Journal of Fractional Calculus and Applications
Vol. 13(1) Jan. 2022, pp. 130-140.
ISSN: 2090-5858.
http://math-frac.org/Journals/JFCA/

ON CERTAIN SUBCLASSES OF PASCU TYPE ALPHA CLOSE-TO-STAR FUNCTIONS

GAGANDEEP SINGH, GURCHARANJIT SINGH

Abstract. In this paper, certain generalized subclasses of analytic functions are introduced by unifying the close-to-star and close-to-convex functions in the open unit disc $E = \{z : |z| < 1\}$. We establish the coefficient estimates, distortion theorems, growth theorems, argument theorems and radius of starlikeness for the functions belonging to these classes. Various known results are shown to follow upon specializing the parameters involved in the results of this paper.

1. Introduction

Let U denote the class of Schwarzian functions of the form $w(z) = \sum_{k=1}^{\infty} c_k z^k$, that are analytic in the open unit disc $E = \{z : |z| < 1\}$ and with the conditions $w(0) = 0, |w(z)| < 1$. By A, we denote the class of functions f which are analytic in E, normalized by $f(0) = f'(0) - 1 = 0$ and having the Taylor series expansion of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1}$$

The well known classes of univalent, starlike and convex functions are denoted by S, S^* and K respectively. For two analytic functions f and g in E, we say that f is subordinate to g, if there exists a Schwarzian function $w(z) \in U$ such that $f(z) = g(w(z))$, denoted by $f \prec g$. If g is univalent in E, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(E) \subset g(E)$. The concept of subordination was introduced by Littlewood [9] and Reade [19].

The class consisting of the functions $p(z)$ analytic in E with $p(0) = 1$ and subordinate to $\dfrac{1 + Cz}{1 + Dz}, (-1 \leq D < C \leq 1)$, is denoted by $P[C, D]$. This class was established by Janowski [6] and so the functions in this class are known as Janowski-type functions.

2010 *Mathematics Subject Classification.* 30C45, 30C50.

Key words and phrases. Univalent functions, Analytic functions, Starlike functions, Convex functions, close-to-convex functions, close-to-star functions, Subordination.

Submitted April 2, 2021. Revised June 30, 2021.

A function $f \in A$ is said to be close-to-convex function if there exists a starlike function g such that $Re\left(\frac{zf'(z)}{g(z)}\right) > 0, z \in E$. The class of close-to-convex functions is denoted by C and was introduced by Kaplan [7]. Subsequently, Reade [19] introduced the class CS^* of close-to-star functions consisting of the functions $f \in A$ and satisfying the condition that $Re\left(\frac{f(z)}{g(z)}\right) > 0, g \in S^*, z \in E$.

Subclasses of close-to-convex and close-to-star functions were extensively studied by various authors. We choose to recall here the following classes:
(i)$CS^*(A, B; C, D)$, the subclass of close-to-star functions studied by Mehrok and Singh [13].
(ii) $CS^*(C, D)$, a subclass of close-to-star functions introduced and studied by Mehrok et al. [14].
(iii) $CS_1^*(C, D)$, the subclass of close-to-star functions introduced and studied by Mehrok et al. [15].
(iv) $C(A, B; C, D)$ and $C_1(A, B; C, D)$, the subclasses of close-to-convex functions studied by Singh and Mehrok [21].
(v) $C(C, D)$, the subclass of close-to-convex functions introduced and studied by Mehrok [11].
(vi) $C_1(C, D)$, a subclass of close-to-convex functions introduced and studied by Mehrok and Singh [12].
(vii) C_1, the subclass of close-to-convex functions studied by Abdel Gawad and Thomas [1].

Mocanu [16], established the class $M_\alpha(0 \leq \alpha \leq 1)$ of alpha-convex functions $f \in A$ with $f(z)f'(z) \neq 0$ and satisfying the condition

$$Re\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)}\right\} > 0, z \in E.$$

Obviously $M_0 \equiv S^*$ and $M_1 \equiv K$. The class M_α unify the classes S^* and K and it was shown in [10], that all alpha-convex functions are univalent

By considering the concept of alpha-convex functions, Paravatham and Srinivasan [18] introduced the class $CS(\alpha)(0 \leq \alpha \leq 1)$ consisting of functions $f \in A$ with the condition that

$$Re\left[(1-\alpha)\frac{f(z)}{g(z)} + \alpha\frac{zf'(z)}{g(z)}\right] > 0, g \in S^*, z \in E.$$

It is obvious that $CS(0) \equiv CS^*$ and $CS(1) \equiv C$. Clearly $CS(\alpha)$ is a linear combination of the classes CS^* and C.

