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AN APPLICATION OF THE FIRST INTEGRAL METHOD FOR TIME M-FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Analytic behavior of fractional differential equations is often seeming perplexing, thus determination comprehensive methods for solving them could mostly have a high importance. In the present study, Feng's first integral method which is attributed to the ring theory of commutative algebra, is developed for analytic conduct fractional differential equations in accordance with truncated M-fractional derivative. Furthermore, some important nonlinear M-fractional differential equations, such as Burgers-KdV, Klein-Gordon, Sharma-Tasso-Olever, KdV-Zakharov-Kuznetsev, and Zakharov-Kuznetsov equations are solved by the proposed procedure.

1. Introduction

Over the past few decades, researchers results are showed that fractional calculus is used to achieve more accurate results in studies and applications of differential equations. Although it has only been highlighted since 1974, after the first international conference on fractional accounts, it has been shown that in modeling problems at natural phenomena are more accurate. Fractional differential equations are studied in various fields of physics and engineering, specifically in signal processing, control engineering, electromagnetism, biosciences, fluid mechanics, electrochemistry, diffusion processes, dynamic of viscoelastic material, continuum and statistical mechanics and propagation of spherical flames. For this development, the help of famous mathematicians such as Lagrange, Abel, Euler, Liouville, Riemann, as well as recently Caputo, etc. was needed. It is possible to define various derivatives and integrals fractional. Each definition has its own strengths and weaknesses and its own properties and thus any of them have a valuable fractional calculus in theory and applications. At present, there are countless and important definitions of types of fractional derivatives, each with its own characteristics and applications [1–3]. Some types of fractional derivatives that so far have been introduced, Riemann-Liouville, Caputo, Hadamard, Caputo-Hadamard, Riesz, among others [4–10]. Many of these derivatives are defined from the fractional integral in

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the Riemann-Liouville sense. Recently, Katugampola, has presented a new fractional integral unifying six existing fractional integrals, named Riemann-Liouville, Hadamard, Erdlyi-Kober, Katugampola, Weyl and Liouville [5–7]. In 2014, Khalil et al. proposed the so-called conformable fractional derivative of order integer to generalize the classical properties of calculus [8]. More recently, in 2014, Katugampola has also proposed an alternative fractional derivative with classical properties, which refers to the Leibniz and Newton calculus, similar to the conformable fractional derivative [9]. In 2017, Sousa and et al., introduced an M-fractional derivative involving a Mittag-Leffler function with one parameter that also satisfies the properties of integer-order calculus [10, 11]. In this sense, Sousa and Oliveira introduced a truncated M-fractional derivative type that unifies four existing fractional derivative types mentioned above and which also satisfied the classical properties of integer-order calculus [12]. During the past three decades, the Burgers equation, Korteweg-de Vrise (KdV) equation and Burgers-Korteweg-de Vrise equation (Burgers-KdV) have attracted a lot of attention from a rather diverse group of scientist such as physicists and mathematicians, because these three equations not only arise from realistic physical phenomena, but also can be widely applied to many physically significant fields such as plasma physics, crystal lattice theory, nonlinear circuit theory and astrophysics [13–16]. Feng presented first integral method for obtaining travelling wave solutions these equations [13]. The study of fractional differential equations has demonstrated very valuable over time. Solving fractional differential equations is very important, due to this fact, finding an exact solution and an approximate solution of fractional differential equations is clearly an important task. The author of this article suggest to dear researchers that refer to the articles cited to see some useful methods for solving fractional differential equations [14–39]. In this article, is applied first integral method for solving M-fractional Burgers-Kdv differential equations, and some other important M-fractional differential equations. The organization of this paper is as follows: In Section 2, truncated M-fractional derivative is described. In Section 3, the first integral method for M-fractional differential equations is presented. In Section 4, to illustrate the proposed approach some important nonlinear M-fractional differential equations are solved. Finally, conclusion is presented, in section 5.

2. The truncated M-fractional derivative

Definition 1. With $\beta > 0$, and $z \in \mathbb{C}$, the truncated Mittag-Leffler function of one parameter is defined by [10],[12]

$$ {}_i\mathbb{E}_\beta(z) = \sum_{k=0}^{i} \frac{z^k}{\Gamma(k\beta+1)}. \tag{1}$$

Definition 2. Given a function $f : [0, \infty) \to \mathbb{R}$. Then the truncated M-fractional derivative of f of order α is defined by

$$ {}_i\mathcal{D}_M^{\alpha,\beta} f(t) = \lim_{\varepsilon \to 0} \frac{f(t\ {}_i\mathbb{E}_\beta(\varepsilon t^{-\alpha})) - f(t)}{\varepsilon}, \tag{2}$$

for all $t > 0$, $\alpha \in (0, 1)$, where ${}_i\mathbb{E}_\beta(.)$, $\beta > 0$ is the Mittag-Leffler function with one parameter as defined by in Eq. (1) [12].

