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SOME NEW NOTES ON THE BICOMPLEX SEQUENCE SPACES $l_p(\mathbb{BC})$

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ABSTRACT. In this study, relationships among different bicomplex sequence spaces $l_p(\mathbb{BC})$ are examined. Also, using the property of completeness, it is obtained that the spaces $l_p(\mathbb{BC})$ are Banach $\mathbb{BC}-$module for $1 \leq p \leq \infty$ and the spaces $l_p(\mathbb{BC})$ are $p-$Banach $\mathbb{BC}-$module for $0 < p < 1$. Moreover, some topological properties of bicomplex sequence spaces such as solidity, seperability etc. are properly investigated. Our proofs and results obtained are well involved and significant.

1. Introduction

In 1892 Segre [1] had introduced the concept of bicomplex numbers. The main contribution in bicomplex analysis was the pioneering works of Price [2] and Alpay et al. [3]. Price [2] introduced the multicomplex spaces and functions. Functional analysis in $\mathbb{BC}$, a substantially new subject, is not only relevant from a mathematical point of view, but also has significant applications in physics and engineering. Alpay et al. [3] developed a general theory of functional analysis with bicomplex scalars.

Sequence spaces play a central role in many areas of mathematics. The most popular sequence spaces are the spaces l_p which consist of absolutely $p-$summable complex sequences having a lot of useful applications. Since they also have rich topological and geometric properties, researchers are motivated to use them to obtain new results in different sequence spaces. Recent works noted in [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16] are some examples on topological properties of some sequence spaces.

Sager and Sağır [17] introduced bicomplex sequence spaces with Euclidean norm in the set of bicomplex numbers and in [18] established the quasi-Banach algebra $\mathbb{BC}(N)$ by defining non-Newtonian bicomplex numbers as a generalization of both bicomplex numbers and non-Newtonian complex numbers. Also they examined the validity of non-Newtonian bicomplex version of the well-known Hölder's and Minkowski's inequalities for sums.

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Following the same line, our aim in this study is to extend inclusion relations and topological properties in the spaces l_p to bicomplex sequence spaces $l_p\left(\mathbb{BC}\right)$. Since $l_p \subset l_p\left(\mathbb{BC}\right)$ our results are more general.

2. Preliminaries

This section deals with some necessary definitions and results which are used in this research.

Definition 1. [2] *Let i and j be independent imaginary units such that $i^2 = j^2 = -1$, $ij = ji$ and $\mathbb{C}\left(i\right)$ be the set of complex numbers with the imaginary unit i. The set of bicomplex numbers $\mathbb{BC}$ is defined by $\mathbb{BC} = \left\{z = z_1 + jz_2 : z_1, z_2 \in \mathbb{C}\left(i\right)\right\}$.*

Theorem 1. [2] *The set $\mathbb{BC}$ forms a Banach space and a ring with respect to the addition, scalar multiplication and norm for all $z = z_1 + jz_2, w = w_1 + jw_2 \in \mathbb{BC}$ and for all $\lambda \in \mathbb{R}$ defined by*

$$\begin{aligned}
z + w &= (z_1 + jz_2) + (w_1 + jw_2) = (z_1 + w_1) + j\left(z_2 + w_2\right), \\
z \times w &= zw = (z_1 + jz_2)\left(w_1 + jw_2\right) = (z_1w_1 - z_2w_2) + j\left(z_1w_2 + z_2w_1\right), \\
\lambda . z &= \lambda z = \lambda . \left(z_1 + jz_2\right) = \lambda z_1 + j\lambda z_2, \\
\|.\| &: \mathbb{BC} \to \mathbb{R}, z \to \|z\| = \sqrt{\left|z_1\right|^2 + \left|z_2\right|^2}.
\end{aligned}$$

Remark 1. [2] *The numbers $e_1 = \frac{1+ij}{2}$ and $e_2 = \frac{1-ij}{2}$ form idempotent basis of bicomplex numbers and hence any bicomplex number $z = z_1 + jz_2$ is uniquely written as $z = \beta_1 e_1 + \beta_2 e_2$ where $\beta_1 = z_1 - iz_2$, $\beta_2 = z_1 + iz_2 \in \mathbb{C}\left(i\right)$. This formula is called the idempotent representation of z.*

Definition 2. [19] *Let X be a linear space over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, $0 < p \leq 1$ and $|||.||| : X \to \mathbb{R}$ be a mapping such that the following properties hold:*
(i) $|||x||| \geq 0$ for all $x \in X$.
(ii) If $|||x||| = 0$, then $x = 0$.
(iii) $|||\mu x||| = \left|\mu\right|^p . |||x|||$ for all $x \in X$ and for all $\mu \in \mathbb{F}$.
(iv) $|||x + y||| \leq |||x||| + |||y|||$ for all $x, y \in X$.
Then, we say that $|||.|||$ is a $p-$norm on X and X is a $p-$normed space with the $p-$norm $|||.|||$.

If a $p-$normed space is complete, then it is said to be a $p-$Banach space [13].