Further, Singh [20] introduced the class $CS(\alpha; C, D)$ which was studied further recently by Altintas and Kilic [2]. $CS(\alpha; C, D)(0 \leq \alpha \leq 1, -1 \leq D < C \leq 1)$ consisting of the functions $f \in A$ and satisfying the condition

$$(1-\alpha)\frac{f(z)}{g(z)} + \alpha\frac{zf'(z)}{g(z)} \prec \frac{1+Cz}{1+Dz}, g \in S^*, z \in E.$$

In particular, $CS(\alpha;1,-1) \equiv CS(\alpha)$, $CS(0;C,D) \equiv CS^*(C,D)$, $CS(0;1,-1) \equiv CS^*$, $CS(1;C,D) \equiv C(C,D)$ and $CS(1;1,-1) \equiv C$.

Throughout this investigation, we assume that

$$-1 \leq D \leq B < A \leq C \leq 1, 0 \leq \alpha \leq 1, z \in E.$$

Getting motivated by the above mentioned work, now we are able to define the classes which are to study in this paper;

Definition 1 $CS^*(\alpha; A, B; C, D)$ be the class of functions $f \in A$ which satisfy the condition

$$(1-\alpha)\frac{f(z)}{g(z)} + \alpha\frac{zf'(z)}{g(z)} \prec \frac{1+Cz}{1+Dz},$$

where $g(z) = z + \sum_{k=2}^{\infty} d_k z^k \in S^*(A, B)$.
The following observations are obvious:
(i) $CS^*(\alpha; 1, -1; C, D) \equiv CS(\alpha; C, D)$.
(ii) $CS^*(\alpha; 1, -1; 1, -1) \equiv CS(\alpha)$.
(iii)$CS^*(0; A, B; C, D) \equiv CS^*(A, B; C, D)$.
(iv) $CS^*(0; 1, -1; C, D) \equiv CS^*(C, D)$.
(v) $CS^*(0; 1, -1; 1, -1) \equiv CS^*$.
(vi) $CS^*(1; A, B; C, D) \equiv C(A, B; C, D)$.
(vii) $CS^*(1; 1, -1; C, D) \equiv C(C, D)$.
(viii) $CS^*(1; 1, -1; 1, -1) \equiv C$.

Definition 2 Let $CS_1^*(\alpha; A, B; C, D)$ denote the class of functions $f \in A$ and satisfying the condition that

$$(1-\alpha)\frac{f(z)}{h(z)} + \alpha\frac{zf'(z)}{h(z)} \prec \frac{1+Cz}{1+Dz},$$

where $h(z) = z + \sum_{k=2}^{\infty} b_k z^k \in K(A, B)$.
We have the following observations:
(i) $CS_1^*(0; 1, -1; C, D) \equiv CS_1^*(C, D)$.
(ii) $CS_1^*(1; A, B; C, D) \equiv C_1(A, B; C, D)$.
(iii) $CS_1^*(1; 1, -1; C, D) \equiv C_1(C, D)$.
(iv) $CS_1^*(1; 1, -1; 1, -1) \equiv C_1$.

The present investigation deals with the study of the classes $CS^*(\alpha; A, B; C, D)$ and $CS_1^*(\alpha; A, B; C, D)$. We establish the coefficient estimates, distortion theorems, growth theorems, argument theorems and radius of starlikeness for the functions in these classes. For particular values of the parameters α, A, B, C and D, the results of some earlier works follows as special cases.

2. Preliminary Results

Lemma 1 [4] If $P(z) = \dfrac{1+Cw(z)}{1+Dw(z)} = 1 + \sum_{k=1}^{\infty} p_k z^k$, then

$$|p_n| \leq (C-D), n \geq 1.$$

The bound is sharp for the function $P_n(z) = \dfrac{1 + C\delta z^n}{1 + D\delta z^n}, |\delta| = 1.$

Lemma 2 [3] If $g(z) \in S^*(A, B)$, then for $A - (n-1)B \geq (n-2), n \geq 3,$

$$|d_n| \leq \frac{1}{(n-1)!} \prod_{j=2}^{n} (A - (j-1)B).$$

Equality sign is attained for $g_0(z) = z(1 + B\delta z)^{\frac{A-B}{B}}, |\delta| = 1.$

