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CERTAIN UNIFIED INTEGRALS PERTAINING TO KIRYAKOVA MULTIINDEX- MITTAG-LEFFLER FUNCTION

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Abstract. The aim of the present paper is to establish three new integrals whose integrands involve the product of multivariable H-function and the multi-index Mittag-Leffler function due to Kiryakova having general argument. A number of results follow as particular cases of integrals established here.

1. Introduction

The Mittag-Leffler function [10] first introduced by Swedish mathematician Gsta Mittag-Leffler, is of great importance and attracted a lot of focus of researchers. This function can interpolate between exponential and hypergeometric function. The function plays an active role in solution of various differential equation of fractional order arose in the field of engineering physics, mathematical biology and applied sciences problems.

In the first section we define the functions involved in our main results. Second section comprises of three integrals evaluated here. Some particular cases are specified in section three and in last the work of this paper has concluded.

The multivariable H-function introduced by Srivastava and Panda[5] is defined and represented as follows:

$$H\left[z_1, \cdots, z_{r+1}\right]$$

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$$= H^{0,\lambda:m_1,n_1;\cdots;m_{r+1},n_{r+1}}_{A,C:p_1,q_1;\cdots;p_{r+1},q_{r+1}} \left[\begin{array}{c|c} z_1; & \left(a_j;\theta',\cdots,\theta_j^{(r)},0\right)_{1,A} \left(b',\gamma_j'\right)_{1,p_1};\cdots\left(b_j^{(r+1)},\gamma_j^{(r+1)}\right)_{1,p_{r+1}} \\ \vdots & \vdots \\ z_{r+1}; & \left(c_j;\psi',\cdots,\psi_j^{(r)},0\right)_{1,C} \left(d',\delta_j'\right)_{1,q_1};\cdots\left(d_j^{(r+1)},\delta_j^{(r+1)}\right)_{1,q_{r+1}} \end{array} \right]$$

$$= \frac{1}{(2\pi \mathrm{i})^{\mathrm{r}}} \int_{\mathrm{L}_1} \ldots \int_{\mathrm{L}_\mathrm{r}} \phi\left(\mathrm{s}_1,\ldots,\mathrm{s}_{\mathrm{r}+1}\right) \theta_1\left(\mathrm{s}_1\right) \ldots \theta_{\mathrm{r}}\left(\mathrm{s}_{\mathrm{r}+1}\right) \mathrm{z}_1^{\mathrm{s}_1} \ldots \mathrm{z}_{r+1}^{\mathrm{s}_\mathrm{r}} \mathrm{ds}_1 \ldots \mathrm{ds}_{\mathrm{r}+1} \tag{1}$$

where

$$\theta_i\left(\mathrm{s}_i\right) = \frac{\prod\limits_{\mathrm{j}=1}^{m_i} \Gamma(\mathrm{d}_\mathrm{j}^{(i)}-\delta_\mathrm{j}^{(i)}\mathrm{s}_\mathrm{i}) \prod\limits_{\mathrm{j}=1}^{n_i} \Gamma(1-\mathrm{b}_\mathrm{j}^{(i)}+\phi_\mathrm{j}^{(i)}\mathrm{s}_i)}{\prod\limits_{\mathrm{j}=\mathrm{m}_i+1}^{\mathrm{q}_i} \Gamma(1-\mathrm{d}_\mathrm{j}^{(i)}+\lambda_\mathrm{j}^{(i)}\mathrm{s}_i) \prod\limits_{\mathrm{j}=\mathrm{n}_i+1}^{\mathrm{p}_i} \Gamma(\mathrm{b}_\mathrm{j}^{(i)}-\phi_\mathrm{j}^{(i)}\mathrm{s}_i)} \quad \forall\{\mathrm{i}=1,\ldots,\mathrm{r}\} \tag{2}$$

$$\psi\left(t_1,\ldots,t_{r+1}\right) = \frac{\prod\limits_{j=1}^{\delta} \Gamma(1-a_j+\sum\limits_{i=1}^{r}\theta_j^{(i)}t_j+0\times t_{r+1})}{\prod\limits_{j=1}^{C} \Gamma(1-c_j+\sum\limits_{i=1}^{r}\psi_j^{(i)}t_j+0\times t_{r+1}) \prod\limits_{j=\delta+1}^{A} \Gamma(a_j-\sum\limits_{i=1}^{r}\theta_j^{(i)}t_j+0\times t_{r+1})} \tag{3}$$

For the nature of contours, various sets of convergence conditions of the integral given by (1) and the other details about this function we may refer to [6].
We will note

$$U = \mathrm{m}_1, n_1;\ \ldots,\ \mathrm{m}_{r+1}, n_{r+1}\,; V = p_1, q_1;\ \ldots,\ p_{r+1}, q_{r+1} \tag{4}$$

$$\bar{A} = \left(\mathrm{a}_j;\theta'_j,\ldots,\theta_j^{(\mathrm{r})},0\right)_{1,A}; \bar{C} = \left(\mathrm{c}_j;\psi'_j,\ldots,\psi_j^{(\mathrm{r})},0\right)_{1,C} \tag{5}$$

$$B = \left(\mathrm{b}'_j,\gamma'_j\right)_{1,p_1};\ldots;\left(\mathrm{b}_j^{(r+1)},\gamma_j^{(r+1)}\right)_{1,p_{r+1}}; D = \left(d'_j,\delta'_j\right)_{1,q_1},\ldots,\left(d_j^{(r+1)},\delta_j^{(r+1)}\right)_{1,q_{r+1}} \tag{6}$$

