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RELATIVE $(p, q, t)L$-TH ORDER AND RELATIVE $(p, q, t)L$-TH LOWER ORDER ORIENTED GROWTH PROPERTIES OF COMPOSITE ENTIRE FUNCTIONS

TANMAY BISWAS

Abstract. In the paper we study some maximum term and maximum modulus oriented growth properties of composite entire functions on the basis of their relative $(p, q, t)L$-th order and relative $(p, q, t)L$-th lower order of entire function with respect to another entire function.

1. Introduction, Definitions and Notations.

Let $\mathbb{C}$ be the set of all finite complex numbers and f be an entire function defined on $\mathbb{C}$. The maximum modulus function and maximum term of $f = \sum\limits_{n=0}^{\infty} a_n z^n$ on $|z| = r$ are respectively denoted as $M_f(r)$ and $\mu_f(r)$ and defined as max ($|f(z)| : |z| = r$) and $\max\limits_{n\geq 0}(|a_n|r^n)$. When f is non-constant, then $M_f(r)$ is strictly increasing and continuous and its inverse $M_f{}^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists and is such that $\lim\limits_{s\to+\infty} M_f{}^{-1}(s) = \infty$. Analogously, $\mu_f^{-1}(r)$ is also an increasing function of r. For $x \in [0, \infty)$ and $k \in \mathbb{N}$, we define $\exp^{[k]} x = \exp(\exp^{[k-1]} x)$ and $\log^{[k]} x = \log(\log^{[k-1]} x)$ where $\mathbb{N}$ is the set of all positive integers. We also denote $\log^{[0]} x = x$, $\log^{[-1]} x = \exp x$, $\exp^{[0]} x = x$ and $\exp^{[-1]} x = \log x$. Further we assume that throughout the present paper p, q, m, n, l always denote positive integers and $t \in \mathbb{N} \cup (-1, 0)$. Recently Shen et al.[19] introduce the definitions of the (m, n)-φ order and (m, n)-φ lower order of a meromorphic function. For detail about meromorphic function, one may see [11]. Using the inequality $T_f(r) \leq \log M_f(r) \leq 3T_f(2r)$ {cf.[11]}, for an entire function f, one may easily verify the following definition:

Definition 1 Let $\varphi : [0, +\infty) \to (0, +\infty)$ be a non-decreasing unbounded function. The (m, n)-φ order $\rho^{(m,n)}(f, \varphi)$ and (m, n)-φ lower order $\lambda^{(m,n)}(f, \varphi)$ of an entire function f are defined as:

$$\begin{matrix} \rho^{(m,n)}(f,\varphi) \\ \lambda^{(m,n)}(f,\varphi) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[m]} M_f(r)}{\log^{[n]} \varphi(r)},$$

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where $m \geq n$.

If we take $m = p$, $n = 1$ and $\varphi(r) = \log^{[q-1]} r$, then the above definition reduces to the following definition:

Definition 2 The (p,q)-th order and (p,q)-th lower order of an entire function f are defined as:

$$\begin{matrix} \rho^{(p,q)}(f) \\ \lambda^{(p,q)}(f) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r}.$$

Definition 2 avoids the restriction $p \geq q$ of the original definition of (p,q)-th order (respectively (p,q)-th lower order) of entire functions introduced by Juneja et al. [12].

However the above definition is very useful for measuring the growth of entire functions. If $p = l$ and $q = 1$ then we write $\rho^{(l,1)}(f) = \rho^{(l)}(f)$ and $\lambda^{(l,1)}(f) = \lambda^{(l)}(f)$ where $\rho^{(l)}(f)$ and $\lambda^{(l)}(f)$ are respectively known as generalized order and generalized lower order of entire function f. For details about generalized order one may see [14]. Also for $p = 2$ and $q = 1$, we respectively denote $\rho^{(2,1)}(f)$ and $\lambda^{(2,1)}(f)$ by $\rho(f)$ and $\lambda(f)$ which are classical growth indicators such as order and lower order of entire function f.