Note that if f is α-differentiable in some $(0, \alpha), \alpha > 0$, and $\lim_{x\to 0^+} {}_i\mathcal{D}_M^{\alpha,\beta} f(t)$ exists, then one can define [12]

$$ {}_{i}\mathcal{D}_M^{\alpha,\beta} f(0) = \lim_{t\to 0^+} {}_{i}\mathcal{D}_M^{\alpha,\beta} f(t). $$

If the M-fractional derivative of f of order α exists, then we simply say that f is α-differentiable [12].
One can easily show that truncated M-fractional derivative satisfies all the following properties [12]:
Let $\alpha \in (0,1)$ and f, g be functions α-differentiable at a point $x > 0$, Then

- **A:** (Linearity Rule) For $a, b \in \mathbb{R}$, ${}_{i}\mathcal{D}_M^{\alpha,\beta}(af + bg) = a({}_{i}\mathcal{D}_M^{\alpha,\beta} f) + b({}_{i}\mathcal{D}_M^{\alpha,\beta} g)$,
- **B:** For all $\rho \in \mathbb{R}$, ${}_{i}\mathcal{D}_M^{\alpha,\beta} t^\rho = \frac{\rho}{\Gamma(\beta+1)} t^{\rho-\alpha}$,
- **C:** For all constant functions $f(t) = \lambda$, ${}_{i}\mathcal{D}_M^{\alpha,\beta}\lambda = 0$,
- **D:** (Product Rule) ${}_{i}\mathcal{D}_M^{\alpha,\beta}(f.g) = g.({}_{i}\mathcal{D}_M^{\alpha,\beta} f) + f.({}_{i}\mathcal{D}_M^{\alpha,\beta} g)$,
- **E:** (Quotient Rule) ${}_{i}\mathcal{D}_M^{\alpha,\beta}(\frac{f}{g}) = \frac{g.({}_{i}\mathcal{D}_M^{\alpha,\beta} f) - f.({}_{i}\mathcal{D}_M^{\alpha,\beta} g)}{g^2}$,
- **F:** (Chain Rule) If function f, ordinary differentiable at $g(x)$, then ${}_{i}\mathcal{D}_M^{\alpha,\beta}(fog) = f'(g(x).({}_{i}\mathcal{D}_M^{\alpha,\beta} g))$,
- **G:** ${}_{i}\mathcal{D}_M^{\alpha,\beta} f(t) = \frac{t^{1-\alpha}}{\Gamma(\beta+1)} \frac{df}{dt}$.

Definition .3 Let $\beta > 0, \alpha \in (n, n+1]$, for some $n \in \mathbb{N}$, and f, n times differentiable (in the classical of sense) for $t > 0$. Then the local M-derivative of order n, of function f is defined by

$$ {}_{i}\mathcal{D}_M^{\alpha,\beta,n} f(t) = \lim_{\varepsilon\to 0} \frac{f^{(n)}(t\ {}_{i}\mathbb{E}_\beta(\varepsilon t^{-\alpha})) - f^{(n)}(t)}{\varepsilon}, \tag{3} $$

if and only if the limit exists [12].

3. Time M-Fractional first integral method

Consider the time M-fractional differential equation with independent variables x, y, z, t and a dependent variable u as the following form

$$ F\Big(u, u_x, u_y, u_z, {}_{i}^{t}\mathcal{D}_M^{\alpha,\beta} u, u_{xx}, u_{yy}, u_{zz}, {}_{i}^{t}\mathcal{D}_M^{\alpha,\beta}({}_{M}^{t}\mathcal{T}_\alpha^{\alpha,\beta} u), u_{xy}, u_{x,z}, u_{yz}, \cdots\Big) = 0, \tag{4} $$

which ${}_{i}^{t}\mathcal{D}_M^{\alpha,\beta}(.)$, is meant M-fractional derivative relative to the time-variable t.
Using the variable transformation [13],[16]

$$ u(x,y,z,t) = U(\xi),\ \ \xi = x + y + z - \frac{\upsilon\Gamma(\beta+1)}{\alpha} t^\alpha, $$

where υ, is constant to be determined later, the Eq. (4) is reduced to an ordinary differential equation

$$ H = (U(\xi), U'(\xi), U''(\xi), \cdots). \tag{5} $$

Suppose that ODE (5), has a solution as follows

$$ U(\xi) = X(\xi), $$

and so

$$ Y(\xi) = X'(\xi). $$

Thus ordinary differential equation (5), will be changed into a system of equations as the following form

$$ \begin{cases} X'(\xi) = Y(\xi) \\ Y'(\xi) = G(X(\xi), Y(\xi)). \end{cases} \tag{6} $$

By using the Division theorem which is based on the Hilbert-Nullstellensatz theorem [40], it will be assumed to obtain one first integral to Eq. (6), which reduces Eq. (5) to a first-order integrable ordinary differential equation. Finally, an exact solution to Eq. (4) will be obtained, through solving the resulting first-order integrable differential equation [13, 16].

Division Theorem. Suppose that $P(x, y)$ and $Q(x, y)$ are polynomials of two variables x and y and $P(x, y)$ is irreducible in $C[x, y]$. If $Q(x, y)$ vanishes at all zero points of $P(x, y)$, then there exists a polynomial $T(x, y)$ in $C[x, y]$ such that [16, 40]

$$Q(x, y) = P(x, y)T(x, y).$$

4. Examples

In this section, some important M-fractional differential equations are solved by the suggested method.