The Mittag-Leffler function $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$ defined as

$$E_\alpha\left(z\right) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma\left(\alpha k+1\right)}$$

and

$$E_{\alpha,\beta}\left(z\right) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma\left(\alpha k+\beta\right)}, \quad \alpha \succ 0, \beta \succ 0$$

are extensions of exponential and trigonometric functions and also satisfy ordinary differential equations of the type $D^n y\left(\lambda z\right) = \lambda^n y\left(\lambda z\right) \quad (n=1,2)$ of I and II order.
From then, the M-L function has been generalized several times by taking additional parameters. For these generalizations one can refer to ([1], [2],[11]).
Then Kiryakova [12], [13] introduced and studied the multi-index Mittag Leffler

function defined as

$$E_{\left(\frac{1}{\rho_i}\right),(\mu_i)}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma\left(\mu_1+\frac{k}{\rho_1}\right)...\Gamma\left(\mu_m+\frac{k}{\rho_m}\right)} \quad (7)$$

which is a multi-index analog of $E_{\alpha,\beta}(z)$ (also see [14]). The Mellin branes integral of multi-indices Mittag Leffler function is

$$E_{\left(\frac{1}{\rho_i}\right),(\mu_i)}(z)=\frac{1}{2\pi i}\int_L\frac{\Gamma(s)\Gamma(1-s)(-z)^{-s}}{\prod_{\mathrm{j}=1}^{m}\Gamma\left(\mu_i-\frac{\mathrm{s}}{\rho_i}\right)}ds \quad z\neq 0 \quad (8)$$

Struve function introduced by [7] in 1882 possess power series representation of the form ([4], [8])

$$\mathrm{H}_\nu[z]=\sum_{m=0}^{\infty}\frac{(-1)^m}{\Gamma\left(m+\frac{3}{2}\right)\Gamma\left(m+\nu+\frac{3}{2}\right)}\left(\frac{z}{2}\right)^{\nu+2m+1} \quad (9)$$

where $\nu\in R$ and $z\in C$. This function is a particular solution of well-known non-Homogenous Bessel differential equation ([9], page 341)

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(x^2-p^2\right)y(x)=\frac{4\left(\frac{x}{2}\right)^{p+1}}{\sqrt{\pi}\Gamma\left(p+\frac{1}{2}\right)} \quad (10)$$

where Γ is the classical gamma function.

2. Main Results:

In this section we establish three integrals concerning Struve function, multi-indices Mittag-Leffler function and multivariable H-function as their integrands using three known results [3], ([8], p. 3.264, 3.278).

Lemma 1

$$\int_0^{\infty}z^{\mu-1}\left\{z+\alpha+\left(z^2+2\alpha z\right)^{\frac{1}{2}}\right\}^{-\lambda}dz=2\lambda\alpha^{-\lambda}\left(\frac{\alpha}{2}\right)^{\mu}\frac{\Gamma(2\mu)\Gamma(\lambda-\mu)}{\Gamma(1+\lambda+\mu)} \quad (11)$$

Theorem 1

$$\int_0^{\infty}t^{\lambda-1}\left\{t+p+\left(t^2+2pt\right)^{\frac{1}{2}}\right\}^{-\upsilon}H_\gamma\left\{t+p+\left(t^2+2pt\right)^{\frac{1}{2}}\right\}^{-\delta}$$

$$E_{\left\{\left(\frac{1}{\rho_j}\right),(\mu_j)\right\}}\left[z\left\{t+p+\left(t^2+2pt\right)^{\frac{1}{2}}\right\}^{-\beta}\right]$$

$$H\left(z_1\left\{t+a+\left(t^2+2pt\right)^{\frac{1}{2}}\right\}^{-\alpha_1},\cdots,z_r\left\{t+a+\left(t^2+2pt\right)^{\frac{1}{2}}\right\}^{-\alpha_r}\right)dt$$

$$=2\left(\frac{p}{2}\right)^{\lambda}\Gamma(2\lambda)p^{-\upsilon}\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{1}{2p^\delta}\right)^{(\gamma+2n+1)}p^{-\delta(1+\gamma+2n)}}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\gamma+\frac{3}{2}\right)}$$

$$H^{0,\lambda+2;U;(1,1)}_{A+2,C+2;V;(1,1),(m+1,1)}\left[\begin{array}{c|c} z_1/p^{\alpha_1} & A_1,\ \bar{A}:B;(1,1) \\ \vdots & \vdots \\ z_r/p^{\alpha_r} & \\ -1/(zp)^{\beta} & \bar{C},C_1:D;(1,1)\,(m+1,1) \end{array}\right] \tag{12}$$

where

$$A_1=(-\upsilon-\delta\left(\gamma+2n+1\right);\alpha_1,...,\alpha_r,\beta),(1+\lambda-\upsilon-\delta\left(\gamma+2n+1\right);\alpha_1,...,\alpha_r,\beta) \tag{13}$$

$$C_1=(1-\upsilon-\delta\left(\gamma+2n+1\right);\alpha_1,...,\alpha_r,\beta),(-\lambda-\upsilon-\delta\left(\gamma+2n+1\right);\alpha_1,...,\alpha_r,\beta) \tag{14}$$

and existence conditions are given by

$$\alpha_1,...,\alpha_r\succ 0,\mathrm{Re}\left(\lambda,\upsilon,\delta,\beta\right)\succ 0 \qquad \text{and}$$
$$\mathrm{Re}\left(\lambda\right)-\mathrm{Re}\left(\upsilon\right)-\mathrm{Re}\left(\delta\right)+\beta-\sum_{i=1}^{r}\alpha_i\min_{1\le j\le U^{(i)}}\mathrm{Re}\left(\frac{d_j}{\delta_j}\right)\prec 0. \tag{15}$$