Definition 3. [21] *Let X be a topological space. Then we say that X is seperable if and only if there is a countable subset of X which is dense in X.*

Definition 4. [17]

$$\begin{aligned}
l_\infty\left(\mathbb{BC}\right) &: = \left\{ s = (s_k) \in w\left(\mathbb{BC}\right) : \sup_{k\in\mathbb{N}} \|s_k\|_{\mathbb{BC}} < \infty \right\}, \\
l_p\left(\mathbb{BC}\right) &: = \left\{ s = (s_k) \in w\left(\mathbb{BC}\right) : \sum_{k=1}^{\infty} \|s_k\|_{\mathbb{BC}}^p < \infty \right\} \text{ for } 0 < p < \infty,
\end{aligned}$$

where $w\left(\mathbb{BC}\right)$ denotes the spaces of all bicomplex sequences.

Theorem 2. [17] $l_\infty(\mathbb{BC})$ *is a Banach space with the norm* $\|.\|_{l_\infty(\mathbb{BC})}$ *characterized by*

$$\|s\|_{l_\infty(\mathbb{BC})} = \sup_{k\in\mathbb{N}} \|s_k\|_{\mathbb{BC}}$$

for all $s = (s_k) \in l_\infty(\mathbb{BC})$.

Theorem 3. [17] *The space* $l_p(\mathbb{BC})$ *is a Banach space for* $1 \leq p < \infty$ *with the norm* $\|.\|_{l_p(\mathbb{BC})}$ *defined by*

$$\|s\|_{l_p(\mathbb{BC})} = \left(\sum_{k=1}^{\infty} \|s_k\|_{\mathbb{BC}}^p\right)^{\frac{1}{p}}$$

for all $s = (s_k) \in l_p(\mathbb{BC})$, and the space $l_p(\mathbb{BC})$ is a $p-$Banach space for $0 < p < 1$ with the $p-$norm $|||.|||_{l_p(\mathbb{BC})}$ defined by

$$|||s|||_{l_p(\mathbb{BC})} = \sum_{k=1}^{\infty} \|s_k\|_{\mathbb{BC}}^p$$

for all $s = (s_k) \in l_p(\mathbb{BC})$.

3. Main Results

This section deals with the inclusion relations of the spaces $l_\infty(\mathbb{BC})$ and $l_p(\mathbb{BC})$ for $0 < p < \infty$. Also it is shown that $l_p(\mathbb{BC})$ are Banach $\mathbb{BC}-$module with its norm and certain topological properties are examined here.

Theorem 4. *For* $0 < p < q < \infty$, *we have the inclusion* $l_p(\mathbb{BC}) \subset l_q(\mathbb{BC})$. *Also, this inclusion strictly holds, where* $1 \leq p < q < \infty$.

Proof. It is obvious that for $0 < p < q < \infty$ the inclusion $l_p(\mathbb{BC}) \subset l_q(\mathbb{BC})$ holds. Let $\zeta = (\zeta_n) \in l_p(\mathbb{BC})$. This implies that there exists a $n_0(\varepsilon) \in \mathbb{N}$ such that $\|\zeta_n\|_{\mathbb{BC}} < 1$ for all $n \geq n_0$. Then we can write $\|\zeta_n\|_{\mathbb{BC}}^{q-p} < 1$ for all $n \geq n_0$. Therefore, if we take $M = \max\left\{\|\zeta_1\|_{\mathbb{BC}}^{q-p}, \|\zeta_2\|_{\mathbb{BC}}^{q-p}, ..., \|\zeta_{n_0}\|_{\mathbb{BC}}^{q-p}, 1\right\}$, we obtain that

$$\sum_{n=1}^{\infty} \|\zeta_n\|_{\mathbb{BC}}^q = \sum_{n=1}^{\infty} \|\zeta_n\|_{\mathbb{BC}}^p \|\zeta_n\|_{\mathbb{BC}}^{q-p} < M\sum_{n=1}^{\infty} \|\zeta_n\|_{\mathbb{BC}}^p < \infty$$

and hence $\sum\limits_{n=1}^{\infty} \|\zeta_n\|_{\mathbb{BC}}^q < \infty$ which means that $\zeta = (\zeta_n) \in l_q(\mathbb{BC})$.

We now want to indicate that the inclusion is strict for $1 \leq p < q < \infty$. Set the sequence $\zeta = (\zeta_n)$ characterized by $\zeta_n = j\frac{1}{n^{\frac{1}{p}}}$ for all $n \in \mathbb{N}$ where j is a bicomplex number. Then, since

$$\sum_{n=1}^{\infty} \|\zeta_n\|_{\mathbb{BC}}^q = \sum_{n=1}^{\infty} \left(\sqrt{\frac{1}{n^{\frac{2}{p}}}}\right)^q = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{q}{p}}}$$

and $\frac{q}{p} > 1$, the series $\sum\limits_{n=1}^{\infty} \|\zeta_n\|_{\mathbb{BC}}^q$ converges. This implies that $\zeta = (\zeta_n) \in l_q(\mathbb{BC})$. Besides, since

$$\sum_{n=1}^{\infty} \|\zeta_n\|_{\mathbb{BC}}^p = \sum_{n=1}^{\infty} \left(\sqrt{\frac{1}{n^{\frac{2}{p}}}}\right)^p = \sum_{n=1}^{\infty} \frac{1}{n}$$

and $\sum\limits_{n=1}^{\infty}\frac{1}{n}$ is divergent, the series $\sum\limits_{n=1}^{\infty}\|\zeta_n\|_{\mathbb{BC}}^p$ doesn't converge, and so $\zeta=(\zeta_n)\notin l_p(\mathbb{BC})$. Thus, the sequence $\zeta=(\zeta_n)$ is in $l_q(\mathbb{BC})$, but not in $l_p(\mathbb{BC})$. So, we conclude that $l_p(\mathbb{BC})\subset l_q(\mathbb{BC})$ is a strict inclusion for $1\le p<q<\infty$. □