Example 1. Consider the time M-fractional Burgers-Korteweg-de Vrise differential equation, as follows

$$ {}_{i}^{t}\mathcal{D}_M^{\alpha,\beta} u + \delta u u_x + \eta u_{xx} + \gamma u_{xxx} = 0, \tag{7}$$

where δ, η and γ are real constants with $\delta\eta\gamma \neq 0$.
Assume that equation (7), has travelling wave solutions as the following form [16]

$$u(x, t) = U(\xi), \ \ \xi = \frac{\upsilon\Gamma(\beta + 1)}{\alpha} t^{\alpha} - x, \ \ \upsilon \in \mathbb{R}.$$

Substitution of above equation into Eq. (5) and integrating once we have

$$U''(\xi) - rU'(\xi) - bU^2(\xi) - bU(\xi) - d = 0, \tag{8}$$

where $r = -\frac{\eta}{\gamma}, a = -\frac{\delta}{2\gamma}, b = \frac{\upsilon}{\gamma}$, and d is an arbitrary integration constant. The second-order ordinary differential equation (8) is exactly the same converted equation according to first integral method for solving differential equations [13, 16], then an exact solution of Eq. (8) is as follows
in the case $b = \frac{6r^2}{25}$,

$$\begin{aligned} u_{\alpha,\beta}(x, t) = & \frac{3\eta^2}{25\delta\gamma} sech^2\Big[\frac{1}{2}(-\frac{\eta}{5\gamma}x + \frac{6\eta^3\Gamma(\beta+1)}{125\gamma^2\alpha}t^{\alpha})\Big] \\ & - \frac{6\eta^2}{25\delta\gamma} tanh\Big[\frac{1}{2}(-\frac{\eta}{5\gamma}x + \frac{6\eta^3\Gamma(\beta+1)}{125\gamma^2\alpha}t^{\alpha})\Big] + \frac{6\eta^2}{25\delta\gamma}, \end{aligned}$$

in the case $b = -\frac{6r^2}{25}$,

$$\begin{aligned} u_{\alpha,\beta}(x, t) = & \frac{3\eta^2}{25\delta\gamma} sech^2\Big[\frac{1}{2}(-\frac{\eta}{5\gamma} - \frac{6\eta^3\Gamma(\beta+1)}{125\gamma^2\alpha}t^{\alpha})\Big] \\ & - \frac{6\beta^2}{25\delta\gamma} tanh\Big[\frac{1}{2}(-\frac{\eta}{5\gamma} - \frac{6\eta^3\Gamma(\beta+1)}{125\gamma^2\alpha}t^{\alpha})\Big] - \frac{6\eta^2}{25\delta\gamma}. \end{aligned}$$

Clearly

$$\lim_{\alpha,\beta \to 1} u_{\alpha,\beta}(x, t) = u(x, t),$$

$u(x, t)$, is an exact solution of Burgers-Korteweg-de Vries equation, which is solved by first integral method [13, 16]. The 3D graph of $u_{\alpha,\beta}(x, t)$, is illustrated in Fig. 1.

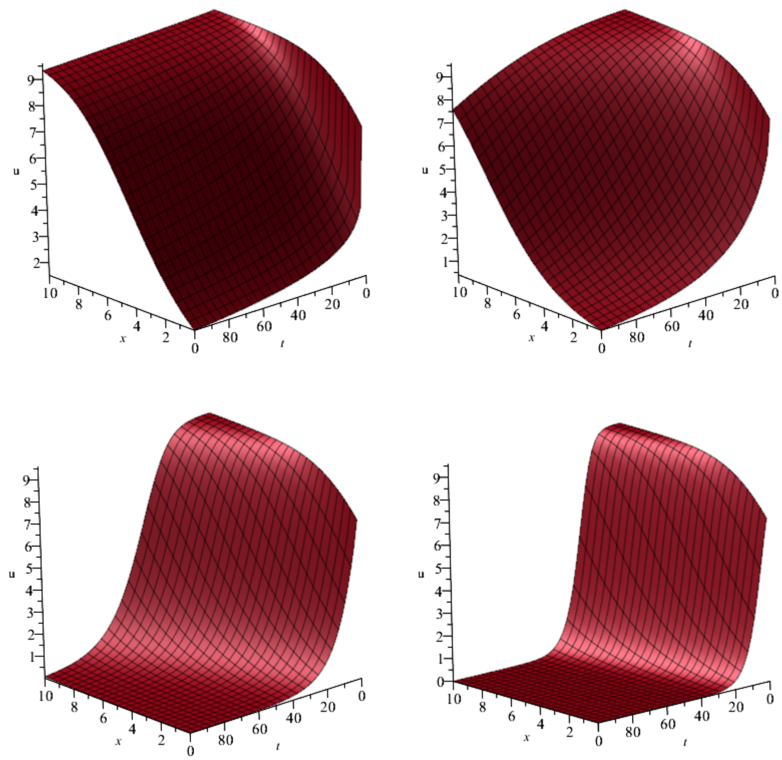


FIGURE 1. Graph of the $u_{\alpha,\beta}(x,t)$, corresponding to the values $\alpha = 0.3, 0.6, 0.9, 1$, $\beta = 1$, from left to right when $\eta = 1$, $\delta = 0.1, \gamma = 0.5$, respectively.