Proof In order to prove integral (12), we first express both of the Multivariable H-function and multiindices Mittag-Leffler function in their Mellin Barnes type contour integral representation and Struve function in its series form using equations (1) and (7) and (9) respectively and changing the order of integration, we have

$$=\frac{1}{(2\pi\omega)^{r+1}}\int_{L_1}\cdots\int_{L_r}\int_{L}\psi\left(t_1,\cdots,t_1\right)\prod_{i=1}^{r}\xi_i\left(t_i\right)z_i^{t_i}\frac{\Gamma\left(s\right)\Gamma\left(1-s\right)\left(\frac{-1}{z}\right)^s}{\prod_{j=1}^{m}\Gamma\left(\mu_j-\frac{s}{\rho_j}\right)}$$
$$\times\sum_{n=0}^{\infty}\frac{(-1)^n\,2}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\gamma+\frac{3}{2}\right)}$$
$$\times\int_0^{\infty}t^{\lambda-1}\left[t+p+\left(t^2+2pt\right)^{\frac{1}{2}}\right]^{-\upsilon-\delta(2n+\gamma+1)-\sum_{i=1}^{r}\alpha_i s_i-\beta s}dtdt_1\cdots dt_{r+1}$$

On applying lemma 1, we get

$$=\frac{1}{(2\pi\omega)^{r+1}}\int_{L_1}\cdots\int_{L_r}\int_{L}\psi\left(t_1,\cdots,t_1\right)\prod_{i=1}^{r}\xi_i\left(t_i\right)z_i^{t_i}\frac{\Gamma\left(s\right)\Gamma\left(1-s\right)\left(\frac{-1}{z}\right)^s}{\prod_{j=1}^{m}\Gamma\left(\mu_j-\frac{s}{\rho_j}\right)}$$
$$\sum_{n=0}^{\infty}\frac{(-1)^n\,2}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\gamma+\frac{3}{2}\right)}2\left(\frac{p}{2}\right)^{\lambda}\Gamma\left(2\lambda\right)p^{-\upsilon-\delta(2n+\gamma+1)-\sum_{i=1}^{r}\alpha_i s_i-\beta s}$$
$$\times\left(-\upsilon-\delta\left(2n+\gamma+1\right)-\sum_{i=1}^{r}\alpha_i s_i-\beta s\right)$$

Now using the property $a=\frac{\Gamma(a+1)}{\Gamma(a)}, \quad a\neq 0,\,-1,\,-2\cdots$ we get

$$\int_0^\infty t^{\lambda-1}\left\{t+p+\left(t^2+2pt\right)^{\frac{1}{2}}\right\}^{-\upsilon} H_\gamma\left\{t+p+\left(t^2+2pt\right)^{\frac{1}{2}}\right\}^{-\delta}$$

$$E_{\left\{\left(\frac{1}{\rho_j}\right),(\mu_j)\right\}}\left[z\left\{t+p+\left(t^2+2pt\right)^{\frac{1}{2}}\right\}^{-\beta}\right]$$

$$H\left(z_1\left\{t+a+\left(t^2+2pt\right)^{\frac{1}{2}}\right\}^{-\alpha_1},\cdots,z_r\left\{t+a+\left(t^2+2pt\right)^{\frac{1}{2}}\right\}^{-\alpha_r}\right)dt$$

$$=2\left(\frac{p}{2}\right)^\lambda\Gamma(2\lambda)\,p^{-\upsilon}\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{1}{2p^\delta}\right)^{(2n+\gamma+1)}}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\gamma+\frac{3}{2}\right)}\frac{1}{(2\pi\omega)^{r+1}}\int_{L_1}\cdots\int_{L_r}\int_L\psi(t_1,\cdots,t_1)$$

$$\prod_{i=1}^{r}\xi_i(t_i)\,z_i^{t_i}\frac{\Gamma(s)\Gamma(1-s)\left(\frac{-1}{zp^\beta}\right)^s}{\prod_{j=1}^{m}\Gamma\left(\mu_j-\frac{s}{\rho_j}\right)}p^{-\delta(2n+\gamma+1)-\sum_{i=1}^{r}\alpha_i s_i-\beta s}$$

$$\frac{\Gamma\left(1+\upsilon+\delta(2n+\gamma+1)+\sum_{i=1}^{r}\alpha_i s_i+\beta s\right)}{\Gamma\left(\upsilon+\delta(2n+\gamma+1)+\sum_{i=1}^{r}\alpha_i s_i+\beta s\right)}$$

$$\frac{\Gamma\left(\upsilon-\lambda+\delta(2n+\gamma+1)+\sum_{i=1}^{r}\alpha_i s_i+\beta s\right)}{\Gamma\left(1+\upsilon+\lambda+\delta(2n+\gamma+1)+\sum_{i=1}^{r}\alpha_i s_i+\beta s\right)}dt\,dt_1\cdots dt_{r+1}$$

Interpreting the above multiple integrals, we find the R. H. S. of (12).

Lemma 2

$$\int_0^\infty t^{\lambda-1}\left(1+at^p\right)^{-\mu}\left(1+bt^p\right)^{-\upsilon}dt=\frac{1}{p}a^{\frac{-\lambda}{p}}B\left(\frac{\lambda}{p},\mu+\upsilon-\frac{\lambda}{p}\right){}_2F_1\left(\upsilon,\frac{-\lambda}{p};\mu+\upsilon;1-\frac{b}{a}\right) \tag{16}$$