Theorem 5. *For $0<p<\infty$, we have the inclusion $l_p(\mathbb{BC})\subset l_\infty(\mathbb{BC})$. Also, this inclusion strictly holds, where $1\le p<\infty$.*

Proof. Let $\zeta=(\zeta_n)\in l_p(\mathbb{BC})$. Then, we have $\sum\limits_{n=1}^{\infty}\|\zeta_n\|_{\mathbb{BC}}^p<\infty$ and so, this implies that there exists a $n_0(\varepsilon)\in\mathbb{N}$ such that $\|\zeta_n\|_{\mathbb{BC}}<1$ for all $n\ge n_0$. If we take $M=\max\{\|\zeta_1\|_{\mathbb{BC}},\|\zeta_2\|_{\mathbb{BC}},...,\|\zeta_{n_0}\|_{\mathbb{BC}},1\}$, we obtain that $\sup\{\|\zeta_n\|_{\mathbb{BC}}:n\in\mathbb{N}\}\le M<\infty$ which means that $\zeta=(\zeta_n)\in l_\infty(\mathbb{BC})$.

Now we have to verify the strictness of the inclusion for $1\le p<\infty$. Set the sequence $\zeta=(\zeta_n)$ characterized by $\zeta_n=j\frac{1}{n^{\frac{1}{p}}}$ for all $n\in\mathbb{N}$. Then, since

$$\sup\{\|\zeta_n\|_{\mathbb{BC}}:n\in\mathbb{N}\}=\sup\left\{\left\|j\frac{1}{n^{\frac{1}{p}}}\right\|_{\mathbb{BC}}:n\in\mathbb{N}\right\}=\sup\left\{\frac{1}{n^{\frac{1}{p}}}:n\in\mathbb{N}\right\}\le 1$$

we have $\zeta=(\zeta_n)\in l_\infty(\mathbb{BC})$. Furthermore, since $\sum\limits_{n=1}^{\infty}\|\zeta_n\|_{\mathbb{BC}}^p=\sum\limits_{n=1}^{\infty}\frac{1}{n}$ and $\sum\limits_{n=1}^{\infty}\frac{1}{n}$ is divergent, $\sum\limits_{n=1}^{\infty}\|\zeta_n\|_{\mathbb{BC}}^p$ doesn't converge, and so $\zeta=(\zeta_n)\notin l_p(\mathbb{BC})$. From this, $\zeta\in l_\infty(\mathbb{BC})\setminus l_p(\mathbb{BC})$ for $1\le p<\infty$. This completes the proof.

Firstly, we state that the set $w(\mathbb{BC})$ defined by $\{\zeta=(\zeta_n):\forall n\in\mathbb{N},\ \zeta_n\in\mathbb{BC}\}$ is a $\mathbb{BC}-$module. □

Theorem 6. *The set $w(\mathbb{BC})$ forms a $\mathbb{BC}-$module with the operations addition and bicomplex scalar multiplication as follows:*

$$\begin{aligned} + &: w(\mathbb{BC})\times w(\mathbb{BC})\to w(\mathbb{BC}),\ (s,t)\to s+t=(s_n+t_n)\\ \cdot &: \mathbb{BC}\times w(\mathbb{BC})\to w(\mathbb{BC}),\ (\lambda,s)\to\lambda\cdot s=\lambda s=(\lambda s_n)\end{aligned}$$

for all $s=(s_n)$, $t=(t_n)\in w(\mathbb{BC})$ and for all $\lambda\in\mathbb{BC}$.

Proof. The proof of this theorem is direct applications of definitions. □

Definition 5. *Let A be a normed algebra over $\mathbb{F}$, and let M be a $p-$normed space over $\mathbb{F}$. M is called a $p-$normed left (right) $A-$module if M is a left (right) $A-$module and there is a positive contant K such that $|||am|||\le K\|a\|^p|||m|||$ ($|||ma|||\le K|||m|||\,\|a\|^p$) for all $a\in A$ and for all $m\in M$. A $p-$normed $A-$module is both a $p-$normed left $A-$module and a $p-$normed right $A-$module. A $p-$normed left (right) $A-$module is called a $p-$Banach left (right) $A-$module if it is complete as a $p-$normed space. A $p-$Banach $A-$module is both a $p-$Banach left $A-$module and a $p-$Banach right $A-$module.*

Now, we obtain that $l_\infty(\mathbb{BC})$ is a Banach $\mathbb{BC}-$module with the norm $\|.\|_{l_\infty(\mathbb{BC})}$, $l_p(\mathbb{BC})$ is a $p-$Banach $\mathbb{BC}-$module by defining as above with the $p-$norm $|||.|||_{l_p(\mathbb{BC})}$ for $0<p<1$ and $l_p(\mathbb{BC})$ is Banach $\mathbb{BC}-$module with the norm $\|.\|_{l_p(\mathbb{BC})}$ for $1\le p<\infty$.

