = a_1 = a_0 = 0$ then from 9 and g is non-constant meromorphic function, we get $h^{n+\lambda} - 1 = 0$ implies $h^{n+\lambda} = 1$
If $a_n(\not\equiv 0)$ and there exist $a_i \neq 0\,[i \in \{0, 1, 2, ...n-1\}]$. Supose that $h^{n+\lambda} \neq 1$, from 9, we have $T(r, g) = S(r, g)$.
Which is contradiction with transcendental function g.
Then $h^{n+\lambda} = 1$, similar to this discussion we can see that $h^{n+\lambda} = 1$, where $a_j \not\equiv 0$, for some $j = 0, 1, 2...n$.
Thus we have $f(z) = tg(z)$, for a constant t such that $t^l = 1$, where $l = GCD(\lambda + \gamma_0, \lambda + \gamma_1, .., \lambda + \gamma_n)$

$$\gamma_j = \begin{cases} j+1 & \text{if } a_j \neq 0, \\ n+1 & \text{if } a_j = 0 \end{cases}$$

Case 2. Suppose $h(z)$ is not constant, then $f(z)$ and $g(z)$ satisifies the algebraic equation $R(f(z), g(z)) = 0$, where

$$R(w_1, w_2) = P(w_1)\prod_{j=1}^{d} w_1(q_jz+c_j)^{v_j} - P(w_2)\prod_{j=1}^{d} w_2(q_jz+c_j)^{v_j}.$$

Note that, when $f(z)$ and $g(z)$ are transcendental entire functions, we have $N(r, F) = 0 = N(r, G)$. By computing similarly to the case of meromorphic functions, we easily obtain the conclusion of Theorem 1.1 with $n \geq 2m(k+1) + 4\lambda$. □

3.2. **Proof of Corollary 1.1.**

Proof. By considering $P(f) = (f-\alpha)^n$ and proceeding as in the lines of proof of Theorem 1.1 we get the proof of Corollary. □

3.3. **Proof of the Theorem 1.2.**

Proof. Let $F(z) = P(f)\prod_{j=1}^{d} f(q_j z + c_j)^{v_j}$ and $F(z)^{(k)} = (P(f)\prod_{j=1}^{d} f(q_j z + c_j)^{v_j})^{(k)}$ and $G(z) = P(g)\prod_{j=1}^{d} g(q_j z + c_j)^{v_j}$ and $G(z)^{(k)} = (P(g)\prod_{j=1}^{d} g(q_j z + c_j)^{v_j})^{(k)}$. Since $F^k(z)$ and $G^{(k)}(z)$ share $a(z)$ IM. If (1) of lemma 2.6 holds, then using the lemma 2.7, we obtain

$$\begin{aligned}
T\left(r, F^{(k)}\right) \leq\ & N_2\left(r, \frac{1}{F^{(k)}}\right) + N_2(r, F^{(k)}) + N_2\left(r, \frac{1}{G^{(k)}}\right) + N_2(r, G^{(k)}) \\
& + 2\left(\overline{N}\left(r, \frac{1}{F^{(k)}}\right) + \overline{N}(r, F^{(k)})\right) + \left(\overline{N}\left(r, \frac{1}{G^{(k)}}\right) + \overline{N}(r, G^{(k)})\right) \\
& + S(r, G) + S(r, F), \\
& \leq N_2(r, F^{(k)}) + T\left(r, F^{(k)}\right) - T(r, F) + N_{k+2}\left(r, \frac{1}{F}\right) + N_{k+2}\left(r, \frac{1}{G}\right) \\
& + k\overline{N}(r, G) + N_2(r, G^{(k)}) + 2\left(N_{k+1}\left(r, \frac{1}{F}\right) + k\overline{N}(r, F) + \overline{N}\left(r, F^{(k)}\right)\right) \\
& + N_{k+1}\left(r, \frac{1}{G}\right) + k\overline{N}(r, G) + \overline{N}\left(r, G^{(k)}\right) + S(r, f) + S(r, g),
\end{aligned}$$

Therefore,

$$\begin{aligned}
T(r, F) \leq\ & 2\overline{N}(r, F) + N_{k+2}\left(r, \frac{1}{G}\right) + N_{k+2}\left(r, \frac{1}{F}\right) + (2k+3)\overline{N}(r, G) \\
& + 2N_{k+1}\left(r, \frac{1}{F}\right) + (2k+2)\overline{N}(r, F) + N_{k+1}\left(r, \frac{1}{G}\right) + S(r, f) + S(r, g).
\end{aligned}$$

$$\begin{aligned}
T(r, F) \leq\ & (2k+4)\,\overline{N}(r, F) + N_{k+2}\left(r, \frac{1}{F}\right) + 2N_{k+1}\left(r, \frac{1}{F}\right) + (2k+3)\overline{N}(r, G) \\
{}' \quad & N_{k+2}\left(r, \frac{1}{G}\right) + N_{k+1}\left(r, \frac{1}{G}\right) + S(r, f) + S(r, g).
\end{aligned} \tag{10}$$

