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THE J-GENERALIZED P - K MITTAG-LEFFLER FUNCTION

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Abstract. We know that the classical Mittag-Leffler function plays an important role as solution of fractional order differential and integral equations. We introduce the j-generalized p - k Mittag-leffler function. Also, we prove some elementary properties and differentiation. Finally, we derive some particular cases.

1. Introduction

The two parameter pochhammer symbol is recently introduce by Gehlot [7], equation (2.1), as follows.

Definition 1 Let $x \in C; k, p \in R^+ - \{0\}$ and $Re(x) > 0, n \in N$, the p - k Pochhammer Symbol (i.e. Two Parameter Pochhammer Symbol), ${}_p(x)_{n,k}$ is given by

$$ {}_p(x)_{n,k} = (\frac{xp}{k})(\frac{xp}{k}+p)(\frac{xp}{k}+2p).........(\frac{xp}{k}+(n-1)p). \quad (1)$$

And the Two Parameter Gamma Function is given by Gehlot [7], some of it's result as follows.

Definition 2 For $x \in C/kZ^-; k, p \in R^+ - \{0\}$ and $Re(x) > 0, n \in N$, the p - k Gamma Function (i.e. Two Parameter Gamma Function), ${}_p\Gamma_k(x)$ as

$$ {}_p\Gamma_k(x) = \frac{1}{k}\lim_{n\to\infty}\frac{n!p^{n+1}(np)^{\frac{x}{k}}}{{}_p(x)_{n+1,k}}, \quad (2)$$

$$ {}_p\Gamma_k(x) = \frac{1}{k}\lim_{n\to\infty}\frac{n!p^{n+1}(np)^{\frac{x}{k}-1}}{{}_p(x)_{n,k}}. \quad (3)$$

The integral representation of p - k Gamma Function is given by

$$ {}_p\Gamma_k(x) = \int_0^\infty e^{-\frac{t^k}{p}}t^{x-1}dt, \quad (4)$$

$$ {}_p\Gamma_k(x) = (\frac{p}{k})^{\frac{x}{k}}\Gamma_k(x) = \frac{p^{\frac{x}{k}}}{k}\Gamma(\frac{x}{k}), \quad (5)$$

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$$_p(x)_{n,k} = (\frac{p}{k})^n (x)_{n,k} = (p)^n (\frac{x}{k})_n. \tag{6}$$

Also for Generalized p - k Pochhammer Symbol, we have

$$_p(x)_{nq,k} = (\frac{p}{k})^{nq} \, (x)_{nq,k} = (p)^{nq} \, (\frac{x}{k})_{nq} = (pq)^{nq} \, \prod_{r=1}^{q} (\frac{\frac{x}{k} + r - 1}{q})_n, \tag{7}$$

$$_p(x)_{n,k} = \frac{_p\Gamma_k(x + nk)}{_p\Gamma_k(x)}, \tag{8}$$

$$_p\Gamma_k(x + k) = \frac{xp}{k} \, _p\Gamma_k(x), \tag{9}$$

$$np \, _p(x)_{n-1,k} = \, _p(x)_{n,k} - \, _p(x - k)_{n,k}, \tag{10}$$

and

$$_p(x)_{n+j,k} = \, _p(x)_{j,k} \times \, _p(x + jk)_{n,k}. \tag{11}$$

The Mittag-Leffler function $E_\alpha(z)$ introduced by Gosta Mittag-Leffler [4] in 1903, defined as

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \tag{12}$$

where $z \in C, \alpha \geq 0$.

Wiman [2] generalized $E_\alpha(z)$ in 1905 and gave $E_{\alpha,\beta}(z)$ known as Wiman function, defined as

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \tag{13}$$

where $z, \alpha, \beta \in C; Re(\alpha) > 0, Re(\beta) > 0$.

Prabhakar [11] in 1971, gave next generalization of Mittag-Leffler function and denoted as $E^\gamma_{\alpha,\beta}(z)$ and defined as

$$E^\gamma_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \tag{14}$$

where $z, \alpha, \beta, \gamma \in C; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0$.