With the help of the inequalities $\mu_f(r) \leq M_f(r) \leq \frac{R}{R-r}\mu_f(R)$ ($cf.$ [15]), for $0 \leq r < R$ one may verify that

$$\begin{matrix} \rho^{(p,q)}(f) \\ \lambda^{(p,q)}(f) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} \mu_f(r)}{\log^{[q]} r}.$$

Throughout the present paper we shall fix the function $L \equiv L(r)$ is a positive continuous function increasing slowly i.e.,$L(ar) \sim L(r)$ as $r \to +\infty$ for every positive constant 'a' i.e., $\lim\limits_{r\to+\infty} \frac{L(ar)}{L(r)} = 1$ where $L \equiv L(r)$ is a positive continuous function increasing slowly. Considering $L(r) = \log r$ and $a = 10^{20}$, one can easily verify that $\lim\limits_{r\to+\infty} \frac{L(ar)}{L(r)} = 1$. In this connection, Somasundaram and Thamizharasi [13] introduced the notions of L-order and L-lower order for entire functions. The more generalized concept of L-order and L-lower order for entire function are $(p,q,t)L$-th order and $(p,q,t)L$-th lower order. If we take $m = p$, $n = 1$ and $\varphi(r) = \log^{[q-1]} r \cdot \exp^{[t+1]} L(r)$, then Definition 1 turn into the definitions of $(p,q,t)L$-th order and $(p,q,t)L$-th lower order of an entire function f which are as follows (see [4, p.4]):

$$\begin{matrix} \rho_f^L(p,q,t) \\ \lambda_f^L(p,q,t) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r + \exp^{[t]} L(r)}.$$

Using the inequalities $\mu_f(r) \leq M_f(r) \leq \frac{R}{R-r}\mu_f(R)$ ($cf.$ [15]), for $0 \leq r < R$ one may verify that

$$\begin{matrix} \rho_f^L(p,q,t) \\ \lambda_f^L(p,q,t) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} \mu_f(r)}{\log^{[q]} r + \exp^{[t]} L(r)}.$$

Mainly the growth investigation of entire functions has usually been done through their maximum moduli in comparison with those of exponential function. But if one is paying attention to evaluate the growth rate of any entire with respect to a new entire function, the notions of relative growth indicators [1, 2] will come. Extending this notion, one may introduce the definitions of relative $(p,q,t)L$-th order and relative $(p,q,t)L$-th lower order of an entire function f with respect to another entire function g in the following way:

Definition 3 [4] Let f and g be entire functions. The relative $(p,q,t)L$-order denoted as $\rho_g^{(p,q,t)L}(f)$ and relative $(p,q,t)L$-lower order denoted as $\lambda_g^{(p,q,t)L}(f)$ of an entire function f with respect to another entire function g are define by

$$\begin{matrix}\rho_g^{(p,q,t)L}(f) \\ \lambda_g^{(p,q,t)L}(f)\end{matrix} = \lim_{r\to+\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[p]} M_g^{-1}(M_f(r))}{\log^{[q]} r + \exp^{[t]} L(r)}.$$

In [8] an alternative definition of relative $(p,q,t)L$-th order and relative $(p,q,t)L$-th lower order of f with respect to g in terms of their maximum terms is given in the following way:

Definition 4 [8] The growth indicators $\rho_g^{(p,q,t)L}(f)$ and $\lambda_g^{(p,q,t)L}(f)$ of an entire function $f(z)$ with respect to another entire function $g(z)$ are defined as:

$$\begin{matrix}\rho_g^{(p,q,t)L}(f) \\ \lambda_g^{(p,q,t)L}(f)\end{matrix} = \lim_{r\to+\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[p]} \mu_g^{-1}(\mu_f(r))}{\log^{[q]} r + \exp^{[t]} L(r)}.$$

In fact, equivalence of Definition 3 and Definition 4 is established in[8].

