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A FAMILY OF THETA-FUNCTION IDENTITIES BASED UPON R_m-FUNCTIONS RELATED TO JACOBI'S TRIPLE-PRODUCT IDENTITY

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Abstract. The authors establish a set of five new relationships involving R_m-functions, which are based upon a q-product identities and Jacobi's celebrated triple-product identity. The present work is motivated and based upon recent findings of Srivastva *et al* (see [18]).

1. Introduction and Definitions

Throughout this article, we denote by $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{C}$ the set of positive integers, the set of integers and the set of complex numbers, respectively. We also let

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \cdots\}$$

Several q-series identities, which emerge naturally from Jacobi's triple-product identity, are worthy of note here (see, for details, [5, pp. 36–37, Entry 22]):

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2\sum_{n=1}^{\infty} q^{n^2}$$
$$= \left\{(-q;q^2)_\infty\right\}^2 (q^2;q^2)_\infty = \frac{(-q;q^2)_\infty\,(q^2;q^2)_\infty}{(q;q^2)_\infty\,(-q^2;q^2)_\infty}; \tag{1}$$

$$\psi(q) := \mathfrak{f}(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}; \tag{2}$$

$$f(-q) := \mathfrak{f}(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n\, q^{\frac{n(3n-1)}{2}}$$
$$= \sum_{n=0}^{\infty}(-1)^n\, q^{\frac{n(3n-1)}{2}} + \sum_{n=1}^{\infty}(-1)^n\, q^{\frac{n(3n+1)}{2}} = (q;q)_\infty. \tag{3}$$

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Equation (3) is known as Euler's *Pentagonal Number Theorem*. Remarkably, the following q-series identity:

$$(-q;q)_\infty = \frac{1}{(q;q^2)_\infty} = \frac{1}{\chi(-q)} \tag{4}$$

provides the analytic equivalent form of Euler's famous theorem (see, for details, [2] and [4]).

By introducing the general family $R(s,t,l,u,v,w)$, Andrews *et al.* [3] investigated a number of interesting double-summation hypergeometric q-series representations for several families of partitions and further explored the rôle of double series in combinatorial-partition identities:

$$R(s,t,l,u,v,w) := \sum_{n=0}^{\infty} q^{s\binom{n}{2}+tn}\, r(l,u,v,w;n), \tag{5}$$

where

$$r(l,u,v,w:n) := \sum_{j=0}^{[\frac{n}{u}]} (-1)^j \, \frac{q^{uv\binom{j}{2}+(w-ul)j}}{(q;q)_{n-uj}\,(q^{uv};q^{uv})_j}. \tag{6}$$

We also recall the following interesting special case of (5) (see, for details, [3, p. 106, Theorem 3]; see also [15]):

$$R(m,m,1,1,1,2) = \frac{(q^{2m};q^{2m})_\infty}{(q^m;q^{2m})_\infty}. \tag{7}$$

Recently, Srivastva *et al* (see [18]) has introduced three notations:

$$R_\alpha = R(2,1,1,1,2,2); R_\beta = R(2,2,1,1,2,2); R_m = R(m,m,1,1,1,2)\ (m \in \mathbb{N}). \tag{8}$$

for multivaraite R-functions, which we shall use for computation of our main results in section 2.

Ever since the year 2015, several new advancements and generalizations of the existing results were made in regard to combinatorial partition-theoretic identities (see, for example, [15] to [17]).

Here, in this paper, our main objective is to establish set of five new relationships involving R_m-functions which are based upon q-product identities and Jacobi's celebrated triple-product identity.

Each of the following preliminary results will be needed for the demonstration of our main results in this paper (see [1]):

Ramanujan continued fraction is defined as (see [1, p. 489])

$$c(q) = \frac{1}{1+}\ \frac{2q}{1-q^2+}\ \frac{q^2(1+q^2)^2}{1-q^6+}\ \frac{q^4(1+q^4)^2}{1-q^{10}+\cdots} \quad (|q|<1). \tag{9}$$

A product representation for $c(q)$ is given as (see [1, p. 490, Theorem 1.2])

$$c(q) = \frac{\phi(q^4)}{\phi(q)}. \tag{10}$$

Let $u = c(q)$, $v = c(-q)$ and $w = c(q^2)$. Then we have (see [1, p. 492, Theorem 3.2])

$$u + v = 2uv, \tag{11}$$

and

$$(u-v)^2(2w-1)^2 + 8uvw(w-1) = 0. \tag{12}$$

Also, let $u = c(q)$ and $v = c(q^3)$. Then we recall (see [1, p. 494, Corollary 4.2])

$$(u^2 + v^2)^2 - 12uv(u + v)^2 - 4uv[(1 + 2uv)(2uv - 3(u + v)) + 1] = 0. \tag{13}$$

2. A Set of Main Results

In this section, we introduce a set of five new relationships involving R_m-functions, which are based upon q-product identities and Jacobi's celebrated triple-product identity.