Lemma 3 [3] If $g(z) \in S^*(A, B)$, then for $|z| = r < 1,$

$$r(1 - Br)^{\frac{A-B}{B}} \leq |g(z)| \leq r(1 + Br)^{\frac{A-B}{B}}, B \neq 0;$$

$$re^{-Ar} \leq |g(z)| \leq re^{Ar}, B = 0.$$

Equality holds for the function defined as

$$g_1(z) = \begin{cases} z(1 + B\delta z)^{\frac{A-B}{B}}, & \text{if } B \neq 0, \\ ze^{A\delta z}, & \text{if } B = 0, |\delta| = 1. \end{cases}$$

Lemma 4 [3] If $g(z) \in S^*(A, B)$, then for $|z| = r < 1,$

$$\left| arg \frac{g(z)}{z} \right| \leq \frac{(A - B)}{B} sin^{-1}(Br), B \neq 0;$$

$$\left| arg \frac{g(z)}{z} \right| \leq Ar, B = 0.$$

Lemma 5 [21] If $h(z) \in K(A, B)$, then for $A - (n-1)B \geq (n-2), n \geq 3,$

$$|b_n| \leq \frac{1}{n!} \prod_{j=2}^{n} (A - (j-1)B).$$

Result is sharp for the function $h_0(z) = \frac{1}{A}[(1 + B\delta z)^{\frac{A}{B}} - 1], |\delta| = 1.$

Lemma 6 [21] If $h(z) \in K(A, B)$, then for $|z| = r < 1,$

$$\frac{1}{A} \left[1 - (1 - Br)^{\frac{A}{B}} \right] \leq |h(z)| \leq \frac{1}{A} \left[(1 + Br)^{\frac{A}{B}} - 1 \right], B \neq 0;$$

$$\frac{1}{A} \left[1 - e^{-Ar} \right] \leq |h(z)| \leq \frac{1}{A} \left[e^{Ar} - 1 \right], B = 0.$$

Equality holds for the function given by

$$h_1(z) = \begin{cases} \frac{1}{A}[(1 + B\delta z)^{\frac{A}{B}} - 1], & \text{if } B \neq 0, \\ \frac{1}{A}[e^{A\delta z} - 1], & \text{if } B = 0, |\delta| = 1. \end{cases}$$

Lemma 7 [21] If $h(z) \in K(A, B)$, then for $|z| = r < 1,$

$$\left| arg \frac{h(z)}{z} \right| \leq \frac{A}{B} sin^{-1}(Br), B \neq 0;$$

$$\left| arg \frac{h(z)}{z} \right| \leq Ar, B = 0.$$

3. The class $CS^*(\alpha; A, B; C, D)$

Theorem 1 If $f(z) \in CS^*(\alpha; A, B; C, D)$, then for $A-(n-1)B \geq (n-2), n \geq 2$,

$$|a_n| \leq \frac{1}{[1+(n-1)\alpha]}\left\{\frac{1}{(n-1)!}\prod_{j=2}^{n}(A-(j-1)B)\right.$$

$$\left. + (C-D)\left[1+\sum_{k=2}^{n-1}\frac{1}{(k-1)!}\prod_{j=2}^{k}(A-(j-1)B)\right]\right\}. \quad (2)$$

The bound is sharp.

Proof. By using Principle of subordination in Definition 1, we have

$$(1-\alpha)f(z)+\alpha z f'(z) = g(z)\left(\frac{1+Cw(z)}{1+Dw(z)}\right), w(z) \in U. \quad (3)$$

After expanding (3), it yields

$$(1-\alpha)[z+a_2z^2+a_3z^3+...+a_nz^n+...]+\alpha[z+2a_2z^2+3a_3z^3+...+na_nz^n+...]$$

$$= (z+d_2z^2+d_3z^3+...+d_nz^n+...)(1+p_1z+p_2z^2+...+p_nz^n+...). \quad (4)$$

On equating the coefficients of z^n on both sides of (4), we obtain

$$[1+(n-1)\alpha]a_n = d_n + p_1d_{n-1} + p_2d_{n-2}... + p_{n-2}d_2 + p_{n-1}. \quad (5)$$

Applying triangle inequality and using Lemma 1 in (5), it gives

$$[1+(n-1)\alpha]|a_n| \leq |d_n| + (C-D)\left[|d_{n-1}|+|d_{n-2}|...+|d_2|+1\right]. \quad (6)$$

On Using Lemma 2 in (6), the result (2) is obvious.

For $n=2$, equality sign in (2) holds for the function $f_n(z)$ defined as

$$(1-\alpha)f_n(z)+\alpha z f_n'(z)$$

$$= z(1+B\delta_1 z)^{\frac{(A-B)}{B}}\left(\frac{1+C\delta_2 z^n}{1+D\delta_2 z^n}\right), B \neq 0, |\delta_1|=1, |\delta_2|=1. \quad (7)$$

Remark 1

(i) For $A=1, B=-1$, Theorem 1 agrees with the result due to Altintas and Kilic [2].