However, Song et al. [18] have proved that if f and g are transcendental entire functions with $0 < \lambda(f) \leq \rho(f) < \infty$, then

$$\lim_{r\to+\infty} \frac{\log^{[2]} M_{f\circ g}(r)}{\log^{[2]} M_f(r)} = +\infty.$$

Singh et al. [17] have proved the following theorems:

Theorem A. Let f and g be entire functions of positive lower order and of finite order, then

$$\lim_{r\to+\infty} \frac{\log^{[2]} M_{f\circ g}(r)}{\log^{[2]} M_f(r^A)} = +\infty$$

for every positive constant A.

Theorem B. Let f and g be entire functions of finite order wit h $0 < \rho(g) \leq \rho(f) < \infty$. Then

$$\lim_{r\to+\infty} \frac{\log^{[2]} M_{f\circ g}(r)}{\log^{[2]} M_f(\exp r^{\rho(f)})} = 0.$$

In the paper we extend the above results under some different conditions and study some maximum term and maximum modulus oriented growth properties of composite entire functions on the basis of their relative $(p,q,t)L$-th order and relative $(p,q,t)L$-th lower order of entire function with respect to another entire function. In fact some recent works related to the growth of composite entire functions have also been explored in [3] to [8]. We do not explain the standard definitions and notations in the theory of entire functions as those are available in [20].

2. Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 [15] Let f and g be entire functions. Then for every $\alpha > 1$ and $0 < r < R$,

$$\mu_{f\circ g}(r) \leq \frac{\alpha}{\alpha-1}\mu_f\Big(\frac{\alpha R}{R-r}\mu_g(R)\Big).$$

Lemma 2 [16] If f and g are any two entire functions. Then for all sufficiently large values of r,

$$\mu_{f\circ g}(r) \geq \frac{1}{2}\mu_f\Big(\frac{1}{16}\mu_g\Big(\frac{r}{4}\Big)\Big).$$

Lemma 3 [9] Let f and g are any two entire functions with $g(0) = 0$. Also let β satisfy $0 < \beta < 1$ and $c(\beta) = \frac{(1-\beta)^2}{4\beta}$. Then for all sufficiently large values of r,

$$M_f(c(\beta)M_g(\beta r)) \leq M_{f\circ g}(r) \leq M_f(M_g(r)).$$

In addition if $\beta = \frac{1}{2}$, then for all sufficiently large values of r,

$$M_{f\circ g}(r) \geq M_f\Big(\frac{1}{8}M_g\Big(\frac{r}{2}\Big)\Big).$$

Lemma 4 [10] If f is entire and $\alpha > 1$, $0 < \beta < \alpha$, then for all sufficiently large r,

$$\mu_f(\alpha r) \geq \beta\mu_f(r).$$

3. Theorems.

In this section we present the main results of the paper.

Theorem 1 Let f, g and h be entire functions such that $\rho_g^L(m,n,t) < \lambda_h^{(p,q,t)L}(f) \leq \rho_h^{(p,q,t)L}(f) < +\infty$. Then for any $\beta > 1$,

$$(i)\ \lim_{r\to+\infty}\frac{\log^{[p]}\mu_h^{-1}(\mu_{f\circ g}(r))}{\log^{[p+q-m]}\mu_h^{-1}(\mu_f(r))} = 0 \text{ if } q+1 \leq m \text{ and}$$

$\exp^{[t]} L(\mu_g(\beta r)) = o(\exp^{[m-q-1]}((\log^{[q-1]} r)\exp^{[t+1]} L(r))^{\alpha})$ as $r \to +\infty$ and for some $\alpha < \lambda_h^{(p,q,t)L}(f)$
and

$$(ii)\ \lim_{r\to+\infty}\frac{\log^{[p]}\mu_h^{-1}(\mu_{f\circ g}(r))}{\log^{[p-1]}\mu_h^{-1}(\mu_f(r))} = 0 \text{ if } q+1 > m \text{ and}$$

$\exp^{[t]} L(\mu_g(\beta r)) = o(((\log^{[q-1]} r)\exp^{[t+1]} L(r))^{\alpha})$ as $r \to +\infty$ and for some $\alpha < \lambda_h^{(p,q,t)L}(f)$.