Example 2. Consider the nonlinear time M-fractional Klein-Gordon equation, subject to the initial condition as the following form

$$
\begin{aligned}
&{}_{i}^{t}\mathcal{D}_{M}^{\alpha,\beta}({}_{i}^{t}\mathcal{D}_{M}^{\alpha,\beta} u) = u_{xx} + au + cu^3, \\
&u(x,0) = \sqrt{\frac{a}{c}} \tan\Big[\sqrt{\frac{a}{2(\upsilon^2 - l^2)}} lx\Big].
\end{aligned} \tag{9}
$$

Assume that equation (9), has travelling wave solutions as follows [16]

$$u(x,t) = U(\xi), \;\; \xi = lx - \frac{\upsilon\Gamma(\beta+1)}{\alpha} t^{\alpha}, \;\; \upsilon \in \mathbb{R}.$$

where l and U are constant. Substitution of above transformation into Eq. (9) we obtain

$$U''(\xi) = \frac{a}{\upsilon^2 - l^2} U(\xi) + \frac{c}{\upsilon^2 - l^2} U^3(\xi). \tag{10}$$

The second-order nonlinear ordinary differential equation (10) is exactly the same transformed equation according to first integral method for solving nonlinear fractional differential equations [14, 16], then an exact solution of Eq. (9), with the initial condition is given as follows

$$u_{\alpha,\beta}(x,t) = \sqrt{\frac{a}{c}}\tan\Big[\sqrt{\frac{a}{2(\upsilon^2-l^2)}}(lx - \frac{\upsilon\Gamma(\beta+1)}{\alpha}t^\alpha)\Big].$$

where

$$\lim_{\alpha,\beta\to 1} u_{\alpha,\beta}(x,t) = u(x,t),$$

$u(x,t)$, is an exact solution of nonlinear fractional Klein-Gordon equation, which is solved with first integral method [14, 16]. The 3D display of $u_{\alpha,\beta}(x,t)$, is exemplified in Fig. 2.

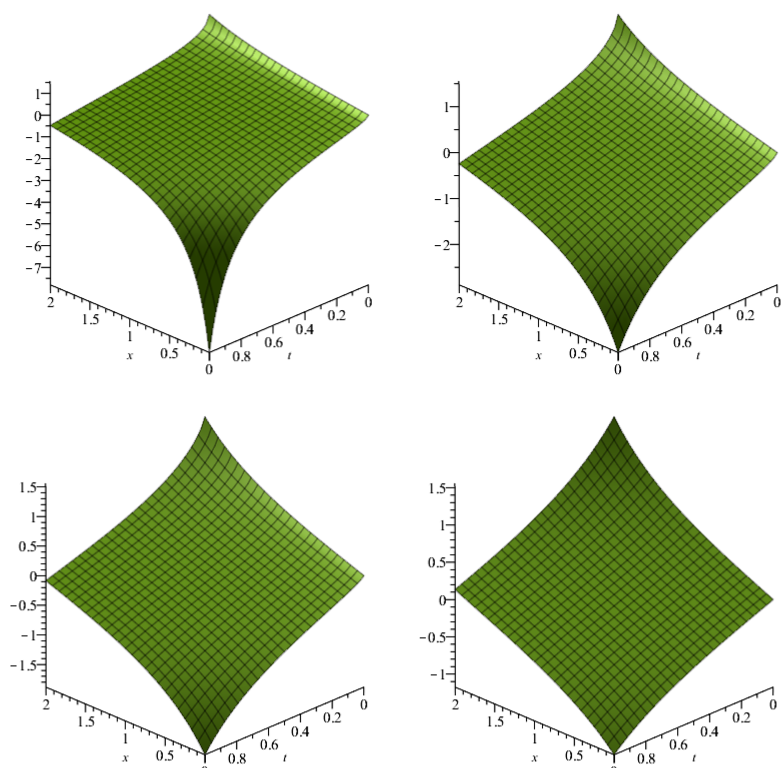


FIGURE 2. Graph of the $u_{\alpha,\beta}(x,t)$, corresponding to the values $\alpha = 0.5, 0.6, 0.8, 1$, $\beta = 1$, from left to right when $\nu = -1$, $l = a = c = 1$, respectively.

Example 3. Consider the nonlinear time M-fractional Sharma-Tasso-Olever equation subject to the initial condition as the following form

$$ {}_i^t\mathcal{D}_M^{\alpha,\beta}u + 3au_x^2 + 3au^2u_x + 3auu_{xx} + au_{xxx} = 0, \ t > 0 \tag{11}$$

$$u(x,0) = -\sqrt{2B_0}\tan(\frac{\sqrt{2B_0}}{2}x),$$

where a and B_0 are arbitrary constants.
Assume that equation (12), has travelling wave solutions as the following form [16]

$$u(x,t) = U(\xi), \;\; \xi = x - \frac{\upsilon\Gamma(\beta+1)}{\alpha}t^{\alpha}, \;\; \upsilon \in \mathbb{R}.$$

where υ is constant. Substitution of above transformation into Eq. (11) we acquire

$$aU''(\xi) + aU^3(\xi) + 3aU(\xi)U'(\xi) - \upsilon U(\xi) + c = 0, \tag{12}$$

where c is constant. The second-order nonlinear ordinary differential equation (12) is exactly the same transformed equation according to first integral method for solving nonlinear fractional differential equations [14, 16], then an exact solution of Eq. (11), with the initial condition is given as follows

$$u_{\alpha,\beta}(x,t) = -2\sqrt{2B_0}\tan\Big(\frac{\sqrt{2B_0}}{2}(x - \frac{a(A_0^2+B_0)\Gamma(\beta+1)}{\alpha}t^{\alpha})\Big),$$

where A_0 is constant and

$$\lim_{\alpha,\beta\to 1} u_{\alpha,\beta}(x,t) = u(x,t),$$

$u(x,t)$, is an exact solution of nonlinear fractional Sharma-Tasso-Olever equation, which is solved with first integral method [14, 16].