Shukla and Prajapati [1] in 2007, gave second generalization of Mittag-Leffler function and denoted it as $E^{\gamma,q}_{\alpha,\beta}(z)$ and defined as,

$$E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq}}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \tag{15}$$

where $z, \alpha, \beta, \gamma \in C; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0$ and $q \in (0, 1) \cup N$.
The function $E^{\gamma,q}_{\alpha,\beta}(z)$ converges absolutely for all z if $q < Re(\alpha) + 1$ and for $|z| < 1$ if $q = Re(\alpha) + 1$. It is entire function of order $\frac{1}{Re(\alpha)}$.

Gehlot [6], introduce Generalized k- Mittag-Leffler function in 2012, denoted as $GE^{\gamma,q}_{k,\alpha,\beta}(z)$ and defined for $k \in R; z, \alpha, \beta, \gamma \in C; GE^{\gamma,q}_{k,\alpha,\beta}(z)$ and defined for $k \in R; z, \alpha, \beta, \gamma \in C; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0$ and $q \in (0, 1) \cup N$, as

$$GE^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k} \, z^n}{\Gamma_k(n\alpha + \beta)(n!)}, \tag{16}$$

where$(\gamma)_{nq,k}$ is the k- pochhammer symbol and $\Gamma_k(x)$ is the k-gamma function given by [10].
The generalized Pochhammer symbol is given as,

$$(\gamma)_{nq} = \frac{\Gamma(\gamma+nq)}{\Gamma(\gamma)} = q^{qn}\prod_{r=1}^{q}(\frac{\gamma+r-1}{q})_n,\ if\ q\in N. \tag{17}$$

Gehlot [7], introduce the p- k Mittag-Leffler function in 2018, is denoted by ${}_pE^{\gamma,q}_{k,\alpha,\beta}(z)$ and defined for $k,p\in R^+-\{0\};\alpha,\beta,\gamma\in C/kZ^-;Re(\alpha)>0,Re(\beta)>0,Re(\gamma)>0$ and $q\in(0,1)\cup N$.

$$_pE^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{_p(\gamma)_{nq,k}}{_p\Gamma_k(n\alpha+\beta)}\frac{z^n}{n!}, \tag{18}$$

where ${}_p(\gamma)_{nq,k}$ is two parameter Pochhammer symbol given by equation (1) and ${}_p\Gamma_k(x)$ is the two parameter Gamma function given by equation (3).

Luque [9] in the year 2019, introduce the L-mittag-Leffler function defined for $\alpha,\beta,\gamma\in C;Re(\alpha)>0,Re(\beta)>0,Re(\gamma)>0,j\in N_0$ by the series

$$L^{\gamma,j}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{(\gamma)_{n+j}}{\Gamma(n\alpha+\beta)}\frac{z^n}{(n+j)!},\qquad (z\in C). \tag{19}$$

Throughout this paper let $C,R^+,Re(),Z^-,N_0$ and N be the sets of complex numbers, positive real numbers, real part of complex number, negative integer, whole number and natural numbers respectively.

2. The j-generelized p - k Mittag-Leffler function

In this section we introduce the j-generelized p - k Mittag-Leffler function and prove some of its properties.

Definition 3 Let $k,p\in R^+-\{0\};\alpha,\beta,\gamma\in C/kZ^-;Re(\alpha)>0,Re(\beta)>0,Re(\gamma)>0$, $j\in N_0$ and $q\in(0,1)\cup N$. The j-generelized p - k Mittag-Leffler function is denoted by ${}^j_pE^{\gamma,q}_{k,\alpha,\beta}(z)$ and defined as

$${}^j_pE^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{_p(\gamma)_{(n+j)q,k}}{_p\Gamma_k(n\alpha+\beta)}\frac{z^n}{(n+j)!},\qquad z\in C, \tag{20}$$

where ${}_p(\gamma)_{nq,k}$ is two parameter Pochhammer symbol given by equation (1) and ${}_p\Gamma_k(x)$ is the two parameter Gamma function given by equation (3).