(ii) For $A=1, B=-1, C=1, D=-1$, Theorem 1 agrees with the result due to Paravatham and Srinivasan [18].

(iii) On putting $\alpha=0$, Theorem 1 gives the result due to Mehrok and Singh [13].

(iv) For $\alpha=0, A=1, B=-1$, the result proved by Mehrok at al. [14] is obvious from Theorem 1.

(v) Taking $\alpha=0, A=1, B=-1, C=1, D=-1$, Theorem 1 agrees with the result given by Reade [19].

(vi) On putting $\alpha=1$, the result established by Singh and Mehrok [21] follows from Theorem 1.

(vii) By considering $\alpha=1, A=1, B=-1$, Theorem 1 gives the result proved by Mehrok [11].

(viii) Putting $\alpha=1, A=1, B=-1, C=1, D=-1$, the result due to Reade [19] follows from Theorem 1.

Theorem 2 Let $f(z) \in CS^*(\alpha; A, B; C, D)$, then for $|z| = r, 0 < r < 1$, we have for $\alpha = 0, B \neq 0$,

$$r(1-Br)^{\frac{A-B}{B}}\left(\frac{1-Cr}{1-Dr}\right) \leq |f(z)| \leq r(1+Br)^{\frac{A-B}{B}}\left(\frac{1+Cr}{1+Dr}\right); \tag{8}$$

for $\alpha = 0, B = 0$,

$$re^{-Ar}\left(\frac{1-Cr}{1-Dr}\right) \leq |f(z)| \leq re^{Ar}\left(\frac{1+Cr}{1+Dr}\right), \tag{9}$$

and for $0 < \alpha \leq 1, B \neq 0$,

$$\frac{1}{\alpha}\int_0^r \left[(1-Bt)^{\frac{A-B}{B}}\left(\frac{1-Ct}{1-Dt}\right)\right] dt \leq |f(z)| \leq \frac{1}{\alpha}\int_0^r \left[(1+Bt)^{\frac{A-B}{B}}\left(\frac{1+Ct}{1+Dt}\right)\right] dt; \tag{10}$$

for $0 < \alpha \leq 1, B = 0$,

$$\frac{1}{\alpha}\int_0^r \left[e^{-At}\left(\frac{1-Ct}{1-Dt}\right)\right] dt \leq |f(z)| \leq \frac{1}{\alpha}\int_0^r \left[e^{At}\left(\frac{1+Ct}{1+Dt}\right)\right] dt. \tag{11}$$

Estimates are sharp.

Proof. From (3), we have

$$|(1-\alpha)f(z) + \alpha z f'(z)| = |g(z)| \left|\frac{1+Cw(z)}{1+Dw(z)}\right|. \tag{12}$$

It is easy to show that the transformation

$$\frac{(1-\alpha)f(z) + \alpha z f'(z)}{g(z)} = \frac{1+Cw(z)}{1+Dw(z)}$$

maps $|w(z)| \leq r$ onto the circle

$$\left|\frac{(1-\alpha)f(z) + \alpha z f'(z)}{g(z)} - \frac{1-CDr^2}{1-D^2r^2}\right| \leq \frac{(C-D)r}{(1-D^2r^2)}, |z| = r.$$

This implies that

$$\frac{1-Cr}{1-Dr} \leq \left|\frac{1+Cw(z)}{1+Dw(z)}\right| \leq \frac{1+Cr}{1+Dr}. \tag{13}$$

Let $F(z) = (1-\alpha)f(z) + \alpha z f'(z)$.

As $g(z) \in S^*(A, B)$, so using (13) and Lemma 3 in (12), it yields

$$\begin{cases} r(1-Br)^{\frac{A-B}{B}}\left(\frac{1-Cr}{1-Dr}\right) \leq |F(z)| \leq r(1+Br)^{\frac{A-B}{B}}\left(\frac{1+Cr}{1+Dr}\right), & \text{if } B \neq 0; \\ re^{-Ar}\left(\frac{1-Cr}{1-Dr}\right) \leq |F(z)| \leq re^{Ar}\left(\frac{1+Cr}{1+Dr}\right), & \text{if } B = 0. \end{cases} \tag{14}$$