Theorem 7. *$l_\infty(\mathbb{BC})$ is a $\mathbb{BC}-$submodule of $w(\mathbb{BC})$.*

Proof. It has been showed that $l_\infty(\mathbb{BC})$ is a subspace of $w(\mathbb{BC})$ in [17]. Also, we get

$$\begin{aligned} \sup\{\|\lambda s_n\|_{\mathbb{BC}} : n\in\mathbb{N}\} &\leq \sup\left\{\sqrt{2}\,\|\lambda\|_{\mathbb{BC}}\,\|s_n\|_{\mathbb{BC}} : n\in\mathbb{N}\right\} \\ &= \sqrt{2}\,\|\lambda\|_{\mathbb{BC}}\sup\{\|s_n\|_{\mathbb{BC}} : n\in\mathbb{N}\} \\ &< \infty \end{aligned} \tag{3.1}$$

for all $\lambda\in\mathbb{BC}$ and for all $s\in l_\infty(\mathbb{BC})$ and so, $\lambda s\in l_\infty(\mathbb{BC})$. That is to say that $l_\infty(\mathbb{BC})$ is a $\mathbb{BC}-$submodule of $w(\mathbb{BC})$. □

Theorem 8. *$l_\infty(\mathbb{BC})$ is a Banach $\mathbb{BC}-$module with the norm $\|.\|_{l_\infty(\mathbb{BC})}$.*

Proof. From inequality (3.1) we write $\|\lambda s\|_{l_\infty(\mathbb{BC})}\leq\sqrt{2}\,\|\lambda\|_{\mathbb{BC}}\,\|s\|_{l_\infty(\mathbb{BC})}$ for all $\lambda\in\mathbb{BC}$ and for all $s\in l_\infty(\mathbb{BC})$. Thus, $l_\infty(\mathbb{BC})$ is a normed $\mathbb{BC}-$module. Also, we know that $l_\infty(\mathbb{BC})$ is a Banach space with the norm $\|.\|_{l_\infty(\mathbb{BC})}$. Therefore, $l_\infty(\mathbb{BC})$ is a Banach $\mathbb{BC}-$module with the norm $\|.\|_{l_\infty(\mathbb{BC})}$. □

Theorem 9. *For $0<p<\infty$, $l_p(\mathbb{BC})$ is a $\mathbb{BC}-$submodule of $w(\mathbb{BC})$.*

Proof. It has been showed that $l_p(\mathbb{BC})$ is a subspace of $w(\mathbb{BC})$ for $0<p<\infty$ in [17]. Also, we obtain that for all $s,t\in l_p(\mathbb{BC})$ and for all $\lambda\in\mathbb{BC}-\{0\}$

$$\begin{aligned} \sum_{k=1}^{\infty}\|\lambda s_k\|_{\mathbb{BC}}^p &\leq \sum_{k=1}^{\infty}\left(\sqrt{2}\right)^p\|\lambda\|_{\mathbb{BC}}^p\|s_k\|_{\mathbb{BC}}^p \\ &= \left(\sqrt{2}\right)^p\|\lambda\|_{\mathbb{BC}}^p\sum_{k=1}^{\infty}\|s_k\|_{\mathbb{BC}}^p<\infty \end{aligned} \tag{3.2}$$

holds for $0<p<1$ and

$$\begin{aligned} \left(\sum_{k=1}^{\infty}\|\lambda s_k\|_{\mathbb{BC}}^p\right)^{\frac{1}{p}} &\leq \left(\sum_{k=1}^{\infty}\left(\sqrt{2}\right)^p\|\lambda\|_{\mathbb{BC}}^p\|s_k\|_{\mathbb{BC}}^p\right)^{\frac{1}{p}} \\ &= \sqrt{2}\,\|\lambda\|_{\mathbb{BC}}\left(\sum_{k=1}^{\infty}\|s_k\|_{\mathbb{BC}}^p\right)^{\frac{1}{p}}<\infty \end{aligned} \tag{3.3}$$

holds for $1\leq p<\infty$. That means $\lambda s\in l_p(\mathbb{BC})$. That is to say that $l_p(\mathbb{BC})$ for $0<p<\infty$ is a $\mathbb{BC}-$submodule of $w(\mathbb{BC})$. □

Theorem 10. *For $0<p<1$, $l_p(\mathbb{BC})$ is a $p-$Banach $\mathbb{BC}-$module with the $p-$norm $|||.|||_{l_p(\mathbb{BC})}$.*

Proof. From inequality (3.2) we write $|||\lambda s|||_{l_p(\mathbb{BC})}\leq\left(\sqrt{2}\right)^p\|\lambda\|_{\mathbb{BC}}^p\,|||s|||_{l_p(\mathbb{BC})}$ for all $\lambda\in\mathbb{BC}$ and for all $s\in l_p(\mathbb{BC})$. Thus, $l_p(\mathbb{BC})$ is a $p-$normed $\mathbb{BC}-$module. Also, we know that $l_p(\mathbb{BC})$ is a $p-$Banach space with the $p-$norm $|||.|||_{l_p(\mathbb{BC})}$. Therefore, $l_p(\mathbb{BC})$ is a $p-$Banach $\mathbb{BC}-$module with the $p-$norm $|||.|||_{l_p(\mathbb{BC})}$. □

Theorem 11. *For $1\leq p<\infty$, $l_p(\mathbb{BC})$ is a Banach $\mathbb{BC}-$module with the norm $\|.\|_{l_p(\mathbb{BC})}$.*

Proof. From inequality (3.3) we write $\|\lambda s\|_{l_p(\mathbb{BC})} \leq \sqrt{2}\,\|\lambda\|_{\mathbb{BC}}\,\|s\|_{l_p(\mathbb{BC})}$ for all $\lambda \in \mathbb{BC}$ and for all $s \in l_p(\mathbb{BC})$. Thus, $l_p(\mathbb{BC})$ is a normed $\mathbb{BC}-$module. Also, we know that $l_p(\mathbb{BC})$ is a Banach space with the norm $\|.\|_{l_p(\mathbb{BC})}$. Therefore, $l_p(\mathbb{BC})$ is a Banach $\mathbb{BC}-$module with the norm $\|.\|_{l_p(\mathbb{BC})}$. □

The following results are devoted to topological properties of bicomplex sequence spaces $l_p(\mathbb{BC})$ for $0 < p \leq \infty$.