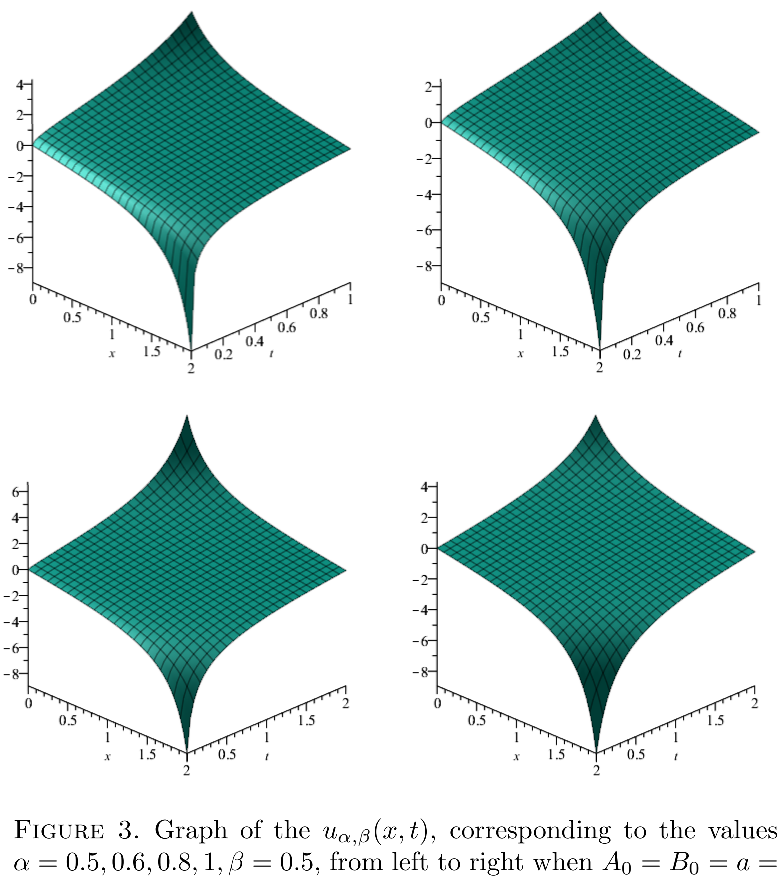


FIGURE 3. Graph of the $u_{\alpha,\beta}(x,t)$, corresponding to the values $\alpha = 0.5, 0.6, 0.8, 1, \beta = 0.5$, from left to right when $A_0 = B_0 = a = 1$, respectively.

The 3D graph of $u_{\alpha,\beta}(x,t)$, is showed in Fig. 3.

Example 4. Consider the (3+1)-dimensional modified M-fractional KdV-Zakharov-Kuznetsev equation

$$ {}_{i}^{t}\mathcal{D}_{M}^{\alpha,\beta}u + \eta u^2 u_x + u_x + u_{xyy} + u_{xzz} = 0, \tag{13}$$

where η is a nonzero constant.

Assume that equation (13), has travelling wave solutions as the following form [16]

$$u(x,y,z,t) = U(\xi),\;\; \xi = x + y + z - \frac{\upsilon\Gamma(\beta+1)}{\alpha}t^{\alpha},\;\; \upsilon \in \mathbb{R}.$$

Appling above transformation into Eq. (13) and integrating once lead to ordinary differential equation as follows

$$U''(\xi) + \frac{1-\upsilon}{2}U(\xi) + \frac{\beta}{\sigma}U^3(\xi) = 0. \tag{14}$$

The second-order nonlinear ordinary differential equation (14) is exactly the same changed equation according to first integral method for solving nonlinear differential equations [15, 16], thus an exact solution of Eq. (13), is as the following form

$$u^1_{\alpha,\beta}(x,y,z,t) = \sqrt{\frac{-3\upsilon}{\eta}}\tan\Big[\frac{\sqrt{6\upsilon}}{6}(x+y+z-\frac{\upsilon\Gamma(\beta+1)}{\alpha}t^{\alpha}+c)\Big],$$

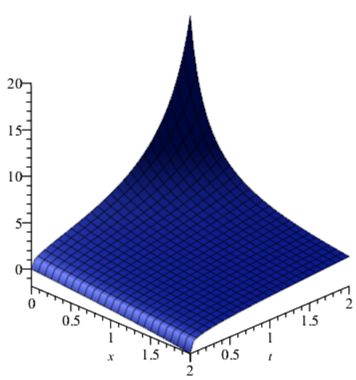
or

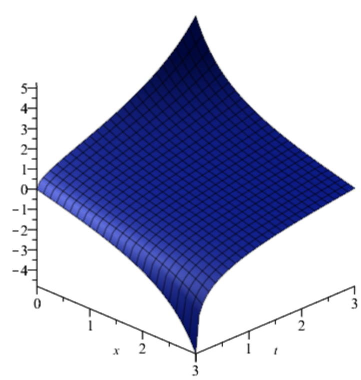
$$u^2_{\alpha,\beta}(x,y,z,t) = -\sqrt{\frac{-3\upsilon}{\eta}}\tan\Big[\frac{\sqrt{6\upsilon}}{6}(x+y+z-\frac{\upsilon\Gamma(\beta+1)}{\alpha}t^{\alpha}+c)\Big].$$