Theorem 1. *Each of the following relationships holds true:*

$$\psi^2(-q) = \frac{2.\, R_1 \{R_4\}^2}{R_8} - \frac{\{R_1\}^3}{R_2}. \tag{14}$$

Equation (14) gives inter-relationships between R_1, R_2, R_4 and R_8.

$$\frac{\{R_4\}^2}{R_8}\left(\frac{R_2}{\{R_1\}^2} + \frac{R_1}{\psi^2(-q)} - \frac{2.R_2\{R_4\}^2}{R_1 R_8 \psi^2(-q)}\right) = 0. \tag{15}$$

Equation (15) gives inter-relationships between R_1, R_2, R_4 and R_8.

$$\frac{4.R_8}{R_{16}}\left(\frac{\{R_2\}^2}{\{R_1\}^4} + \frac{\{R_1\}^2}{\psi^4(-q)}\right) \cdot \left(\frac{\{R_8\}^3}{R_{16}} - \{R_4\}^2\right) + \frac{\{R_4\}^4}{\{R_8\}^2}\left(\frac{R_2}{\{R_1\}^2} - \frac{R_1}{\psi^2(-q)}{}^2\right) = 0. \tag{16}$$

Equation (16) gives inter-relationships between R_1, R_2, R_4, R_8 and R_{16}.

$$\begin{aligned}
&\frac{\{R_4\}^8\{R_2\}^4}{\{R_1\}^8\{R_8\}^4} + \frac{\{R_{12}\}^8\{R_6\}^4}{\{R_3\}^8\{R_{24}\}^4} - \frac{30.\{R_4\}^4\{R_2\}^2}{\{R_1\}^4\{R_8\}^2} \cdot \frac{\{R_{12}\}^4\{R_6\}^2}{\{R_3\}^4\{R_{24}\}^2} \\
&-\frac{12.\{R_4\}^2 R_2}{\{R_1\}^2 R_8} \cdot \frac{\{R_{12}\}^2 R_6}{\{R_3\}^2 R_{24}}\left(\frac{\{R_4\}^4\{R_2\}^2}{\{R_1\}^4\{R_8\}^2} + \frac{\{R_{12}\}^4\{R_6\}^2}{\{R_3\}^4\{R_{24}\}^2} - \frac{\{R_4\}^2 R_2}{\{R_1\}^2 R_8}\right. \\
&\left.-\frac{\{R_{12}\}^2 R_6}{\{R_3\}^2 R_{24}} + \frac{2.\{R_4\}^4\{R_2\}^2}{\{R_1\}^4\{R_8\}^2} \cdot \frac{\{R_{12}\}^2 R_6}{\{R_3\}^2 R_{24}} + \frac{2.\{R_4\}^2 R_2}{\{R_1\}^2 R_8} . \frac{\{R_{12}\}^4\{R_6\}^2}{\{R_3\}^4\{R_{24}\}^2}\right) \\
&-\frac{4.\{R_4\}^2 R_2}{\{R_1\}^2 R_8} \cdot \frac{\{R_{12}\}^2 R_6}{\{R_3\}^2 R_{24}}\left(\frac{4.\{R_4\}^4\{R_2\}^2}{\{R_1\}^4\{R_8\}^2} \cdot \frac{\{R_{12}\}^4\{R_6\}^2}{\{R_3\}^4\{R_{24}\}^2} + 1\right) = 0.
\end{aligned} \tag{17}$$

Equation (17) gives inter-relationships between $R_1, R_2, R_3, R_4, R_6, R_8, R_{12}$ and R_{24}.

$$\frac{R_2}{\{R_1\}^2}\psi^2(-q) = -R_1 + \frac{2.R_2\{R_4\}^2}{R_1 R_8}. \tag{18}$$

Equation (18) gives inter-relationships between R_1, R_2, R_4 and R_8.

It is assumed that each member of the assertions (14) to (18) exists.

Proof: First of all, in order to prove the assertion (14) of Theorem 1, let $u = c(q),\ v = c(-q)$ and $w = c(q^2)$; using (10), (4), (1) and (8), we have:

$$u = c(q) = \frac{\{R_4\}^2 R_2}{\{R_1\}^2 R_8}, \tag{19}$$

$$v = c(-q) = \frac{\{R_4\}^2 R_1}{R_8 \psi^2(-q)}, \tag{20}$$

and

$$w = c(q^2) = \frac{\{R_8\}^3}{\{R_4\}^2 R_{16}}. \tag{21}$$

Now combining values of u and v from (19) and (20) into (11), we are led to the first assertion (14) of Theorem 1.
Secondly, we prove the second relationship (15) of Theorem 1; by using values of u, v and w from (19), (20) and (21) into (12), we are led to the second assertion (15) of Theorem 1.
Thirdly, we prove the third relationship (16) of Theorem 1. Let $v = c(q^3)$; and using identities (10), (4), (1) and (7), we have;

$$v = c(q^3) = \frac{\{R_{12}\}^2 R_6}{\{R_3\}^2 R_{24}}. \tag{22}$$

Using values of u and v from (19) and (22) into (13), we are led to the third assertion (16) of Theorem 1.
We can prove identities (17) and (18) easily, by using similar argument.
We thus have completed our proof of the above Theorem 1.

3. Concluding Remarks and Observations

The present investigation was motivated by several recent developments dealing essentially with theta-function identities and combinatorial partition-theoretic identities. Here, in this article, we have established a family of five presumably new theta-function identities which depict the inter-relationships that exist among derivatives of R_m-functions. We have also considered several closely-related identities such as (for example) q-product identities and Jacobi's triple-product identities. And, with a view to further motivating researches involving theta-function identities and combinatorial partition-theoretic identities,we have chosen to indicate rather briefly a number of recent developments on the subject-matter of this article.
The list of citations, which we have included in this article, is believed to be potentially useful for indicating some of the directions for further researches and related developments on the subject-matter which we have dealt with here. In particular, the recent works by Cao *et al.* [6], Chaudhary *et al.* (see [7] – [10]) and Srivastava *et al.* (see [15]–[18]) are worth mentioning here.

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