Particular cases : For some particular values of the parameters $j,p,q,k,\alpha,\beta,\gamma$ we can obtain certain defined and undefined Mittag-Leffler functions:

(i) For $j=0$, equation (20) reduces to the p-k Mittag-Leffler functions defined by [8]as

$${}^0_pE^{\gamma,q}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{_p(\gamma)_{nq,k}}{_p\Gamma_k(n\alpha+\beta)}\frac{z^n}{n!},\qquad z\in C. \tag{21}$$

(ii) For $q = 1$, equation (20) reduces to j form of p-k Mittag-Leffler functions defined as

$$ {}_p^jE_{k,\alpha,\beta}^{\gamma,1}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{(n+j),k}}{{}_p\Gamma_k(n\alpha+\beta)} \frac{z^n}{(n+j)!}, \qquad z \in C. \tag{22}$$

(iii) For $q = 1$and $p = k$, equation (20) reduces to j form of k- Mittag-Leffler functions defined as

$$ {}_k^jE_{k,\alpha,\beta}^{\gamma,1}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{(n+j),k}}{\Gamma_k(n\alpha+\beta)} \frac{z^n}{(n+j)!}, \qquad z \in C. \tag{23}$$

(iv) For $q = 1$ and $j = 0$, equation (20) reduces to generalized form of k- Mittag-Leffler functions defined as

$$ {}_pE_{k,\alpha,\beta}^{\gamma,1}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{n,k}\, z^n}{{}_p\Gamma_k(n\alpha+\beta)(n!)}. \tag{24}$$

(v) For $p = k$ and $j = 0$, equation (20) reduces to Generalized k- Mittag-Leffler functions defined by [6]

$$ {}_kE_{k,\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{{}_k(\gamma)_{nq,k}\, z^n}{{}_k\Gamma_k(n\alpha+\beta)(n!)} = GE_{k,\alpha,\beta}^{\gamma,q}(z). \tag{25}$$

(vi) For $p = k, q = 1$ and $j = 0$, equation (20) reduces to k - Mittag-Leffler functions defined by [3]

$$ {}_kE_{k,\alpha,\beta}^{\gamma,1}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}\, z^n}{\Gamma_k(n\alpha+\beta)(n!)} = E_{k,\alpha,\beta}^{\gamma}(z). \tag{26}$$

(vii) For $p = k, k = 1$ and $j = 0$, equation (20) reduces to Mittag-Leffler functions defined by [1]

$$ {}_1E_{1,\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq}\, z^n}{\Gamma(n\alpha+\beta)(n!)} = E_{\alpha,\beta}^{\gamma,q}(z). \tag{27}$$

(viii) For $p = k = q = 1$, equation (20) reduces to L-Mittag-Leffler functions defined by [9]

$$ {}_1E_{1,\alpha,\beta}^{\gamma,1}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n+j}\, z^n}{\Gamma(n\alpha+\beta)(n+j)!} = L_{\alpha,\beta}^{\gamma,j}(z). \tag{28}$$

(ix) For $p = k, q = 1, j = 0$ and $k = 1$, equation (20) reduces to Mittag-Leffler functions defined by [11]

$$ {}_1E_{1,\alpha,\beta}^{\gamma,1}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n}\, z^n}{\Gamma(n\alpha+\beta)(n!)} = E_{\alpha,\beta}^{\gamma}(z). \tag{29}$$

(x) For $p = k, q = 1, k = 1, j = 0$ and $\gamma = 1$, equation (20) reduces to Mittag-Leffler functions defined by [3]

$$ {}_1E_{1,\alpha,\beta}^{1,1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+\beta)} = E_{\alpha,\beta}(z). \tag{30}$$

(xi) For $p = k, q = 1, k = 1, \gamma = 1, j = 0$ and $\beta = 1$, equation (20) reduces to Mittag-Leffler functions defined by [4]

$$ {}_1E_{1,\alpha,1}^{1,1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)} = E_{\alpha}(z). \tag{31}$$