Clearly

$$\lim_{\alpha,\beta\to1} u^1_{\alpha,\beta}(x,y,z,t) = u^1(x,y,z,t),$$
$$\lim_{\alpha,\beta\to1} u^2_{\alpha,\beta}(x,y,z,t) = u^2(x,y,z,t),$$

$u^1(x,y,z,t)$, and $u^2(x,y,z,t)$ are an exact solution of the (3+1)-dimensional modified KdV-Zakharov-Kuznetsev equation, which is solved with first integral method [15, 16]. The 3D display of $u^1_{\alpha,\beta}(x,y,z,t)$, is revealed in Fig. 4.





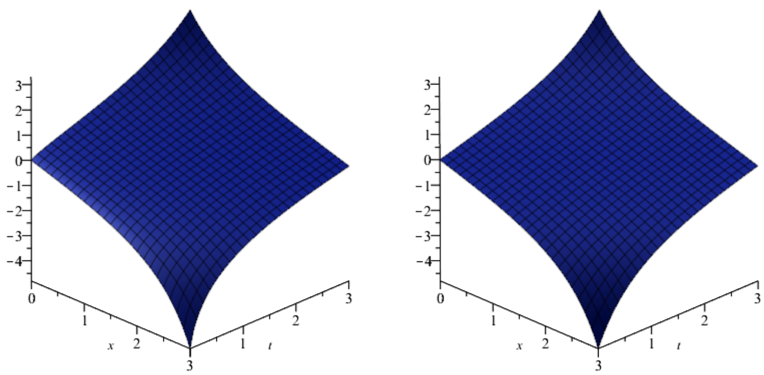


FIGURE 4. Graph of the $u^1_{\alpha,\beta}(x,y,z,t)$, corresponding to the values $\alpha = 0.3, 0.5, 0.8, 1, \beta = 0.5$, from left to right when $\nu = 1$, $\eta = 1, c = 0, y = z = 0$, respectively.

Example 5. Consider the (3+1)-dimensional M-fractional Zakharov-Kuznetsov equation, as the following

$$ {}_i^t\mathcal{D}_M^{\alpha,\beta} u + \eta u u_x + u_{xx} + u_{yy} + u_{zz} = 0, \tag{15} $$

Assume that equation (15), has travelling wave solutions as follows [16]

$$ u(x,y,z,t) = U(\xi), \;\; \xi = x + y + z - \frac{\upsilon\Gamma(\beta+1)}{\alpha} t^{\alpha}, \;\; \upsilon \in \mathbb{R}. $$

Appling above transformation into Eq. (15), we obtain the ordinary differential equation as the following form

$$ U''(\xi) - \frac{\upsilon}{3} U'(\xi) + \frac{\eta}{3} U(\xi)U'(\xi) = 0. \tag{16} $$

The second-order nonlinear ordinary differential equation (16) is exactly the same reformed equation according to first integral method for solving differential equations [15, 16], thus an exact solution of Eq. (15), is as the following

case 1. suppose $m = 1$,

$$ u^1_{\alpha,\beta}(x,y,z,t) = \frac{\upsilon}{\eta} - \frac{\sqrt{6c\eta - \upsilon^2}}{\eta} \tan\Big[\frac{\sqrt{6c\eta - \upsilon^2}}{\eta} (x + y + z - \frac{\upsilon\Gamma(\beta+1)}{\alpha} t^{\alpha} + d) \Big], $$

case 2. assume $m = 2$,

$$ u^2_{\alpha,\beta}(x,y,z,t) = \frac{\upsilon}{\eta} - \frac{\sqrt{-3c\eta \pm 3\eta\sqrt{c^2 - 4d} - \upsilon^2}}{\eta} \tan\Big[\frac{\sqrt{-3c\eta \pm 3\eta\sqrt{c^2 - 4d} - \upsilon^2}}{\eta} (x + y + z - \frac{\upsilon\Gamma(\beta+1)}{\alpha} t^{\alpha} + e) \Big], $$

where c, d, and e are an arbitrary integration constant.
Clearly

$$ \lim_{\alpha,\beta\to 1} u^1_{\alpha,\beta}(x,y,z,t) = u^1(x,y,z,t), $$
$$ \lim_{\alpha,\beta\to 1} u^2_{\alpha,\beta}(x,y,z,t) = u^2(x,y,z,t), $$

$u^1(x,y,z,t)$, and $u^2(x,y,z,t)$ are an exact solution of the (3+1)-dimensional Zakharov-Kuznetsov equation, which is solved with first integral method [15, 16]. The 3D graph of $u^1_{\alpha,\beta}(x,y,z,t)$, is demonstrated in Fig. 5.