Therefore, we have

$$\begin{cases} r(1-Br)^{\frac{A-B}{B}}\left(\frac{1-Cr}{1-Dr}\right) & \\ \leq |(1-\alpha)f(z) + \alpha z f'(z)| \leq r(1+Br)^{\frac{A-B}{B}}\left(\frac{1+Cr}{1+Dr}\right), & \text{if } B \neq 0; \\ re^{-Ar}\left(\frac{1-Cr}{1-Dr}\right) \leq |(1-\alpha)f(z) + \alpha z f'(z)| \leq re^{Ar}\left(\frac{1+Cr}{1+Dr}\right), & \text{if } B = 0. \end{cases} \tag{15}$$

For $\alpha = 0$, the resluts (8) and (9) are obvious from (15).
Also for $0 < \alpha \leq 1$, the results (10) and (11) can be easily obtained on integrating (15) from 0 to r.
Sharpness follows for the function defined as

$$F(z) = \begin{cases} z(1+B\delta_2 z)^{\frac{A-B}{B}}\left(\dfrac{1+C\delta_1 z}{1+D\delta_1 z}\right) & \text{if } B \neq 0, \\ ze^{A\delta_2 z}\left(\dfrac{1+C\delta_1 z}{1+D\delta_1 z}\right) & \text{if } B = 0, |\delta_1| = 1, |\delta_2| = 1. \end{cases}$$

Remark 2
(i) For $\alpha = 0, A = 1, B = -1$, the result proved by Mehrok et al. [14] is obvious from Theorem 2.
(ii) Taking $\alpha = 0, A = 1, B = -1, C = 1, D = -1$, Theorem 2 agrees with the result given by Goel and Sohi [5].
(i) For $\alpha = 1$, the result due to Singh and Mehrok [21] is obvious from Theorem 2.
(iv) By considering $\alpha = 1, A = 1, B = -1$, Theorem 2 gives the result proved by Mehrok [11].

Theorem 3 If $f(z) \in CS^*(\alpha; A, B; C, D)$ and let $F(z) = (1-\alpha)f(z) + \alpha z f'(z)$, then

$$\left|arg\frac{F(z)}{z}\right| \leq sin^{-1}\left(\frac{(C-D)r}{1-CDr^2}\right) + \frac{(A-B)}{B}sin^{-1}(Br), B \neq 0; \tag{16}$$

$$\left|arg\frac{F(z)}{z}\right| \leq sin^{-1}\left(\frac{(C-D)r}{1-CDr^2}\right) + Ar, B = 0. \tag{17}$$

Proof. (3) can be expressed as

$$F(z) = g(z)\left(\frac{1+Cw(z)}{1+Dw(z)}\right).$$

Therefore, we have

$$\left|arg\frac{F(z)}{z}\right| \leq \left|arg\left(\frac{1+Cw(z)}{1+Dw(z)}\right)\right| + \left|arg\frac{g(z)}{z}\right|. \tag{18}$$

As in Theorem 2, it is clear that

$$\left|\frac{F(z)}{g(z)} - \frac{1-CDr^2}{1-D^2r^2}\right| \leq \frac{(C-D)r}{(1-D^2r^2)}.$$

So, it yields

$$\left|arg\left(\frac{1+Cw(z)}{1+Dw(z)}\right)\right| \leq sin^{-1}\left(\frac{(C-D)r}{1-CDr^2}\right). \tag{19}$$

On using Lemma 4 and inequality (19) in (18), the results (16) and (17) are obvious.
Results are sharp for the function defined in Theorem 3 for which

$$\delta_1 = \frac{r}{z}\left[\frac{-(C+D)r + i((1-C^2r^2)(1-D^2r^2))^{\frac{1}{2}}}{(1+CDr^2)}\right], \delta_2 = \frac{r}{z}\left[-Dr + i(1-D^2r^2)^{\frac{1}{2}}\right].$$

Remark 3
(i) For $\alpha = 0, A = 1, B = -1$, the result proved by Mehrok et. al. [14] is obvious from Theorem 3.
(ii) Taking $\alpha = 0, A = 1, B = -1, C = 1, D = -1$, Theorem 3 agrees with the result given by Goel and Sohi [5].

(iii) On putting $\alpha = 1$, Theorem 3 gives the result due to Singh and Mehrok [21].
(iv) By considering $\alpha = 1, A = 1, B = -1$, Theorem 3 gives the result proved by Mehrok [11].
(v) By considering $\alpha = 1, A = 1, B = -1, C = 1, D = -1$, Theorem 3 gives the result proved by Ogawa [17] and Krzyz [8].