Theorem 2

$$\int_0^\infty t^{\lambda-1}\left(1+at^p\right)^{-\mu}\left(1+bt^p\right)^{-\upsilon}H_\gamma\left\{\left(1+at^p\right)^{-\mu}\left(1+bt^p\right)^{-\upsilon}\right\}$$

$$E_{\left\{\frac{1}{\rho_j},\mu_j\right\}}\left[z\left\{\left(1+at^p\right)^{\frac{1}{2}}\right\}^{-\alpha}\left\{\left(1+bt^p\right)^{\frac{1}{2}}\right\}^{-\beta}\right]H\begin{bmatrix}z_1\left\{\left(1+at^p\right)^{-\alpha_1}\left(1+bt^p\right)^{-\beta_1}\right\}\\ \vdots\\ z_r\left\{\left(1+at^p\right)^{-\alpha_r}\left(1+bt^p\right)^{-\beta_r}\right\}\end{bmatrix}dt$$

$$=\frac{1}{p}a^{\frac{-\lambda}{p}}\sum_{m,n=0}^{\infty}\frac{1}{m!}\left(1-\frac{b}{a}\right)^m\frac{(-1)^n\left(\frac{1}{2}\right)^{\gamma+2n+1}\Gamma\left(\frac{\lambda}{p}+m\right)}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\gamma+\frac{3}{2}\right)}$$

$$H^{0,\lambda+2;U;(1,1)}_{A+2,C+2;V;(m+1,1)}\left[\begin{matrix}z_1\\ \vdots\\ z_r\\ (-z)^{-1}\end{matrix}\left|\begin{matrix}A_2,\bar{A}:B;(1,1)\\ \vdots\\ \bar{C},C_2;D;(1,1),(m+1,1)\end{matrix}\right.\right] \tag{17}$$

where

$$A_2=\left(1+\frac{\lambda}{p}-\mu-\upsilon-(s+\rho)(\gamma+2n+1);\alpha_1+\beta_1,...,\alpha_r+\beta_r,\alpha+\beta\right),\\ (1-m-\upsilon-\rho(\gamma+2n+1);\beta_1,...,\beta_r,\beta) \tag{18}$$

$$C_2=(1-m-\upsilon-\mu-(s+\rho)(\gamma+2n+1);\alpha_1+\beta_1,...,\alpha_r+\beta_r,\alpha+\beta),\\ \left(1-\upsilon-\rho(\gamma+2n+1);\beta_{\mathrm{j}}^{'},...,\beta_{\mathrm{j}}^{r},\beta\right) \tag{19}$$

under the conditions

$$p\succ 0,|\arg b|\prec\pi,0\prec Re(\lambda)\prec 2Re\left[\mu+\delta(\gamma+2n+1)+\alpha+\min_{1\le j\le U^{(i)}}\left\{Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right\}\right] \tag{20}$$

Proof In order to prove integral (17), we first express both of the Multivariable H-function and multiindices Mittag-Leffler function in their Mellin Barnes type contour integral representation and Struve function in its series form using equations (1) and (7) and (9) respectively

$$=\frac{1}{p}a^{\frac{-\lambda}{p}}\sum_{m,n=0}^{\infty}\frac{1}{m!}\left(1-\frac{b}{a}\right)^m\frac{(-1)^n\left(\frac{1}{2}\right)^{\gamma+2n+1}\Gamma\left(\frac{\lambda}{p}+m\right)}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\gamma+\frac{3}{2}\right)}\frac{1}{(2\pi\omega)^{r+1}}$$

$$\int_{L_1}\cdots\int_{L_r}\int_{L}\psi(t_1,\cdots,t_1)\prod_{i=1}^{r}\xi_i(t_i)z_i^{t_i}\frac{\Gamma(s)\Gamma(1-s)\left(\frac{-1}{zp^{\beta}}\right)^s}{\prod_{j=1}^{m}\Gamma\left(\mu_j-\frac{s}{\rho_j}\right)}(-z)^{-s}\int_0^{\infty}t^{\lambda-1}$$

$$\times(1+at^p)^{-\mu-\delta(2n+\gamma+1)+\sum_{i=1}^{r}\alpha_is_i+\beta s}(1+bt^p)^{-\upsilon-\delta(2n+\gamma+1)+\sum_{i=1}^{r}\alpha_is_i+\beta s}dtdt_1\cdots dt_rds$$

using lemma 2, we have

$$\int_0^{\infty}t^{\lambda-1}(1+at^p)^{-\mu}(1+bt^p)^{-\upsilon}H_{\gamma}\left\{(1+at^p)^{-\mu}(1+bt^p)^{-\upsilon}\right\}$$

$$E_{\left\{\frac{1}{\rho_j},\mu_j\right\}}\left[z\left\{(1+at^p)^{\frac{1}{2}}\right\}^{-\alpha}\left\{(1+bt^p)^{\frac{1}{2}}\right\}^{-\beta}\right]H\begin{bmatrix}z_1\left\{(1+at^p)^{-\alpha_1}(1+bt^p)^{-\beta_1}\right\}\\ \vdots\\ z_r\left\{(1+at^p)^{-\alpha_r}(1+bt^p)^{-\beta_r}\right\}\end{bmatrix}dt$$

$$=\frac{1}{p}a^{\frac{-\lambda}{p}}\sum_{m,n=0}^{\infty}\frac{1}{m!}\left(1-\frac{b}{a}\right)^m\frac{(-1)^n\left(\frac{1}{2}\right)^{\gamma+2n+1}\Gamma\left(\frac{\lambda}{p}+m\right)}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\gamma+\frac{3}{2}\right)}\frac{1}{(2\pi\omega)^{r+1}}$$

$$\int_{L_1}\cdots\int_{L_r}\int_{L}\psi(t_1,\cdots,t_1)\prod_{i=1}^{r}\xi_i(t_i)z_i^{t_i}(\upsilon+\rho(\gamma+2n+1);\beta_1,...,\beta_r,\beta)_m$$

$$\frac{\Gamma\left(\frac{-\lambda}{p}+\mu+\upsilon+(s+\rho)(\gamma+2n+1);\alpha_1+\beta_1,\cdots\alpha_r+\beta_r,\alpha+\beta\right)}{\Gamma(\mu+\upsilon+(s+\rho)(\gamma+2n+1)+m;\alpha_1+\beta_1,\cdots\alpha_r+\beta_r,\alpha+\beta)}dt_1\cdots dt_rds$$

Now using the relation $(\alpha_m) = \frac{\Gamma(\alpha+m)}{\Gamma(\alpha)}, \quad \alpha \neq m$ and interpreting the result in terms of Mellin-Barnes type integrals, we obtain the R.H. S. of (17).