Definition 6. *Let X be a bicomplex sequence space and*

$$\widetilde{X} := \{(u_n) \in w(\mathbb{BC}) : \exists (x_n) \in X \text{ such that } \|u_n\|_{\mathbb{BC}} \leq \|x_n\|_{\mathbb{BC}} \text{ for all } n \in \mathbb{N}\}.$$

Then, X is said to be bicomplex solid (normal) if and only if $\widetilde{X} \subset X$.

Definition 7. *Let X be a bicomplex sequence space,*

$$A := \{x = (x_n) \in w(\mathbb{BC}) : \forall n \in \mathbb{N},\ x_n \in \{0,1\}\}$$

and $M_0 := spA$. Then, X is called bicomplex monotone if and only if $M_0X \subset X$.

Definition 8. *If X is a Banach bicomplex sequence space and $\zeta_l^{(n)} \to \zeta_l$ $(n \to \infty)$ for all $l \in \mathbb{N}$ whenever $\zeta^{(n)} \to \zeta$ $(n \to \infty)$, X is called a bicomplex $BK-$space.*

Definition 9. *Let X be a bicomplex sequence space and π denote the set of all permutations of $\mathbb{N}$, that is, injective and surjective maps of $\mathbb{N}$. Then, X is called bicomplex symmetric if $x_\sigma = (x_{\sigma_k}) \in X$ whenever $x \in X$ and $\sigma \in \pi$.*

Theorem 12. *$l_\infty(\mathbb{BC})$ is a bicomplex solid space.*

Proof. Let

$$(s_n) \in l_\infty\widetilde{(\mathbb{BC})} := \{(u_n) \in w(\mathbb{BC}) : \exists (x_n) \in l_\infty(\mathbb{BC}),\ \|u_n\|_{\mathbb{BC}} \leq \|x_n\|_{\mathbb{BC}}, \forall n \in \mathbb{N}\}.$$

Then, there is a sequence $(t_n) \in l_\infty(\mathbb{BC})$ such that $\|s_n\|_{\mathbb{BC}} \leq \|t_n\|_{\mathbb{BC}}$ for all $n \in \mathbb{N}$. Therefore, $\sup\{\|t_n\|_{\mathbb{BC}} : n \in \mathbb{N}\} < \infty$ and so, $\sup\{\|s_n\|_{\mathbb{BC}} : n \in \mathbb{N}\} < \infty$. This implies that $(s_n) \in l_\infty(\mathbb{BC})$. Then, we have the inclusion $l_\infty\widetilde{(\mathbb{BC})} \subset l_\infty(\mathbb{BC})$ which means that $l_\infty(\mathbb{BC})$ is bicomplex solid. □

Theorem 13. *$l_\infty(\mathbb{BC})$ is a bicomplex monotone space.*

Proof. Let $(\zeta_n) \in M_0 l_\infty(\mathbb{BC})$. Then, there exist $(s_n) \in M_0$ and $(t_n) \in l_\infty(\mathbb{BC})$ such that $(\zeta_n) = (s_n t_n)$. Therefore, $\{s_n : n \in \mathbb{N}\}$ is finite and so, we have

$$\sup\{\|s_n\|_{\mathbb{BC}} : n \in \mathbb{N}\} < \infty.$$

Then, since

$$\begin{aligned} \sup\{\|s_n t_n\|_{\mathbb{BC}} : n \in \mathbb{N}\} &\leq \sup\left\{\sqrt{2}\,\|s_n\|_{\mathbb{BC}}\,\|t_n\|_{\mathbb{BC}} : n \in \mathbb{N}\right\} \\ &= \sqrt{2}\sup\{\|s_n\|_{\mathbb{BC}} : n \in \mathbb{N}\}\sup\{\|t_n\|_{\mathbb{BC}} : n \in \mathbb{N}\}, \end{aligned}$$

we write $\sup\{\|s_n t_n\|_{\mathbb{BC}} : n \in \mathbb{N}\} < \infty$. This shows that $(\zeta_n) \in l_\infty(\mathbb{BC})$. The proof is completed. □

Theorem 14. *$l_\infty(\mathbb{BC})$ is a bicomplex $BK-$space.*

Proof. Let $\left(\zeta^{(n)}\right) \in l_\infty(\mathbb{BC})$ such that $\zeta^{(n)} \to \zeta$ as $n \to \infty$. Then, for every $\varepsilon > 0$ there is a $n_0(\varepsilon) \in \mathbb{N}$ such that $\left\|\zeta^{(n)} - \zeta\right\|_{l_\infty(\mathbb{BC})} < \varepsilon$ for all $n \geq n_0(\varepsilon)$. Therefore, we have $\sup\left\{\left\|\zeta_l^{(n)} - \zeta_l\right\|_{\mathbb{BC}} : l \in \mathbb{N}\right\} < \varepsilon$ for all $n \geq n_0(\varepsilon)$. So, for any fixed $l \in \mathbb{N}$ and for all $n \geq n_0(\varepsilon)$ we can write $\left\|\zeta_l^{(n)} - \zeta_l\right\|_{\mathbb{BC}} < \varepsilon$. This implies that $\left(\zeta_l^{(n)}\right)$ converges to the bicomplex number ζ_l. Thus, the coordinates are continuous on $l_\infty(\mathbb{BC})$. This completes the proof. □