Proof Let us consider $\beta > 1$, and $\xi > \frac{\alpha}{\alpha-1}$. Now taking $R = \beta r$ in Lemma 1 and in view of Lemma 4 we have for all sufficiently large values of r that

$$\mu_{f\circ g}(r) \leq \mu_f\Big(\frac{\alpha\beta\xi}{(\beta-1)}\mu_g(\beta r)\Big).$$

Since $\mu_h^{-1}(r)$ is an increasing function of r, it follows from above that for all sufficiently large values of r we have

$$\log^{[p]}\mu_h^{-1}(\mu_{f\circ g}(r)) \leq \log^{[p]}\mu_h^{-1}\Big(\mu_f\Big(\frac{\alpha\beta\xi}{(\beta-1)}\mu_g(\beta r)\Big)\Big),$$

$$i.e.,\ \log^{[p]}\mu_h^{-1}(\mu_{f\circ g}(r)) \leq (\rho_h^{(p,q,t)L}(f)+\varepsilon)(\log^{[q]}\mu_g(\beta r) + \exp^{[t]} L(\mu_g(\beta r))) + O(1). \quad (1)$$

Case I. Let $q+1 \leq m$. Then we have from (1) for all sufficiently large values of r that

$$\log^{[p]}\mu_h^{-1}(\mu_{f\circ g}(r)) \leq (\rho_h^{(p,q,t)L}(f)+\varepsilon)\times (\exp^{[m-q-1]}((\log^{[n-1]}\beta r)\exp^{[t+1]} L(r))^{(\rho_g^L(m,n,t)+\varepsilon)} + \exp^{[t]} L(\mu_g(\beta r)) + O(1)). \quad (2)$$

Case II. Let $q+1>m$. Then for all sufficiently large values of r we get from (1) that

$$\log^{[p]}\mu_h^{-1}(\mu_{f\circ g}(r))\leq(\rho_h^{(p,q,t)L}(f)+\varepsilon)\times$$
$$(((\log^{[n-1]}\beta r)\exp^{[t+1]}L(r))^{(\rho_g^L(m,n,t)+\varepsilon)}+\exp^{[t]}L(\mu_g(\beta r))+O(1)). \quad (3)$$

Now we obtain for all sufficiently large values of r that

$$\log^{[p-1]}\mu_h^{-1}(\mu_f(r))\geq((\log^{[q-1]}r)\exp^{[t+1]}L(r))^{(\lambda_h^{(p,q,t)L}(f)-\varepsilon)} \quad (4)$$

$$i.e.,\ \log^{[p+q-m]}\mu_h^{-1}(\mu_f(r))\geq\exp^{[m-q-1]}((\log^{[q-1]}r)\exp^{[t+1]}L(r))^{(\lambda_h^{(p,q,t)L}(f)-\varepsilon)}. \quad (5)$$

Now from (2) and (5) we get for all sufficiently large values of r that

$$\frac{\log^{[p]}\mu_h^{-1}(\mu_{f\circ g}(r))}{\log^{[p+q-m]}\mu_h^{-1}(\mu_f(r))}\leq\frac{(\rho_h^{(p,q,t)L}(f)+\varepsilon)}{\exp^{[m-q-1]}((\log^{[q-1]}r)\exp^{[t+1]}L(r))^{(\lambda_h^{(p,q,t)L}(f)-\varepsilon)}}\times$$

$$(\exp^{[m-q-1]}((\log^{[n-1]}\beta r)\exp^{[t+1]}L(r))^{(\rho_g^L(m,n,t)+\varepsilon)}+\exp^{[t]}L(\mu_g(\beta r))+O(1)). \quad (6)$$

Since $\rho_g^L(m,n,t)<\lambda_h^{(p,q,t)L}(f)$, we can choose $\varepsilon(>0)$ in such a way that

$$\rho_g^L(m,n,t)+\varepsilon<\ \lambda_h^{(p,q,t)L}(f)-\varepsilon. \quad (7)$$

Now let $\exp^{[t]}L(\mu_g(\beta r))=o(\exp^{[m-q-1]}((\log^{[q-1]}r)\exp^{[t+1]}L(r))^{\alpha})$ as $r\to+\infty$ and for some $\alpha<\lambda_h^{(p,q,t)L}(f)$.