Lemma 3

$$\int_0^\infty \left(a+\frac{b}{t^2}\right)\left[\left(a+\frac{b}{t^2}\right)+c\right]^{-p-1} dt = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}+p\right)}{(4ab+c)^{p+\frac{1}{2}}\,\Gamma(1+p)} \tag{21}$$

Theorem 3

$$\begin{aligned}
&\int_0^\infty \left(a+\frac{b}{t^2}\right)\left[\left(at+\frac{b}{t^2}\right)^2+c\right]^{-p-1} H_\gamma\left[z\left\{\left(a+\frac{b}{t^2}\right)^2+c\right\}^{-\alpha}\right] \\
&E_{\left\{\frac{1}{\rho_j},\mu_j\right\}}\left[z\left\{\left(at+\frac{b}{t^2}\right)^2+c\right\}^{-\beta}\right] H\begin{bmatrix} z_1\left\{\left(at+\frac{b}{t^2}\right)^2+c\right\}^{-\alpha_1} \\ \vdots \\ z_r\left\{\left(at+\frac{b}{t^2}\right)^2+c\right\}^{-\alpha_r}\end{bmatrix} dt \\
&= \frac{\sqrt{\pi}}{(4ab+c)^{p+\frac{1}{2}}}\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{z}{2}\right)^{\gamma+2n+1}}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\gamma+\frac{3}{2}\right)}\frac{1}{(4ab+c)^{\alpha(\gamma+2n+1)}} \\
&H^{0,\lambda+1;U;(1,1)}_{A+1,C+1;0,V;(1,1),(m+1,1)}\left[\begin{matrix}\left\{\frac{z_1}{(4ab+c)}\right\}^{-\alpha_1} \\ \vdots \\ \left\{\frac{z_r}{(4ab+c)}\right\}^{-\alpha_r} \\ \left\{\frac{(-z)^{-1}}{(4ab+c)^{\beta}}\right\}\end{matrix}\ \middle|\ \begin{matrix} A_3, \bar{A} : B; (1,1) \\ \vdots \\ \bar{C}, C_3 : D; (1,1), (m+1,1)\end{matrix}\right]
\end{aligned} \tag{22}$$

where

$$A_3 = \left(\frac{1}{2}-\alpha(\gamma+2n+1);\alpha_1,\cdots,\alpha_r,\beta\right); C_3 = \left(-\alpha(\gamma+2n+1);\alpha_1,\cdots,\alpha_r,\beta\right) \tag{23}$$

and existence conditions are

$$a,b \succ 0, c \succ -4ab, \left[p+\delta(\gamma+2n+1)+\beta s+\alpha_i \min_{1\le j\le U^{(i)}}\left\{Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right\}\right] \succ \frac{-1}{2} \tag{24}$$

Proof In order to prove integral (22), we first express both of the Multivariable H-function and multiindices Mittag-Leffler function in their Mellin Barnes type contour integral representation using equations (1) and (7) and Struve function in its series form using equation (9) following by interchanging the order of integrals (which is permissible under the given conditions), we have

$$= \frac{\sqrt{\pi}}{(4ab+c)^{p+\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{\gamma+2n+1}}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\gamma+\frac{3}{2}\right)} \frac{1}{(4ab+c)^{\alpha(\gamma+2n+1)}} \frac{1}{(2\pi\omega)^{r+1}}$$
$$\int_{L_1} \cdots \int_{L_r} \int_L \psi(t_1, \cdots, t_1) \prod_{i=1}^{r} \xi_i(t_i) z_i^{t_i} \frac{\Gamma(s)\Gamma(1-s)}{\prod_{j=1}^{m} \Gamma\left(\mu_j - \frac{s}{\rho_j}\right)} (-z)^{-s}$$
$$\int_0^{\infty} \left[\left(a+\frac{b}{t^2}\right)\left[\left(a+\frac{b}{t^2}\right)+c\right]^{-p-(2n+\gamma+1)-\sum_{i=1}^{r}\alpha_i s_i - \beta s}\right] dt$$

The inner integral can be solved using lemma 3 and we get

$$\int_0^{\infty} \left(a+\frac{b}{t^2}\right)\left[\left(at+\frac{b}{t^2}\right)^2+c\right]^{-p-1} H_\gamma\left[z\left\{\left(a+\frac{b}{t^2}\right)^2+c\right\}^{-\alpha}\right]$$
$$E_{\left\{\frac{1}{\rho_j},\mu_j\right\}}\left[z\left\{\left(at+\frac{b}{t^2}\right)^2+c\right\}^{-\beta}\right] H \begin{bmatrix} z_1\left\{\left(at+\frac{b}{t^2}\right)^2+c\right\}^{-\alpha_1} \\ \vdots \\ z_r\left\{\left(at+\frac{b}{t^2}\right)^2+c\right\}^{-\alpha_r} \end{bmatrix} dt$$
$$= \frac{\sqrt{\pi}}{(4ab+c)^{p+\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{\gamma+2n+1}}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\gamma+\frac{3}{2}\right)} \frac{1}{(4ab+c)^{\alpha(\gamma+2n+1)}} \frac{1}{(2\pi\omega)^{r+1}}$$
$$\int_{L_1} \cdots \int_{L_r} \int_L \psi(t_1, \cdots, t_1) \prod_{i=1}^{r} \xi_i(t_i) z_i^{t_i} \frac{\Gamma(s)\Gamma(1-s)}{\prod_{j=1}^{m} \Gamma\left(\mu_j - \frac{s}{\rho_j}\right)} (-z)^{-s}$$
$$\frac{1}{(4ab+c)^{\alpha(2n+\gamma+1)+\sum_{i=1}^{r}\alpha_i s_i+\beta s}} \frac{\Gamma\left(\frac{1}{2}+\alpha(2n+\gamma+1)+\sum_{i=1}^{r}\alpha_i s_i+\beta s\right)}{\Gamma\left(1+\alpha(2n+\gamma+1)+\sum_{i=1}^{r}\alpha_i s_i+\beta s\right)} dt_1 \cdots dt_r ds$$

On interpreting the above integral in terms of multivariable H-function, we obtain the R.H.S. of (22).