Theorem 15. *$l_\infty(\mathbb{BC})$ is a bicomplex symmetric space.*

Proof. Let $(s_n) \in l_\infty(\mathbb{BC})$ and $\sigma \in \pi$. Then, since $\sigma : \mathbb{N} \to \mathbb{N}$ is an injective and surjective function, we have $\left\{\left\|s_{\sigma(n)}\right\|_{\mathbb{BC}} : n \in \mathbb{N}\right\} = \left\{\|s_n\|_{\mathbb{BC}} : n \in \mathbb{N}\right\}$. Then, the equality $\sup\left\{\left\|s_{\sigma(n)}\right\|_{\mathbb{BC}} : n \in \mathbb{N}\right\} = \sup\left\{\|s_n\|_{\mathbb{BC}} : n \in \mathbb{N}\right\}$ holds. Since $\sup\left\{\|s_n\|_{\mathbb{BC}} : n \in \mathbb{N}\right\} < \infty$, we get $\sup\left\{\left\|s_{\sigma(n)}\right\|_{\mathbb{BC}} : n \in \mathbb{N}\right\} < \infty$. This means that $\left(s_{\sigma(n)}\right) \in l_\infty(\mathbb{BC})$. The proof is completed. □

Theorem 16. *$l_\infty(\mathbb{BC})$ is not a seperable space.*

Proof. Let $E = \{s = (s_n) \in w(\mathbb{BC}) : s_n \in \{0, j\}, \ \forall n \in \mathbb{N}\}$. It is not to hard show that E is not countable. So, we omit the details.

Let $s = (s_n)$, $t = (t_n) \in E$ and $s \neq t$. Then,

$$d_{l_\infty(\mathbb{BC})}(s,t) = \sup\left\{\|s_n - t_n\|_{\mathbb{BC}} : n \in \mathbb{N}\right\} = 1.$$

Consider the open balls $B\left(s, \frac{1}{2}\right)$ for $s \in E$. Since

$$\begin{aligned} B\left(s, \frac{1}{2}\right) &= \left\{t \in l_\infty(\mathbb{BC}) : d_{l_\infty(\mathbb{BC})}(s,t) < \frac{1}{2}\right\} \\ &= \left\{t \in l_\infty(\mathbb{BC}) : d_{l_\infty(\mathbb{BC})}(s,t) = 0\right\} \\ &= \{s\}, \end{aligned}$$

we have $\bigcup_{x \in E} B\left(s, \frac{1}{2}\right) = E$ and $B\left(s, \frac{1}{2}\right) \cap B\left(t, \frac{1}{2}\right) = \varnothing$. Hence, E can be written uncountably infinite union of distinct open balls.

Now, let Y be any dense subset of $l_\infty(\mathbb{BC})$, that is, $\overline{Y} = l_\infty(\mathbb{BC})$. Then, for all $s \in E$, we can write $B\left(s, \frac{1}{2}\right) \cap Y \neq \varnothing$. Since $B\left(s, \frac{1}{2}\right) = \{s\}$, we have $s \in Y$ and hence, $E \subset Y$. Since E is not countable, Y is not countable. Thus, no dense set of the space $l_\infty(\mathbb{BC})$ can be countable. This proves that $l_\infty(\mathbb{BC})$ is not seperable. The proof is completed. □

Theorem 17. *$l_p(\mathbb{BC})$ is a bicomplex solid space for $0 < p < \infty$.*

Proof. Let $(s_n) \in l_p(\widetilde{\mathbb{BC}})$Then, there exists a sequence $(t_n) \in l_p(\mathbb{BC})$ such that $\|s_n\|_{\mathbb{BC}} \leq \|t_n\|_{\mathbb{BC}}$ for all $n \in \mathbb{N}$. So, we can write $\|s_n\|_{\mathbb{BC}}^p \leq \|t_n\|_{\mathbb{BC}}^p$ for all $n \in \mathbb{N}$. Therefore, since the series $\sum_{n=1}^{\infty} \|t_n\|_{\mathbb{BC}}^p$ is convergent, the comparison test implies that the series $\sum_{n=1}^{\infty} \|s_n\|_{\mathbb{BC}}^p$ is comvergent. Then, we obtain that $(s_n) \in l_p(\mathbb{BC})$. Therefore, we have the inclusion $l_p(\widetilde{\mathbb{BC}}) \subset l_p(\mathbb{BC})$ which means that $l_p(\mathbb{BC})$ is bicomplex solid. □

Theorem 18. *$l_p(\mathbb{BC})$ is a bicomplex monotone space for $0 < p < \infty$.*