As $\alpha<\lambda_h^{(p,q,t)L}(f)$ we can choose $\varepsilon(>0)$ in such a way that

$$\alpha<\ \lambda_h^{(p,q,t)L}(f)-\varepsilon. \quad (8)$$

Since $\exp^{[t]}L(\mu_g(\beta r))=o(\exp^{[m-q-1]}((\log^{[q-1]}r)\exp^{[t+1]}L(r))^{\alpha})$ as $r\to+\infty$ we get using (8) that

$$\frac{\exp^{[t]}L(\mu_g(\beta r))}{\exp^{[m-q-1]}((\log^{[q-1]}r)\exp^{[t+1]}L(r))^{\alpha}}\to 0 \text{ as } r\to+\infty$$

$$i.e.,\ \frac{\exp^{[t]}L(\mu_g(\beta r))}{\exp^{[m-q-1]}((\log^{[q-1]}r)\exp^{[t+1]}L(r))^{(\lambda_h^{(p,q,t)L}(f)-\varepsilon)}}\to 0 \text{ as } r\to+\infty. \quad (9)$$

Now in view of (6), (7) and (9) we obtain that

$$\lim_{r\to+\infty}\frac{\log^{[p]}\mu_h^{-1}(\mu_{f\circ g}(r))}{\log^{[p+q-m]}\mu_h^{-1}(\mu_f(r))}=0.$$

Thus the first part of the theorem is established. Similarly in view of (3) and (4), one can easily verify the second part of the theorem.

The following theorem can be carried out in the line of Theorem 1 and with the help of Lemma 3 and therefore its proof is omitted:

Theorem 2 Let f, g and h be entire functions such that $\rho_g^L(m,n,t)<\lambda_h^{(p,q,t)L}(f)\leq\rho_h^{(p,q,t)L}(f)<+\infty$. Then for any $\beta>1$,

$$(i)\ \lim_{r\to+\infty}\frac{\log^{[p]}M_h^{-1}(M_{f\circ g}(r))}{\log^{[p+q-m]}M_h^{-1}(M_f(r))}=0 \text{ if } q+1\leq m \text{ and}$$

$\exp^{[t]} L(M_g(r)) = o(\exp^{[m-q-1]}((\log^{[q-1]} r)\exp^{[t+1]} L(r))^{\alpha})$ as $r \to +\infty$ and for some $\alpha < \lambda_h^{(p,q,t)L}(f)$
and

$$(ii) \lim_{r\to+\infty} \frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(r))}{\log^{[p-1]} M_h^{-1}(M_f(r))} = 0 \text{ if } q+1 > m \text{ and}$$

$\exp^{[t]} L(M_g(r)) = o(((\log^{[q-1]} r)\exp^{[t+1]} L(r))^{\alpha})$ as $r \to +\infty$ and for some $\alpha < \lambda_h^{(p,q,t)L}(f)$.

Remark 1 Theorem 1 and Theorem 2 remain valid with "limit inferior " instead of " limit ", if we take $\lambda_g^L(m,n,t) < \lambda_h^{(p,q,t)L}(f) \leq \rho_h^{(p,q,t)L}(f) < +\infty$ instead of $\rho_g^L(m,n,t) < \lambda_h^{(p,q,t)L}(f) \leq \rho_h^{(p,q,t)L}(f) < +\infty$ and the other conditions remain the same.