3. Special Cases:

Concerning the theorem 1, we suppose the conditions $\lambda = A = C = 0$, in this particular situation, we have $\lambda = A = C = 0 = \bar{A} = \bar{B}$ and we get the result about the product of (r + 1) Foxs H-functions by using the same notations.

Corollary 1.

$$\int_0^{\infty} t^{\lambda-1}\left\{t+p+\left(t^2+2pt\right)^{\frac{1}{2}}\right\}^{-\upsilon} H_\gamma\left\{t+p+\left(t^2+2pt\right)^{\frac{1}{2}}\right\}^{-\delta}$$
$$E_{\left\{\left(\frac{1}{\rho_j}\right),(\mu_j)\right\}}\left[z\left\{t+p+\left(t^2+2pt\right)^{\frac{1}{2}}\right\}^{-\beta}\right] \prod_{i=1}^{r} H_{p_i,q_i}^{m_i,n_i}\left[z_i\left\{t+a+\left(t^2+2pt\right)^{\frac{1}{2}}\right\}^{-\alpha_i}\right] dt$$

$$= 2\left(\frac{p}{2}\right)^{\lambda}\Gamma(2\lambda)\, p^{-\upsilon}\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{1}{2p^{\delta}}\right)^{(\gamma+2n+1)} z^{-\gamma-2n-1}p^{-\delta(1+\gamma+2n)}}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\gamma+\frac{3}{2}\right)}$$

$$H^{0,\lambda+2;U;(1,1)}_{2,2;V;(1,1),(m+1,1)}\left[\begin{array}{c|c} z_1/p^{\alpha_1} & A_1,\ B;(1,1) \\ \vdots & \vdots \\ z_r/p^{\alpha_r} & C_1 : D;(1,1)\,(m+1,1) \\ -1/(zp)^{\beta} & \end{array}\right] \tag{25}$$

under the conditions derived from those mentioned with equation (12).
Taking $\gamma = -\frac{1}{2}$, we have

Corollary 2.

$$\int_0^{\infty} t^{\lambda-1}\left\{t+p+\left(t^2+2pt\right)^{\frac{1}{2}}\right\}^{-\upsilon} J_{\frac{1}{2}}\left\{t+p+\left(t^2+2pt\right)^{\frac{1}{2}}\right\}^{-\delta}$$

$$E_{\left\{\left(\frac{1}{\rho_j}\right),(\mu_j)\right\}}\left[z\left\{t+p+\left(t^2+2pt\right)^{\frac{1}{2}}\right\}^{-\beta}\right] H\left[\begin{array}{c} z_1\left\{t+a+\left(t^2+2pt\right)^{\frac{1}{2}}\right\}^{\alpha_1} \\ \vdots \\ z_r\left\{t+a+\left(t^2+2pt\right)^{\frac{1}{2}}\right\}^{\alpha_r} \end{array}\right] dt$$

$$= 2\left(\frac{p}{2}\right)^{\lambda}\Gamma(2\lambda)\, p^{-\upsilon}\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{1}{2p^{\delta}}\right)^{\left(2n+\frac{1}{2}\right)} p^{-\delta\left(2n+\frac{1}{2}\right)}}{\Gamma\left(n+\frac{3}{2}\right)\Gamma(n+1)}$$

$$H^{0,\lambda+2;U;(1,1)}_{2,2;V;(1,1),(m+1,1)}\left[\begin{array}{c|c} z_1/p^{\alpha_1} & A'_1,\bar{A}\, B;(1,1) \\ \vdots & \vdots \\ z_r/p^{\alpha_r} & \bar{C}, C'_1 : D;(1,1)\,(m+1,1) \\ -1/(zp)^{\beta} & \end{array}\right] \tag{26}$$

where $J\,()$ is Bessels function and

$$A_1 = \left(-\upsilon-\delta(\gamma+2n+1);\alpha_1,\cdots,\alpha_r,\beta\right),\left(1+\lambda-\upsilon-\delta(\gamma+2n+1);\alpha_1,\cdots,\alpha_r,\beta\right) \tag{27}$$

$$c_1 = \left(1-\upsilon-\delta(\gamma+2n+1);\alpha_1,\cdots,\alpha_r,\beta\right),\left(-\lambda-\upsilon-\delta(\gamma+2n+1);\alpha_1,\cdots,\alpha_r,\beta\right) \tag{28}$$

and conditions of existence can be derived from (14).
Now considering Theorem 2 and taking $\gamma = -h - \frac{1}{2}$, we get the following result

Corollary 3.