Proof. Let $(\zeta_n) \in M_0 l_p(\mathbb{BC})$. Then, there exist $(s_n) \in M_0$ and $(t_n) \in l_p(\mathbb{BC})$ such that $(\zeta_n) = (s_n t_n)$. Therefore, $\{s_n : n \in \mathbb{N}\}$ is finite and so, $\sup\{\|s_n\|_{\mathbb{BC}} : n \in \mathbb{N}\} < \infty$ and $\sup\{\|s_n\|_{\mathbb{BC}}^p : n \in \mathbb{N}\} < \infty$. Then, since

$$\|s_n t_n\|_{\mathbb{BC}}^p \leq \left(\sqrt{2}\right)^p \|s_n\|_{\mathbb{BC}}^p \|t_n\|_{\mathbb{BC}}^p \leq \left(\sqrt{2}\right)^p \sup\{\|s_n\|_{\mathbb{BC}}^p : n \in \mathbb{N}\} \|t_n\|_{\mathbb{BC}}^p,$$

it is said that the series $\sum\limits_{n=1}^{\infty} \|s_n t_n\|_{\mathbb{BC}}^p$ is convergent. Thus, we conclude that $(\zeta_n) \in l_p(\mathbb{BC})$. The proof is completed. □

Theorem 19. *$l_p(\mathbb{BC})$ is a bicomplex BK$-$space for* $1 \leq p < \infty$.

Proof. Let $\left(\zeta^{(n)}\right) \in l_p(\mathbb{BC})$ such that $\zeta^{(n)} \to \zeta$ as $n \to \infty$. Then, for every $\varepsilon > 0$ there is a $n_0(\varepsilon) \in \mathbb{N}$ such that $\left\|\zeta^{(n)} - \zeta\right\|_{l_p(\mathbb{BC})} < \varepsilon$ for all $n \geq n_0(\varepsilon)$. Therefore, we have $\left(\sum\limits_{l=1}^{\infty} \left\|\zeta_l^{(n)} - \zeta_l\right\|_{\mathbb{BC}}^p\right)^{\frac{1}{p}} < \varepsilon$ for all $n \geq n_0(\varepsilon)$. Thus, for any fixed $l \in \mathbb{N}$ and for all $n \geq n_0(\varepsilon)$ we can write $\left\|\zeta_l^{(n)} - \zeta_l\right\|_{\mathbb{BC}}^p < \varepsilon^p$ and $\left\|\zeta_l^{(n)} - \zeta_l\right\|_{\mathbb{BC}} < \varepsilon$. This implies that $\left(\zeta_l^{(n)}\right)$ converges to the bicomplex number ζ_l. Thus, the coordinates are continuous on $l_p(\mathbb{BC})$ for $1 \leq p < \infty$. This completes the proof. □

Theorem 20. *$l_p(\mathbb{BC})$ is a bicomplex symmetric space for* $0 < p < \infty$.

Proof. Let $(s_n) \in l_p(\mathbb{BC})$ and $\sigma \in \pi$. Then, since $\sigma : \mathbb{N} \to \mathbb{N}$ is a injective and surjective function, we have $\left\{\left\|s_{\sigma(n)}\right\|_{\mathbb{BC}} : n \in \mathbb{N}\right\} = \{\|s_n\|_{\mathbb{BC}} : n \in \mathbb{N}\}$ and so $\left\{\left\|s_{\sigma(n)}\right\|_{\mathbb{BC}}^p : n \in \mathbb{N}\right\} = \{\|s_n\|_{\mathbb{BC}}^p : n \in \mathbb{N}\}$ hold. So, we can write $\sum\limits_{n=1}^{\infty} \left\|s_{\sigma(n)}\right\|_{\mathbb{BC}}^p = \sum\limits_{n=1}^{\infty} \|s_n\|_{\mathbb{BC}}^p$. Since $\sum\limits_{n=1}^{\infty} \|s_n\|_{\mathbb{BC}}^p$ converges, we conclude that $\sum\limits_{n=1}^{\infty} \left\|s_{\sigma(n)}\right\|_{\mathbb{BC}}^p$ converges. That means $\left(s_{\sigma(n)}\right) \in l_p(\mathbb{BC})$. The proof is completed. □

Theorem 21. *$l_p(\mathbb{BC})$ is a seperable space for* $2 \leq p < \infty$.

Proof. Let $S = \{z \in \mathbb{C} : z = a + ib,\ a, b \in \mathbb{Q}\}$ and

$$Y = \{\zeta \in l_p(\mathbb{BC}) : \zeta = (\zeta_n) = (\zeta_1, \zeta_2, ..., \zeta_n, 0, 0, ...),\ \zeta_l = a_l e_1 + b_l e_2,\ a_l, b_l \in S\}.$$

where We claim that $\overline{Y} = l_p(\mathbb{BC})$ for $2 \leq p < \infty$.

Define the mapping

$$\begin{aligned} f \quad : \quad & S^2 \times S^2 \times ... \times S^2 \to Y, \\ & (a_1, b_1, a_2, b_2, ..., a_n, b_n) \\ \to \quad & f(a_1, b_1, a_2, b_2, ..., a_n, b_n) = (a_1 e_1 + b_1 e_2, a_2 e_1 + b_2 e_2, ..., a_n e_1 + b_n e_2, 0, 0, ...). \end{aligned}$$

It is clear that the mapping f is bijective. Then, the sets $S^2 \times S^2 \times ... \times S^2$ and Y are equivalent. Also, since S is countable, we have that $S^{2n} = S^2 \times S^2 \times ... \times S^2$ is countable. This shows that Y is a countable set.