Theorem 4 Let $F(z) = (1-\alpha)f(z)+\alpha zf'(z)$, where $f(z) \in CS^*(\alpha; A, B; C, D)$, then $F(z)$ is starlike in $|z| < r_0$ where r_0 is the smallest positive root of

$$1 + [2D - A]r + [CD - AC - AD + BC - BD]r^2 - ACDr^3 = 0 \tag{20}$$

in the interval $(0, 1)$.
Proof. As $f(z) \in CS^*(\alpha; A, B; C, D)$, we have

$$(1 - \alpha)f(z) + \alpha zf'(z) = g(z)\left(\frac{1 + Cw(z)}{1 + Dw(z)}\right) = g(z)P(z).$$

Here $F(z) = (1 - \alpha)f(z) + \alpha zf'(z)$. So, we have

$$F(z) = g(z)P(z).$$

On differentiating it logarithmically, we get

$$\frac{zF'(z)}{F(z)} = \frac{zg'(z)}{g(z)} + \frac{zP'(z)}{P(z)}. \tag{21}$$

Now for $g \in S^*(A, B)$, we have

$$Re\left(\frac{zg'(z)}{g(z)}\right) \geq \frac{1 - Ar}{1 - Br}. \tag{22}$$

Also from (13), we have

$$\left|\frac{1 + Cw(z)}{1 + Dw(z)}\right| = |P(z)| \leq \frac{1 + Cr}{1 + Dr},$$

which implies

$$\left|\frac{zP'(z)}{P(z)}\right| \leq \frac{r(C - D)}{(1 + Cr)(1 + Dr)}.$$

So it gives,

$$Re\left(\frac{zF'(z)}{F(z)}\right) \geq Re\left(\frac{zg'(z)}{g(z)}\right) - \left|\frac{zP'(z)}{P(z)}\right|.$$

Therefore, we have

$$Re\left(\frac{zF'(z)}{F(z)}\right) \geq \frac{1 - Ar}{1 - Br} - \frac{r(C - D)}{(1 + Cr)(1 + Dr)}.$$

On simplification, the above inequality can be expressed as

$$Re\left(\frac{zF'(z)}{F(z)}\right) \geq \frac{1 + [2D - A]r + [CD - AC - AD + BC - BD]r^2 - ACDr^3}{(1 - Br)(1 + Cr)(1 + Dr)}.$$

Hence $F(z)$ is starlike in $|z| < r_0$ where r_0 is the smallest positive root of

$$1 + [2D - A]r + [CD - AC - AD + BC - BD]r^2 - ACDr^3 = 0$$

in the interval $(0, 1)$. Sharpness follows for the function $f_n(z)$ defined in (7).

4. The class $CS_1^*(\alpha; A, B; C, D)$

Theorem 5 If $f(z) \in CS_1^*(\alpha; A, B; C, D)$, then for $A-(n-1)B \geq (n-2), n \geq 2$,

$$|a_n| \leq \frac{1}{[1+(n-1)\alpha]}\left\{\frac{1}{n!}\prod_{j=2}^{n}(A-(j-1)B)\right.$$

$$\left.+(C-D)\left[1+\sum_{k=2}^{n-1}\frac{1}{k!}\prod_{j=2}^{k}(A-(j-1)B)\right]\right\}. \quad (23)$$

The bounds are sharp.

Proof. From Definition 2, by Principle of subordination, we have

$$(1-\alpha)f(z)+\alpha z f'(z) = h(z)\left(\frac{1+Cw(z)}{1+Dw(z)}\right), w(z) \in U. \quad (24)$$

Then following the procedure of Theorem 1, using Lemma 1 and Lemma 5, the result (23) can be easily obtained from (24).

For $n = 2$, equality sign in (23) hold for the functions $f_n(z)$ defined as

$$(1-\alpha)f_n(z)+\alpha z f_n'(z) = \frac{1}{A}[(1+B\delta_1 z)^{\frac{A}{B}}-1]\left(\frac{1+C\delta_2 z^n}{1+D\delta_2 z^n}\right), B \neq 0, |\delta_1| = 1, |\delta_2| = 1. \quad (25)$$

Remark 4

(i) For $\alpha = 0, A = 1, B = -1$, the result proved by Mehrok et al. [15] is obvious from Theorem 5.

(ii) For $\alpha = 1$, Theorem 5 agrees with the result proved by Singh and Mehrok [21].

(iii) By considering $\alpha = 1, A = 1, B = -1$, Theorem 5 gives the result proved by Mehrok and Singh [12].

(iv) Putting $\alpha = 1, A = 1, B = -1, C = 1, D = -1$, the result due to Abdel-Gawad and Thomas [1] follows from Theorem 5.