$$\int_0^\infty t^{\lambda-1}\left(1+at^p\right)^{-\mu}\left(1+bt^p\right)^{-\upsilon} J_{h+\frac{1}{2}}\left[\left\{\left(1+at^p\right)^{\frac{1}{2}}\right\}^{\delta}\left\{\left(1+bt^p\right)^{\frac{1}{2}}\right\}^{-\rho}\right]$$

$$E_{\left\{\frac{1}{\rho_j},\mu_j\right\}}\left[z\left\{\left(1+at^p\right)^{\frac{1}{2}}\right\}^{-\alpha}\left\{\left(1+bt^p\right)^{\frac{1}{2}}\right\}^{-\beta}\right] H\begin{bmatrix} z_1\left\{\left(1+at^p\right)^{-\alpha_1}\left(1+bt^p\right)^{-\beta_1}\right\} \\ \vdots \\ z_r\left\{\left(1+at^p\right)^{-\alpha_r}\left(1+bt^p\right)^{-\beta_r}\right\}\end{bmatrix} dt$$

$$= \frac{1}{p} a^{\frac{-\lambda}{p}} \sum_{m,n=0}^{\infty} \frac{1}{m!}\left(1-\frac{b}{a}\right)^m \frac{(-1)^n\left(\frac{1}{2}\right)^{\gamma+2n+1}\Gamma\left(\frac{\lambda}{p}+m\right)}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\gamma+\frac{3}{2}\right)}$$

$$H^{0,\lambda+2;U;(1,1)}_{A+2,C+2;V;(m+1,1)}\left[\begin{array}{c|c} z_1 & A'_{2,}\bar{A}:B;(1,1) \\ \vdots & \vdots \\ z_r & \\ (-z)^{-1} & \bar{C},C'_2;D;(1,1),(m+1,1)\end{array}\right] \tag{29}$$

where

$$A'_2 = \left(1+\frac{\lambda}{p}-\mu-\upsilon-(s+\rho)\left(\frac{1}{2}+2n-h\right);\alpha_1+\beta_1,...,\alpha_r+\beta_r,\alpha+\beta\right),$$
$$\left(1-m-\upsilon-\rho\left(\frac{1}{2}+2n-h\right);\beta_1,...,\beta_r,\beta\right) \tag{30}$$

$$C'_2 = \left(1-m-\upsilon-\mu-(s+\rho)\left(\frac{1}{2}+2n-h\right);\alpha_1+\beta_1,...,\alpha_r+\beta_r,\alpha+\beta\right),$$
$$\left(1-\upsilon-\rho\left(\frac{1}{2}+2n-h\right);\beta_{\mathrm{j}}{}^{'},...,\beta_{\mathrm{j}}{}^{r},\beta\right) \tag{31}$$

under the conditions derived from those mentioned with equation (17).

Now putting $r=1$, we have the following integral involving the H-function of one variable:

Corollary 4.

$$\int_0^\infty t^{\lambda-1}\left(1+at^p\right)^{-\mu}\left(1+bt^p\right)^{-\upsilon} H_{\gamma}\left[\left\{\left(1+at^p\right)^{\frac{1}{2}}\right\}^{\delta}\left\{\left(1+bt^p\right)^{\frac{1}{2}}\right\}^{-\rho}\right]$$

$$E_{\left\{\frac{1}{\rho_j},\mu_j\right\}}\left[z\left\{\left(1+at^p\right)^{\frac{1}{2}}\right\}^{-\alpha_1}\left\{\left(1+bt^p\right)^{\frac{1}{2}}\right\}^{-\beta_1}\right] H\left[z_1\left\{\left(1+at^p\right)^{-\alpha_1}\left(1+bt^p\right)^{-\beta_1}\right\}\right] dt$$

$$= \frac{1}{p} a^{\frac{-\lambda}{p}} \sum_{m,n=0}^{\infty} \frac{1}{m!} \left(1-\frac{b}{a}\right)^m \frac{(-1)^n \left(\frac{1}{2}\right)^{\gamma+2n+1} \Gamma\left(\frac{\lambda}{p}+m\right)}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\gamma+\frac{3}{2}\right)}$$

$$H^{0,\lambda+2;U;(1,1)}_{p'+2,q'+2;(m''+1,1)} \left[\begin{array}{c|c} z_1 & A_{21}, (a_j, \theta_j)_{1,p_j} : B; (1,1) \\ \vdots & \vdots \\ z_r & \\ (-z)^{-1} & (c_j, \psi_j)_{1,q_j}, C_{21}; D; (1,1), (m''+1,1) \end{array} \right] \tag{32}$$

where

$$A_{21} = \left(1+\frac{\lambda}{p} - \mu - \upsilon - (s+\rho)(\gamma+2n+1); \alpha_1+\beta_1, \alpha+\beta\right), \\ (1-m-\upsilon-\rho(\gamma+2n+1); \beta_1, \beta) \tag{33}$$

$$C_{21} = \left(1-m-\upsilon-\mu-(s+\rho)\left(\frac{1}{2}+\gamma+2n\right); \alpha_1+\beta_1, \alpha+\beta\right), \\ \left(1-\upsilon-\rho\left(\frac{1}{2}+\gamma+2n\right); \beta_{\mathrm{j}}, \beta\right) \tag{34}$$

under the conditions derived from those mentioned with equation (17).
Considering the theorem 3, on taking $\lambda = A = C = 0 = \bar{A} = \bar{B}$, we have the following relation about the product of (r + 1) Foxs H-functions.

Corollary 5.

$$\int_0^\infty \left(a+\frac{b}{t^2}\right)\left[\left(at+\frac{b}{t^2}\right)^2+c\right]^{-p-1} H_\gamma\left[z\left\{\left(a+\frac{b}{t^2}\right)^2+c\right\}^{-\alpha}\right]$$

$$E_{\left\{\frac{1}{\rho_j},\mu_j\right\}}\left[z\left\{\left(at+\frac{b}{t^2}\right)^2+c\right\}^{-\beta}\right]\prod_{i=1}^{r} H^{m_i,n_i}_{p_i,q_i}\left[z_i\left\{\left(at+\frac{b}{t^2}\right)+c\right\}^{\alpha_i}\right]dt$$

$$= \frac{\sqrt{\pi}}{(4ab+c)^{p+\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{\gamma+2n+1}}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\gamma+\frac{3}{2}\right)} \frac{1}{(4ab+c)^{\alpha(\gamma+2n+1)}} \tag{35}$$

$$H^{0,1;U;(1,1)}_{1,1;0,V;(1,1),(m+1,1)} \left[\begin{array}{c|c} \left\{\frac{z_1}{(4ab+c)}\right\}^{-\alpha_1} & A_3 : B; (1,1) \\ \vdots & \vdots \\ \left\{\frac{z_r}{(4ab+c)}\right\}^{-\alpha_r} & C_3 : D; (1,1), (m+1,1) \\ \left\{\frac{(-z)^{-1}}{(4ab+c)^{\beta}}\right\} & \end{array} \right]$$

with the same notations and existence conditions of theorem 3.