Theorem 1 The j-generelized p - k Mittag-Leffler function defined by equation (20) is an entire function of order

$$\frac{1}{\rho} = Re(\frac{\alpha}{k}) - q + 1. \tag{32}$$

Proof: Let R be the radius of convergence of the j-generelized p - k Mittag-Leffler function. The asymptotic Striling formula for Gamma function and factorial are given by,[5]

$$\Gamma(az+b) = \sqrt{2\pi}e^{-az}(az)^{az+b-\frac{1}{2}}\Big[1 + o(\frac{1}{z})\Big], (arg(az+b) < \pi; z \to \infty), \tag{33}$$

and

$$n! = \sqrt{2\pi}e^{-n}(n)^{n+\frac{1}{2}}\Big[1 + o(\frac{1}{n})\Big], (n \in N; n \to \infty). \tag{34}$$

From equation (20), we have

$${}_p^jE_{k,\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{(n+j)q,k}}{{}_p\Gamma_k(n\alpha+\beta)} \frac{z^n}{(n+j)!} = \sum_{n=0}^{\infty} C_n z^n,$$

since

$$R = \limsup_{n\to\infty} \Big|\frac{C_n}{C_{n+1}}\Big|,$$

$$\Big|\frac{C_n}{C_{n+1}}\Big| = \Big|\frac{{}_p(\gamma)_{(n+j)q,k}}{{}_p\Gamma_k(n\alpha+\beta)} \frac{1}{(n+j)!} \times \frac{{}_p\Gamma_k(n\alpha+\alpha+\beta)(n+1+j)!}{{}_p(\gamma)_{(n+1+j)q,k}}\Big|$$

using equations (2.19) and (2.20) of [7], we have

$$\Big|\frac{C_n}{C_{n+1}}\Big| = (n+1+j)\Big|p^{\frac{\alpha-qk}{k}}\Big|\Big|\frac{\Gamma(nq+jq+\frac{\gamma}{k})}{\Gamma(nq+jq+q+\frac{\gamma}{k})}\Big|\Big|\frac{\Gamma(\frac{n\alpha+\alpha+\beta}{k})}{\Gamma(\frac{n\alpha+\beta}{k})}\Big|,$$

using equation (2.11) of [7], we have

$$\simeq \Big|p^{\frac{\alpha}{k}-q}\Big|\Big|q^{-q}\Big|\Big|(\frac{\alpha}{k})^{\frac{\alpha}{k}}\Big|\Big|n^{\frac{\alpha}{k}+1-q}\Big| \to \infty$$

when,

$$Re(\frac{\alpha}{k} + 1 - q) > 0,$$

Thus, the j-generelized p - k Mittag-Leffler function is an entire function for $q < Re(\frac{\alpha}{k}) + 1$.
To determine the order ρ,

$$\rho = \limsup_{n\to\infty} \frac{n \ln n}{\ln(\frac{1}{|C_n|})}, \tag{35}$$

$$\Big|\frac{1}{C_n}\Big| = \Big|\frac{{}_p\Gamma_k(n\alpha+\beta)(n+j)!}{{}_p(\gamma)_{(n+j)q,k}}\Big|,$$

using theorem 2.19 and 2.20 of [7], we have

$$\Big|\frac{1}{C_n}\Big| = \frac{(n+j)!}{k}\Big|p^{\frac{\gamma}{k}+\frac{n\alpha+\beta}{k}-\frac{\gamma+(n+j)qk}{k}}\Big|\Big|\frac{\Gamma(\frac{\gamma}{k})\Gamma(\frac{n\alpha+\beta}{k})}{\Gamma(\frac{\gamma}{k}+(n+j)q)}\Big|,$$

By using equation (2.11) and (2.15) of [7], we get

$$\left|\frac{1}{C_n}\right| = k^{-1}(2\pi)^{\frac{1}{2}}\left|p^{(\frac{\alpha-qk}{k})n+\frac{\beta}{k}-jq}\right|\left|(\frac{\alpha}{k})^{\frac{n\alpha}{k}+\frac{\beta}{k}-\frac{1}{2}}\right|\left|n^{\frac{n\alpha}{k}+\frac{\beta}{k}-\frac{\gamma}{k}-nq-jq+n+j+\frac{1}{2}}\right|\left|e^{-nRe(\frac{\alpha}{k}+1-q)}\right|$$