Theorem 6 If $f(z) \in CS_1^*(\alpha; A, B; C, D)$, then for $|z| = r, 0 < r < 1$, we have for $\alpha = 0, B \neq 0$,

$$\frac{1}{A}\left[1-(1-Br)^{\frac{A}{B}}\right]\left(\frac{1-Cr}{1-Dr}\right) \leq |f(z)| \leq \frac{1}{A}\left[(1+Br)^{\frac{A}{B}}-1\right]\left(\frac{1+Cr}{1+Dr}\right); \quad (26)$$

for $\alpha = 0, B = 0$,

$$\frac{1}{A}\left[1-e^{-Ar}\right]\left(\frac{1-Cr}{1-Dr}\right) \leq |f(z)| \leq \frac{1}{A}\left[e^{Ar}-1\right]\left(\frac{1+Cr}{1+Dr}\right), \quad (27)$$

and for $0 < \alpha \leq 1, B \neq 0$,

$$\frac{1}{\alpha}\int_0^r \frac{1}{At}\left[1-(1-Bt)^{\frac{A}{B}}\right]\left(\frac{1-Ct}{1-Dt}\right)dt \leq |f(z)| \leq \frac{1}{\alpha}\int_0^r \frac{1}{At}\left[(1+Bt)^{\frac{A}{B}}-1\right]\left(\frac{1+Ct}{1+Dt}\right)dt; \quad (28)$$

for $0 < \alpha \leq 1, B = 0$,

$$\frac{1}{\alpha}\int_0^r \frac{1}{At}\left[1-e^{-At}\right]\left(\frac{1-Ct}{1-Dt}\right)dt \leq |f(z)| \leq \frac{1}{\alpha}\int_0^r \frac{1}{At}\left[e^{At}-1\right]\left(\frac{1+Ct}{1+Dt}\right)dt. \quad (29)$$

Estimates are sharp.
Proof. From (24), we have

$$|(1-\alpha)f(z)+\alpha zf'(z)| = |h(z)|\left|\frac{1+Cw(z)}{1+Dw(z)}\right|, w(z)\in U. \tag{30}$$

On the lines of Theorem 2 and by using Lemma 6 in (30), the resuls (26)-(29) are obvious.
The results are sharp for the function

$$F(z) = \begin{cases} \left(\dfrac{1+C\delta_1 z}{1+D\delta_1 z}\right)\dfrac{1}{A}\left[(1+B\delta_2 z)^{\frac{A}{B}}-1\right] & \text{if } B\neq 0, \\ \left(\dfrac{1+C\delta_1 z}{1+D\delta_1 z}\right)\dfrac{1}{A}[e^{A\delta_2 z}-1] & \text{if } B=0, |\delta_1|=1, |\delta_2|=1, \end{cases}$$

Remark 5
(i) For $\alpha=0, A=1, B=-1$, the result proved by Mehrok and Singh [15] is obvious from Theorem 6.
(ii) On putting $\delta=1$, Theorem 6 gives the result due to Singh and Mehrok [21].
(iii) By considering $\alpha=1, A=1, B=-1$, Theorem 6 gives the result proved by Mehrok and Singh [12].
(iv) Putting $\alpha=1, A=1, B=-1, C=1, D=-1$, the result due to Abdel-Gawad and Thomas [1] follows from Theorem 6.

Theorem 7 If $f(z)\in CS_1^*(\alpha;A,B;C,D)$ and let $F(z)=(1-\alpha)f(z)+\alpha zf'(z)$, then

$$\left|arg\frac{F(z)}{z}\right| \leq sin^{-1}\left(\frac{(C-D)r}{1-CDr^2}\right)+\frac{A}{B}sin^{-1}(Br), B\neq 0; \tag{31}$$

$$\left|arg\frac{F(z)}{z}\right| \leq sin^{-1}\left(\frac{(C-D)r}{1-CDr^2}\right)+Ar, B=0. \tag{32}$$

Proof. Following the procedure of Theorem 3 and using Lemma 7, the results (31) and (32) can be easily obtained.

Remark 6
(i) For $\alpha=0, A=1, B=-1$, the result proved by Mehrok et al. [15] is obvious from Theorem 7.
(ii) On substituting $\alpha=1$ in Theorem 7, the result due to Singh and Mehrok [21] follows immediately.
(ii) By considering $\alpha=1, A=1, B=-1$, Theorem 7 gives the result proved by Mehrok and Singh [12].
(iii) Putting $\alpha=1, A=1, B=-1, C=1, D=-1$, the result due to Abdel-Gawad and Thomas [1] follows from Theorem 7.