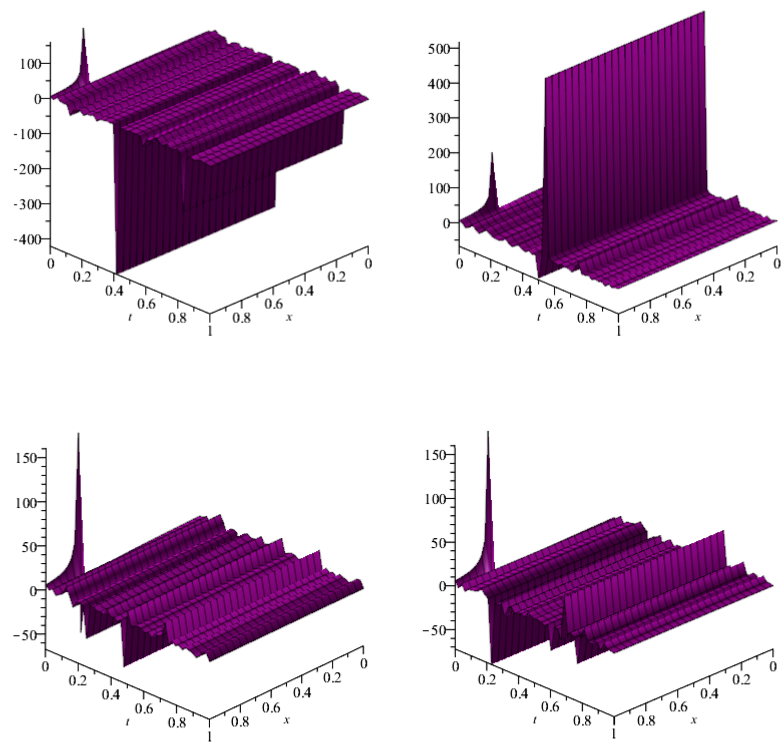


FIGURE 5. Graph of the $u^1_{\alpha,\beta}(x,y,z,t)$, corresponding to the values $(\alpha = 0.3, \beta = 20), (\alpha = 0.5, \beta = 21)$, $(\alpha = 0.8, \beta = 23)$, $(\alpha = 0.9, \beta = 24)$ from left to right when $\nu = \sqrt{2}, \eta = 1, c = 1$, $d = 0, y = z = 0$, respectively.

5. Conclusion

In this paper, the first integral method has been applied to obtain the travelling wave solutions of M-fractional differential equations. To this end, a truncated M-fractional derivative has been used to find the solution. The results showed that the definition is the simplest tool to obtain the approximation solutions of nonlinear M-fractional differential equations in comparison to the other definitions. To show the effectiveness of the approach and simplicity of the definition, some fractional differential equations with form truncated M-fractional derivative, have been solved. It can be concluded from the solved examples that the accuracy and efficiency of the first integral method for solving M-fractional differential equations is the same as differential equations. The solution of M-fractional differential equations are exactly the same as the solution of differential equations, when tendency α, β to 1.

Furthermore, the method presented in the article, can easily be extended to any M-fractional differential equations with any number of independent variables [16].

References

[1] R. Herrmann, Fractional calculus: An Introduction for Physicists, World Scientific Publishing Company, Singapore, 2011.

[2] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, Elsevier, Amsterdam, Vol. 207, 2006.

[3] I. Podlubny, Fractional Differential Equations, Mathematics in Science and Engineering, Academic Press, San Diego, Vol. 198, 1999.

[4] E. Capelas de Oliveira, J. A. Tenreiro Machado, A review of definitions for fractional derivatives and integral, Mathematical Problems in Engineering, 2014, (2014) (238459).

[5] U. N. Katugampola, New fractional integral unifying six existing fractional integrals, arxiv.org/abs/1612.08596, (2016).

[6] R. Figueiredo Camargo, E. Capelas de Oliveira, Fractional Calculus (In Portuguese), Editora Livraria da Fsica, So Paulo, 2015.

[7] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and Applications of the Fractional Differential Equations, Vol. 204, Elsevier, Amsterdam, 2006.

[8] R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, A new definition of fractional derivative, Journal of Computational and Applied Mathematics, 264, (2014) 65-70.

[9] U. N. Katugampola, A new fractional derivative with classical properties, arXiv:1410.6535v2, (2014).

[10] R. Goreno, A. A. Kilbas, F. Mainardi, S. V. Rogosin, Mittag-Leffler Functions, Related Topics and Applications, Springer, Berlin, 2014.

[11] J. Vanterler da C. Sousa, E. Capelas de Oliveira, M-fractional derivative with classical properties, (2017), arXiv:1704.08187.

[12] J. Vanterler da C. Sousa, E. Capelas de Oliveira, A New Truncated M-Fractional Derivative Type Unifying Some Fractional Derivative Types with Classical Properties, International Journal of Analysis and Applications., 16 (1) (2018), 83-96.

[13] Z. Feng, The first integral method to study the Burgers-Korteweg-de Vries equation, Journal of Physics A: Mathematical and General, 35 (2002) 343-349.

[14] Bin Lu, The first integral method for some time fractional differential equations, Journal of Mathematical Analysis and Applications, 395 (2012) 684-693.

[15] K. Hosseini, P. Gholamin, Feng's First Integral Method for Analytic Treatment of Two Higher Dimensional Nonlinear Partial Differential Equations, Differential Equations and Dynamical Systems, 23(3) (2015) 317-325.

[16] M. Ilie, J. Biazar, Z. Ayati, The first integral method for solving some conformable fractional differential equations, Optical and Quantum Electronics, 50 (2) (2018), https://doi.org/10.1007/s11082-017-1307-x.