Similarly,

$$\begin{aligned}
T(r, G) \leq\ & (2k+4)\,\overline{N}(r, G) + N_{k+2}\left(r, \frac{1}{G}\right) + 2N_{k+1}\left(r, \frac{1}{G}\right) + (2k+3)\overline{N}(r, F) \\
& N_{k+2}\left(r, \frac{1}{F}\right) + N_{k+1}\left(r, \frac{1}{F}\right) + S(r, f) + S(r, g).
\end{aligned} \tag{11}$$

We have

$$\overline{N}\,(r, F) \leq (1+d)T(r, f) + S(r, f). \tag{12}$$

$$N_{k+2}\left(r, \frac{1}{F}\right) \leq [m(k+2) + \lambda]\, T(r, f) + S(r, f). \tag{13}$$

$$N_{k+1}\left(r, \frac{1}{F}\right) \leq [m(k+1) + \lambda]\, T(r, f) + S(r, f). \tag{14}$$

Similarly,

$$\overline{N}(r,G) \leq (1+d)T(r,g) + S(r,g). \tag{15}$$

$$N_{k+2}\left(r,\frac{1}{G}\right) \leq [m(k+2)+\lambda]\,T(r,g) + S(r,g). \tag{16}$$

$$N_{k+1}\left(r,\frac{1}{G}\right) \leq [m(k+1)+\lambda]\,T(r,g) + S(r,g). \tag{17}$$

Substituting 12-17 in 10, we get

$$\begin{aligned} T(r,F) \leq\ & (2k+4)(1+d)T(r,f) + (m(k+2)+\lambda)\,T(r,f) + 2\,(m(k+1)+\lambda)\,T(r,f) \\ & + (2k+3)(1+d)T(r,g) + (m(k+2)+\lambda)\,T(r,g) + (m(k+1)+\lambda)\,T(r,g) \\ & + S(r,f) + S(r,g), \end{aligned}$$

$$\begin{aligned} (n-d)T(r,f) \leq\ & [(2k+4)(1+d) + m(k+2) + 2m(k+1) + 3\lambda]\,T(r,f) \\ & + [(2k+3)(1+d) + m(k+2) + m(k+1) + 2\lambda]\,T(r,g) \\ & + S(r,f) + S(r,g), \end{aligned} \tag{18}$$

Similarly,

$$\begin{aligned} (n-d)T(r,g) \leq\ & [(2k+4)(1+d) + m(k+2) + 2m(k+1) + 3\lambda]\,T(r,g) \\ & + [(2k+3)(1+d) + m(k+2) + m(k+1) + 2\lambda]\,T(r,f) \\ & + S(r,f) + S(r,g), \end{aligned} \tag{19}$$

From 18 and 19, we obtain

$$\begin{aligned} (n-d)\,[T(r,f)+T(r,g)] \leq\ & [(4k+7)(1+d) + 2m(k+2) + 3m(k+1) + 5\lambda]\,(T(r,f)+T(r,g)) \\ & + S(r,f) + S(r,g). \end{aligned}$$

Which is contradiction to

$$n > 2m(k+2) + 3m(k+1) + 4k(1+d) + 8d + 5\lambda + 7.$$

Thus by Lemma 2.6, we have $F^{(k)}(z).G^{(k)}(z) \equiv a^2(z)$ or $F^{(k)}(z) \equiv G^{(k)}(z)$

Case 1. Suppose $F^{(k)}(z).G^{(k)}(z) \equiv a^2(z)$, that is $(P(f(z))\prod_{j=1}^{d}(f(q_jz+c_j)^{v_j}))^{(k)}.(P(g(z))\prod_{j=1}^{d}(g(q_jz+c_j)^{v_j}))^{(k)} \equiv a(z)^2$.

This is one of the conclusion of Theorem.

Case 2. Now we consider $F^{(k)}(z) \equiv G^{(k)}(z)$. By an argument as in theorem 1.1, we obtain that $f(z)$ and $g(z)$ satisify one of the following two statement:

(1) $f(z) = tg(z)$ for a constant t with $t^l = 1$, where $l = GCD(\lambda+\gamma_0, \lambda+\gamma_1, .., \lambda+\gamma_n)$, and

$$\gamma_j = \begin{cases} j+1 & \text{if } a_j \neq 0, \\ n+1 & \text{if } a_j = 0, \end{cases}$$

(2) $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f(z), g(z)) = 0$, where

$$R(w_1,w_2) = P(w_1)\prod_{j=1}^{d} w_1(q_jz+c_j)^{v_j} - P(w_2)\prod_{j=1}^{d} w_2(q_jz+c_j)^{v_j}.$$

□

3.4. **Proof of Corollary 1.2.**

Proof. By considering $P(f) = (f - \alpha)^n$ and proceeding as in the lines of proof of Theorem 1.2 we get the proof of Corollary. □

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HARINA P. WAGHAMORE

Department of Mathematics, Jnanabharathi Campus, Bangalore University, Bangalore, India - 560 056

E-mail address: harinapw@gmail.com, harina@bub.ernet.in

Manjunath B. E.

Department of Mathematics, Jnanabharathi Campus, Bangalore University, Bangalore, India - 560 056

E-mail address: manjunath.bebub@gmail.com