taking ln of above equation and put in equation (35), we have the order of j-generelized p - k Mittag-Leffler function is given by

$$\rho = \frac{k}{Re(\alpha) - qk + k}.$$

Theorem 2 The functional relation between the j-generelized p - k Mittag-Leffler function given by equation (20) with p - k Mittag-Leffler function defined by [8] and generelized Mittag-Leffler function defined by [1] are given by

$$ {}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta}(z) = \left(kp^{jq-\frac{\beta}{k}}\right) {}^{j}E^{\frac{\gamma}{k},q}_{\frac{\alpha}{k},\frac{\beta}{k}}(zp^{q-\frac{\alpha}{k}}), \tag{36}$$

$$(\frac{d}{dz})^{l}\left[z^{j} \times {}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta}(z)\right] = {}_{p}(\gamma)_{lq,k}\ z^{j-l}\ {}^{j-l}_{p}E^{\gamma+lqk,q}_{k,\alpha,\beta}(z),\ for\ l < j, \tag{37}$$

$$(\frac{d}{dz})^{l}\left[z^{j} \times {}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta}(z)\right] = {}_{p}(\gamma)_{lq,k}\ {}_{p}E^{\gamma+lqk,q}_{k,\alpha,\beta}(z),\ for\ l = j, \tag{38}$$

$$(\frac{d}{dz})^{l}\left[z^{j} \times {}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta}(z)\right] = {}_{p}(\gamma)_{lq,k}\ {}_{p}E^{\gamma+lqk,q}_{k,\alpha,\beta+l\alpha-j\alpha}(z),\ for\ l > j. \tag{39}$$

Proof of equation (36)
Using equation (5) and (6), we get the desired result.
Proof of equation (37), (38) and (39)
Using the equation (20), in the left hand side of (37), we have

$$\frac{d^{l}}{dz^{l}}\left[z^{j} \times {}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta}(z)\right] = \sum_{n=0}^{\infty}\frac{{}_{p}(\gamma)_{(n+j)q,k}}{{}_{p}\Gamma_{k}(n\alpha+\beta)}\frac{z^{n+j-l}}{(n+j-l)!},$$

using equation (11), we have

$$\frac{d^{l}}{dz^{l}}\left[z^{j} \times {}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta}(z)\right] = \sum_{n=0}^{\infty}\frac{{}_{p}(\gamma)_{lq,k}\ {}_{p}(\gamma+lqk)_{(n+j-l)q,k}}{{}_{p}\Gamma_{k}(n\alpha+\beta)}\frac{z^{n+j-l}}{(n+j-l)!},$$

hence we have,

$$(\frac{d}{dz})^{l}\left[z^{j} \times {}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta}(z)\right] = {}_{p}(\gamma)_{lq,k}\ z^{j-l}\ {}^{j-l}_{p}E^{\gamma+lqk,q}_{k,\alpha,\beta}(z),\ for\ l < j,$$

$$(\frac{d}{dz})^{l}\left[z^{j} \times {}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta}(z)\right] = {}_{p}(\gamma)_{lq,k}\ {}_{p}E^{\gamma+lqk,q}_{k,\alpha,\beta}(z),\ for\ l = j,$$

$$(\frac{d}{dz})^{l}\left[z^{j} \times {}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta}(z)\right] = {}_{p}(\gamma)_{lq,k}\ {}_{p}E^{\gamma+lqk,q}_{k,\alpha,\beta+l\alpha-j\alpha}(z),\ for\ l > j.$$

Theorem 3 The following elementary properties are satisfied by the j-generelized p - k Mittag-Leffler function defined by equation (20),

$$k\ {}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta}(z) = p\beta\ {}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta+k}(z) + zp\alpha\frac{d}{dz}\ {}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta+k}(z), \tag{40}$$

$$pq\ {}_{p}(\gamma)_{q-1,k}\ {}^{j-1}_{p}E^{\gamma+kq-k,q}_{k,\alpha,\beta}(z) = {}^{j}_{p}E^{\gamma,q}_{k,\alpha,\beta}(z) - {}^{j}_{p}E^{\gamma-k,q}_{k,\alpha,\beta}(z), \tag{41}$$

$$\sum_{n=0}^{\infty}(x+y)^n {}_p^jE_{k,0,nk+jk+k}^{nqk+k,q}(xy) = \sum_{r=0}^{\infty} \frac{{}_p\Gamma_k(rqk+k)(xyp)^r}{{}_p\Gamma_k(rk+jk+k)} \times {}_p^jE_{k,qk,k}^{rqk+k,q}(\frac{x+y}{p}). \quad (42)$$