[17] M. Ilie, J. Biazar, Z. Ayati, General solution of second order fractional differential equations, International Journal of Applied Mathematical Research, 7 (2) (2018) 56-61.

[18] M Ilie, J. Biazar, Z. Ayati, Resonant solitons to the nonlinear Schrödinger equation with different forms of nonlinearities, Optik, 164 (2018) 201-209.

[19] M Ilie, J. Biazar, Z. Ayati, Analytical study of exact traveling wave solutions for time-fractional nonlinear Schrödinger equations, Optical and Quantum Electronics, 50 (12) (2018), https://doi.org/10.1007/s11082-018-1682-y.

[20] M. Ilie, J. Biazar, Z. Ayati, General solution of Bernoulli and Riccati fractional differential equations based on conformable fractional derivative, International Journal of Applied Mathematical Research, 6(2) (2017) 49-51.

[21] M. Ilie, J. Biazar, Z. Ayati, Application of the Lie Symmetry Analysis for second-order fractional differential equations, Iranian Journal of Optimization, 9(2) (2017) 79-83.

[22] M. Ilie, J. Biazar, Z. Ayati, Lie Symmetry Analysis for the solution of first-order linear and nonlinear fractional differential equations, International Journal of Applied Mathematical Research, 7 (2) (2018) 37-41.

[23] M. Ilie, J. Biazar, Z. Ayati, Analytical solutions for conformable fractional Bratu-type equations, International Journal of Applied Mathematical Research, 7 (1) (2018) 15-19.

[24] M. Ilie, J. Biazar, Z. Ayati, Neumann method for solving conformable fractional Volterra integral equations, Computational Methods for Differential Equations, 8(1) (2020) 54-68.

[25] M. Ilie, J. Biazar, Z. Ayati, Mellin transform and conformable fractional operator: applications, SeMA Journal, doi.org/10.1007/s40324-018-0171-3.

[26] M. Ilie, J. Biazar, Z. Ayati, Optimal homotopy asymptotic method for conformable fractional Volterra integral equations of the second kind, 49th Annual Iranian Mathematics Conference, August 23-26, 2018, ISC 97180-51902.

[27] M. Ilie, M. Navidi, A. Khoshkenar, Analytical solutions for conformable fractional Volterra integral equations of the second kind, 49th Annual Iranian Mathematics Conference, August 23-26, 2018, ISC 97180-51902.

[28] S. Salahshour, A, Ahmadian, S. Abbasbandy, D. Baleanu, M-fractional derivative under interval uncertainty: Theory, properties and applications, Chaos, Solitons and Fractals, 117 (2018) 84-93.

[29] K. Hosseini, M. Mirzazadeh, M. Ilie, JF. Gómez-Aguilar, BiswasArshed equation with the beta time derivative: optical solitons and other solutions, Optik, 217 (2020) 164801.

[30] K Hosseini, M Ilie, M Mirzazadeh, D Baleanu, A detailed study on a new (2+1)-dimensional mKdV equation involving the Caputo-Fabrizio time-fractional derivative, Advances in Difference Equations, 331 (2020). https://doi.org/10.1186/s13662-020-02789-5.

[31] K. Hosseini, M. Ilie, M. Mirzazadeh, D. Baleanu, An analytic study on the approximate solution of a nonlinear time...fractional Cauchy reaction-diffusion equation with the Mittag-Leffler law, Mathematical Methods in the Applied Sciences, (2021), https://doi.org/10.1002/mma.7059.

[32] K. Hosseini, M. Ilie, M. Mirzazadeh, A. Yusuf, T. A. Sulaiman, D. Baleanu, S. Salahshour, An effective computational method to deal with a time-fractional nonlinear water wave equation in the Caputo sense, Mathematics and Computers in Simulation, 187 (2021) 248-260.

[33] M. Al Horani, R. Khalil, Total fractional differentials with applications to exact fractional differential equations, International Journal of Computer Mathematics, DOI: 10.1080/00207160.2018.1438602.

[34] M. Saffarian, A. Mohebbi, A novel ADI Galerkin spectral element method for the solution of two-dimensional time fractional subdiffusion equation, International Journal of Computer Mathematics, https://doi.org/10.1080/00207160.2020.1792450.

[35] D. Baleanu, M. Jleli, S. Kumar, B. Samet. A fractional derivative with two singular kernels and application to a heat conduction problem. Advances in Difference Equations, 252 (2020). https://doi.org/10.1186/s13662-020-02684-z.

[36] M. Ilie, A. Khoshkenar, General solution of the Bernoulli and Riccati fractional differential Equations via truncated M-fractional derivative, International Journal of Computer Mathematics, submitted 01May 2021.

[37] M. Ilie, J. Biazar, Z. Ayati, Optimal Homotopy Asymptotic Method for first-order conformable fractional differential equations, Journal of Fractional Calculus and Applications, 10 (1) 2019 33-45.

[38] M. Ilie, J. Biazar, Z. Ayati, Analytical solutions for second-order fractional differential equations via OHAM, Journal of Fractional Calculus and Applications, 10 (1) (2019) 105-119.

[39] D. Baleanu, B. Kilic, M. Inc, the first integral method for Wu-Zhang nonlinear system with time-dependent coefficients, Proceedings of the Romanian Academy, Series A, 16(2) (2015) 160-167.

[40] N. Bourbak, Commutative Algebra, Paris Addison-Wesley, 1972.

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