Theorem 8 Let $F(z)=(1-\alpha)f(z)+\alpha zf'(z)$, where $f(z)\in CS_1^*(\alpha;A,B;C,D)$, then $F(z)$ is starlike in $|z|<r_0$ where r_0 is the smallest positive root of

$$1+2Dr+(CD+BC-BD)r^2=0 \tag{33}$$

in the interval $(0,1)$.
Proof. Following the procedure of Theorem 4 and using the inequality that

$$Re\left(\frac{zF'(z)}{F(z)}\right)\geq\frac{1}{1-Br}, h(z)\in K(A,B),$$

the result (33) can be easily obtained.
Sharpness follows for the function $f_n(z)$ defined in (25).

References

[1] H. R. Abdel-Gawad and D. K. Thomas, A subclass of close-to-convex functions, Publ. Inst. Math. (N. S.), Vol. 49, No. 63, 61-66, 1991.

[2] Osman Altintas and Ozur Ozkan Kilic, Coefficient estimates for a class containing quasi-convex functions, Turkish J. Math., Vol. 42, 2819-2825, 2018.

[3] R. M. Goel and B. S. Mehrok, On a class of close-to-convex functions, Indian. J. Pure Appl. Math., Vol. 12, No. 5, 648-658, 1981.

[4] R. M. Goel and B. S. Mehrok, A subclass of univalent functions, Houston J. Math., Vol. 8, No. 3, 343-357, 1982.

[5] R. M. Goel and N. S. Sohi, On certain classes of analytic functions, Indian. J. Pure Appl. Math., Vol. 11, No. 10, 1308-1324, 1980.

[6] W. Janowski, Some extremal problems for certain families of analytic functions, Ann. Polon. Math., Vol. 28, 297-326, 1973.

[7] W. Kaplan, Close-to-convex schlicht functions, Michigan Math. J., Vol. 1, 169-185, 1952.

[8] J. Krzyz, On the derivative of close-to-convex functions, Colloq. Math., Vol. 10, 139-142, 1963.

[9] J. E. Littlewood, On inequalities in the theory of functions, Proc. Lond. Math. Soc., Vol. 23, 481-519, 1925.

[10] S. S. Miller, P. T. Mocanu and M. O. Reade, All α-convex functions are univalent and starlike, Proc. Amer. Math. Soc., Vol. 2, 553-554, 1973.

[11] B. S. Mehrok, A subclass of close-to-convex functions, Bull. Inst. Math. Acad. Sin., Vol. 10, No. 4, 389-398.

[12] B. S. Mehrok and Gagandeep Singh, A subclass of close-to-convex functions, Int. J. Math. Anal., Vol. 4, No. 27, 1319-1327, 2010.

[13] B. S. Mehrok and Gagandeep Singh, A subclass of close-to-star functions, Int. J. Modern Math. Sci., Vol. 4, No. 3, 139-145, 2012.

[14] B. S. Mehrok, Gagandeep Singh and Deepak Gupta, A subclass of analytic functions, Glob. J. Math. Sci.: Theory and Practical, Vol. 2, No. 1, 91-97, 2010.

[15] B. S. Mehrok, Gagandeep Singh and Deepak Gupta, On a subclass of analytic functions, Antarctica J. Math. Vol. 7, No. 4, 447-453, 2010.

[16] P. T. Mocanu, Une propriete de convexite g?n?ralis?e dans la th?orie de la repr?sentation conforme, Mathematica (CLUJ) , Vol. 11, No. 34, 127-133, 1969.

[17] S. Ogawa, A note on close-to-convex functions, J. Nara Gakugei Univ., Vol. 8, 9-10, 1959.

[18] R. Paravatham and S. Srinivasan, On Pascu type α-close-to-star functions, Publ. Inst. Math. (N. S.), Vol. 49, No. 63, 71-75, 1991.

[19] M. O. Reade, On close-to-convex univalent functions, Michigan Math. J., Vol. 3, 59-62, 1955-56.

[20] Gagandeep Singh, Certain subclasses of α-close-to-star functions, Int. J. Modern Math. Sci., Vol. 7, No. 1, 80-89, 2013.

[21] Harjinder Singh and B. S. Mehrok, Subclasses of close-to-convex functions, Tamkang J. Math., Vol. 44, No. 4, 377-386, 2013.

Gagandeep Singh

Department of Mathematics, Khalsa College, Amritsar, Punjab, India.

E-mail address: `kamboj.gagandeep@yahoo.in`

Gurcharanjit Singh

G.N.D.U. College, Chungh(Tarn-Taran), Punjab, India.

E-mail address: `dhillongs82@yahoo.com`