On taking $\theta_j^{(i)}, \psi_j^{(i)}, \gamma_j^{(i)}, \delta_j^{(i)} \to 1\,(i,j=1,...,r+1)$, the multivariable H-function reduces to multivariable Meijers G-function, this gives

Corollary 6.

$$\int_0^\infty \left(a+\frac{b}{t^2}\right)\left[\left(at+\frac{b}{t^2}\right)^2+c\right]^{-p-1} H_\gamma\left[z\left\{\left(a+\frac{b}{t^2}\right)^2+c\right\}^{-\alpha}\right]$$

$$E_{\left\{\frac{1}{\rho_j},\mu_j\right\}}\left[z\left\{\left(at+\frac{b}{t^2}\right)^2+c\right\}^{-1}\right] G\begin{bmatrix} z_1\left\{\left(at+\frac{b}{t^2}\right)^2+c\right\}^{-1} \\ \vdots \\ z_r\left\{\left(at+\frac{b}{t^2}\right)^2+c\right\}^{-1}\end{bmatrix} dt$$

$$= \frac{\sqrt{\pi}}{(4ab+c)^{p+\frac{1}{2}}}\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{z}{2}\right)^{\gamma+2n+1}}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\gamma+\frac{3}{2}\right)}\frac{1}{(4ab+c)^{\alpha(\gamma+2n+1)}} \tag{36}$$

$$G^{0,\lambda+1;U;(1,1)}_{A+1,C+1;0,V;(1,1),(m+1,1)}\left[\begin{matrix}\left\{\frac{z_1}{(4ab+c)}\right\}^{-\alpha_1} \\ \vdots \\ \left\{\frac{z_r}{(4ab+c)}\right\}^{-\alpha_r} \\ \left\{\frac{(-z)^{-1}}{(4ab+c)^{\beta}}\right\}\end{matrix}\left|\begin{matrix} A'_3, A_0 : B_0; (1,1) \\ \vdots \\ C_0, C'_3 : D_0; (1,1), (m+1,1)\end{matrix}\right.\right]$$

where

$$A_0=(a_j;1,\cdots,1,0)_{1,A}; C_0=(c_j;1,\cdots,1,0)_{1,C} \tag{37}$$

$$B_0=\left(b'_j;1\right)_{1,p_1};\cdots;\left(b^{(r+1)}_j;1\right)_{1,p_{r+1}}; d_0=\left(d'_j;1\right)_{1,q_1};\cdots;\left(d^{(r+1)}_j;1\right)_{1,q_{r+1}} \tag{38}$$

$$A'_3=\left(\frac{1}{2}-\alpha(1+\gamma+2n);\underbrace{1,\cdots,1}_{r},1\right); C'_3=\left(-\alpha(1+\gamma+2n);\underbrace{1,\cdots,1}_{r},1\right) \tag{39}$$

and existence conditions are same as given in (24).

4. Conclusion:

In this paper we establish three integrals involving Mittag-Leffler function, Struve function and multivariable H-function. These all functions hold generality in their manifolds. First by changing the parameters, we obtain three special cases, which involves the product of r H-functions, Bessel function and some cosine terms. Further these results can be deduced to many integral relations which comprise G function, Fox H function and special function of Mittag Leffler function.

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References

[1] A. Kilbas, M. Saigo, R. K. Saxena, Generalized Mittag- Leffler functions and generalized fractional calculus operators, Integral Transform Spec. Function, 15, 2004, 31-49.

[2] Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and application of Fractional Differential Equations, Elsevier, Amsterdam, 2006.
[3] F. Oberhettinger, Tables of Mellin Transform, Berlin, Heidelberg, New York: Springer-Verlag, P. 22, 1974.
[4] G. N. Watson, Atreatise on the theory of Bessel funcions, 2/e, Cambridge univ. press, London, 1966.
[5] H. M. Srivastava and R. Panda, some bilateral generating functions for a class of generalized hypergeometric polynomials, J. Reine Angew. Math, 283/284, 1976, 265-274.
[6] H. M. Srivastava, K. C. Gupta and S. P. Goyal, The H-function of one and Two Variables with Applications, South Asian Publishers, New Delhi, 1982.
[7] H. Struve, Beitrag zur Theirie der Differaction an Fernrohren, Ann. Physik Chemie, 17, 1882, 1008-1016.
[8] S. Gradshteyin and I. M. Ryzhik, Tables of Integrals, series and products, 7/e, Academic press, New Delhi, 2001.
[9] Jin, J. M. and Zhang Shan Jjie, Computation of special functions, Wiley, 1996.
[10] Mittag-Leffler, Gsta Magnus, Sur la nouvelle fonction $E_\alpha(x)$, CR Acad. Sci. Paris, 137 (2), 1903, 554-558.
[11] R. Gornfelo, A. Kilbas, S. V. Rogozin, On the generalized Mittag-Leffler type function, Integral Transform Spec. Function, 7, 1998, 215-224.
[12] V. Kiryakova, Multiindex Mittag-Leffler functions related Gelfond-Leontiev operators and Laplace type integral transform, Fract. Calc. Appl. Anal., 2, 1999, 445-462.
[13] V. Kiryakova, Multiple Mittag-Leffler functions and relations to generalized fractional calculus, J. Comput. Appl. Math., 118, 2000.
[14] Yu. Luchko, Operational method in fractional calculus, Fractional calculus and applied analysis, 2, 1999), 463-488.

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