Proof of equation (40)

Consider the right hand side of equation (40),

$$A \equiv p\beta\, {}_p^jE_{k,\alpha,\beta+k}^{\gamma,q}(z) + zp\alpha \frac{d}{dz}\; {}_p^jE_{k,\alpha,\beta+k}^{\gamma,q}(z),$$

using equation (20),

$$A \equiv p\beta \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{(n+j)q,k}}{{}_p\Gamma_k(n\alpha+\beta+k)} \frac{z^n}{(n+j)!} + zp\alpha \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{(n+j)q,k}}{{}_p\Gamma_k(n\alpha+\beta+k)} \frac{nz^{n-1}}{(n+j)!},$$

$$A \equiv p \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{(n+j)q,k}(n\alpha+\beta)}{{}_p\Gamma_k(n\alpha+\beta+k)} \frac{z^n}{(n+j)!},$$

using the equation (9), we have

$$A \equiv k\, {}_p^jE_{k,\alpha,\beta}^{\gamma,q}(z).$$

Proof of equation (41)

Consider the right hand side of (41),

$$A \equiv {}_p^jE_{k,\alpha,\beta}^{\gamma,q}(z) - {}_p^jE_{k,\alpha,\beta}^{\gamma-k,q}(z),$$

using equation (20), we have

$$A \equiv \sum_{n=0}^{\infty} \frac{z^n}{{}_p\Gamma_k(n\alpha+\beta)(n+j)!} \Big[{}_p(\gamma)_{(n+j)q,k} - {}_p(\gamma-k)_{(n+j)q,k} \Big],$$

using equations (10) and (11), we have

$$A \equiv pq\, {}_p(\gamma)_{q-1,k}\; {}_p^{j-1}E_{k,\alpha,\beta}^{\gamma+kq-k,q}(z).$$

Proof of equation (42)

Consider the Left hand side of equation (42),

$$A \equiv \sum_{n=0}^{\infty}(x+y)^n\; {}_p^jE_{k,0,(n+j+1)k}^{nqk+k,q}(xy),$$

using equation (20), we have

$$A \equiv \sum_{n=0}^{\infty}(x+y)^n \sum_{r=0}^{\infty} \frac{{}_p(nqk+k)_{(r+j)q,k}}{{}_p\Gamma_k(nk+jk+k)} \frac{(xy)^r}{(r+j)!}, \quad (43)$$

now simplifying, by using equation (5) and (6), we have

$${}_p(nqk+k)_{(r+j)q,k} = p^{(r+j)q}(nq+1)_{(r+j)q},$$

$$= p^{(r+j)q} \frac{\Gamma(nq+(r+j)q+1)}{\Gamma(nq+1))},$$

$$= p^{(r+j)q} \frac{\Gamma((r+j)q+1+nq)}{\Gamma((r+j)q+1)} \frac{\Gamma((r+j)q+1)}{\Gamma(nq+1)},$$

$$= {}_p\Gamma_k(rqk+k) \frac{{}_p(rqk+k)_{(r+j)q,k}}{{}_p\Gamma_k(nqk+k)},$$

then equation (42) becomes by rearranging the terms, we have

$$A \equiv \sum_{r=0}^{\infty} \frac{{}_p\Gamma_k(rqk+k)(xyp)^r}{{}_p\Gamma_k(rk+jk+k)} \sum_{n=0}^{\infty} \frac{{}_p(rqk+k)_{(n+j)q,k}}{{}_p\Gamma_k(qkn+k)(n+j)!}(\frac{x+y}{p})^n.$$

This completes the proof.

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