Now, let $\zeta = (\zeta_n) \in l_p(\mathbb{BC})$. Then, $\sum\limits_{n=1}^{\infty} \|\zeta_n\|_{\mathbb{BC}}^p$ converges and so, $R_n \to 0$ as $n \to \infty$ where $R_n = \sum\limits_{l=n+1}^{\infty} \|\zeta_l\|_{\mathbb{BC}}^p$. Thus, for every $\varepsilon > 0$ there exists a $n_0(\varepsilon) \in \mathbb{N}$

such that

$$\|R_n - 0\|_{\mathbb{BC}} = \sum_{l=n+1}^{\infty} \|\zeta_l\|_{\mathbb{BC}}^p = \sum_{l=n+1}^{\infty} \|a_l e_1 + b_l e_2\|_{\mathbb{BC}}^p < \frac{\varepsilon^p}{2}$$

for all $n \geq n_0(\varepsilon)$.

Furthermore, since $a_l, b_l \in \mathbb{C} = \overline{S}$ for each $l \in \{1, 2, ..., n\}$, we can write for every $\varepsilon > 0$, $B(a_l, \varepsilon) \cap S \neq \varnothing$ and $B(b_l, \varepsilon) \cap S \neq \varnothing$. This implies that there exist $c_l, d_l \in S$ such that $c_l \in B(a_l, \varepsilon)$ and $d_l \in B(b_l, \varepsilon)$ for every $\varepsilon > 0$. Therefore, $|a_l - c_l| < \frac{\varepsilon}{\sqrt[p]{2n_0}}$ and $|b_l - d_l| < \frac{\varepsilon}{\sqrt[p]{2n_0}}$ for every $\varepsilon > 0$. Thus, we get

$$\sum_{l=1}^{n_0} |a_l - c_l|^p < \sum_{l=1}^{n_0} \left(\frac{\varepsilon}{\sqrt[p]{2n_0}}\right)^p = \sum_{l=1}^{n_0} \frac{\varepsilon^p}{2n_0} = \frac{\varepsilon^p}{2}$$

and

$$\sum_{l=1}^{n_0} |b_l - d_l|^p < \frac{\varepsilon^p}{2}.$$

Also, for $\psi = (c_1 e_1 + d_1 e_2, c_2 e_1 + d_2 e_2, ..., c_{n_0} e_1 + d_{n_0} e_2, 0, 0, ...) \in Y$, we have

$$\begin{aligned}
\|\zeta - \psi\|_{l_p(\mathbb{BC})}^p &= \sum_{n=1}^{\infty} \|\zeta_n - \psi_n\|_{\mathbb{BC}}^p \\
&= \sum_{n=1}^{n_0} \|\zeta_n - \psi_n\|_{\mathbb{BC}}^p + \sum_{n=n_0+1}^{\infty} \|\zeta_n - \psi_n\|_{\mathbb{BC}}^p \\
&= \sum_{n=1}^{n_0} \|\zeta_n - \psi_n\|_{\mathbb{BC}}^p + \sum_{n=n_0+1}^{\infty} \|\zeta_n\|_{\mathbb{BC}}^p \\
&= \sum_{n=1}^{n_0} \|\zeta_n - \psi_n\|_{\mathbb{BC}}^p + R_{n_0} \\
&= \sum_{n=1}^{n_0} \|(a_n e_1 + b_n e_2) - (c_n e_1 + d_n e_2)\|_{\mathbb{BC}}^p + R_{n_0} \\
&= \sum_{n=1}^{n_0} \|(a_n - c_n) e_1 + (b_n - d_n) e_2\|_{\mathbb{BC}}^p + R_{n_0} \\
&= \sum_{n=1}^{n_0} \left(\frac{1}{\sqrt{2}} \sqrt{|a_n - c_n|^2 + |b_n - d_n|^2}\right)^p + R_{n_0} \\
&\leq \sum_{n=1}^{n_0} \frac{1}{(\sqrt{2})^p} 2^{\frac{p-2}{2}} (|a_n - c_n|^p + |b_n - d_n|^p) + R_{n_0} \\
&= \frac{1}{2} \sum_{n=1}^{n_0} (|a_n - c_n|^p + |b_n - d_n|^p) + R_{n_0} \\
&= \frac{1}{2} \sum_{n=1}^{n_0} |a_n - c_n|^p + \frac{1}{2} \sum_{n=1}^{n_0} |b_n - d_n|^p + R_{n_0} \\
&< \frac{1}{2} \frac{\varepsilon^p}{2} + \frac{1}{2} \frac{\varepsilon^p}{2} + \frac{\varepsilon^p}{2} \\
&= \varepsilon^p
\end{aligned}$$

and so, $\|\zeta - \psi\|_{l_p(\mathbb{BC})} < \varepsilon$. Then, Y is dense countable subset of $l_p(\mathbb{BC})$. Thus, $l_p(\mathbb{BC})$ is seperable for $2 \leq p < \infty$. The proof is completed. □

4. Conclusion

Bicomplex sequence spaces $l_p(\mathbb{BC})$ are the generalization of real and complex sequence spaces l_p were studied by many authors. Then, it has been investigated whether inclusion relations and some topological properties in the spaces l_p are provided in the spaces $l_p(\mathbb{BC})$. Also, based on the completeness property of the spaces $l_p(\mathbb{BC})$ proved in [17], it has been examined whether the spaces $l_p(\mathbb{BC})$ satisfy the conditions for being a Banach $\mathbb{BC}-$module. Results are explained by using some illustrative examples. Some crucial properties of the spaces considered in this work may attract further study on other aspects of such spaces.

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