Theorem 3 Let f, g, h and k be entire functions with $\rho_h^{(p,q,t)L}(f) < \infty$, $\lambda_k^{(l,n,t)L}(g) > 0$ and $\rho_g^L(m,n,t) < +\infty$ where $q+1 \geq m$. Then for any $\beta > 1$,
(i) if $\exp^{[t]} L(\mu_g(\beta r)) = o(\log^{[l]} \mu_k^{-1}(\mu_g(r)))$ as $r \to +\infty$, then

$$\limsup_{r\to+\infty} \frac{\log^{[p+1]} \mu_h^{-1}(\mu_{f\circ g}(r))}{\log^{[l]} \mu_k^{-1}(\mu_g(r)) + \exp^{[t]} L(\mu_g(\beta r))} \leq \frac{\rho_g^L(m,n,t)}{\lambda_k^{(l,n,t)L}(g)}$$

and
(ii) if $\exp^{[t]} L(M_g(r)) = o(\log^{[l]} M_k^{-1}(M_g(r)))$ as $r \to +\infty$, then

$$\limsup_{r\to+\infty} \frac{\log^{[p+1]} M_h^{-1}(M_{f\circ g}(r))}{\log^{[l]} M_k^{-1}(M_g(r)) + \exp^{[t]} L(M_g(r))} \leq \frac{\rho_g^L(m,n,t)}{\lambda_k^{(l,n,t)L}(g)}.$$

Proof Since $\log\left(\frac{1+\exp^{[t]} L(\mu_g(\beta r))+O(1)}{\log^{[q]} \mu_g(\beta r)}\right) < \frac{1+\exp^{[t]} L(\mu_g(\beta r))+O(1)}{\log^{[q]} \mu_g(\beta r)}$ we have from (1) for all sufficiently large values of r that

$$\log^{[p+1]} \mu_h^{-1}(\mu_{f\circ g}(r)) \leq \log(\rho_h^{(p,q,t)L}(f) + \varepsilon) + \log^{[q+1]} \mu_g(\beta r) + \log\left(\frac{1+\exp^{[t]} L(\mu_g(\beta r)) + O(1)}{\log^{[q]} \mu_g(\beta r)}\right)$$

$$i.e.,\ \log^{[p+1]} \mu_h^{-1}(\mu_{f\circ g}(r)) \leq \log(\rho_h^{(p,q,t)L}(f) + \varepsilon) + + \log^{[q+1]} \mu_g(\beta r) + \frac{1+\exp^{[t]} L(\mu_g(\beta r)) + O(1)}{\log^{[q]} \mu_g(\beta r)}$$

$$i.e.,\ \log^{[p+1]} \mu_h^{-1}(\mu_{f\circ g}(r)) \leq \log^{[m]} \mu_g(\beta r) + \log(\rho_h^{(p,q,t)L}(f) + \varepsilon) + \frac{1+\exp^{[t]} L(\mu_g(\beta r)) + O(1)}{\log^{[q]} \mu_g(\beta r)}$$

$$i.e.,\ \log^{[p+1]} \mu_h^{-1}(\mu_{f\circ g}(r)) \leq (\rho_g^L(m,n,t) + \varepsilon)[\log^{[n]}(\beta r) + \exp^{[t]} L(\beta r)] + \log(\rho_h^{(p,q,t)L}(f) + \varepsilon) + \frac{1+\exp^{[t]} L(\mu_g(\beta r)) + O(1)}{\log^{[q]} \mu_g(\beta r)}$$

$$i.e.,\ \log^{[p+1]} \mu_h^{-1}(\mu_{f\circ g}(r)) \leq (\rho_g^L(m,n,t) + \varepsilon)[\log^{[n]} r + \exp^{[t]} L(r) + O(1)]$$

$$+\log(\rho_h^{(p,q,t)L}(f)+\varepsilon)+\frac{1+\exp^{[t]}L(\mu_g(\beta r))+O(1)}{\log^{[q]}\mu_g(\beta r)}. \quad (10)$$

Again we have for all sufficiently large values of r that

$$\log^{[l]}\mu_k^{-1}(\mu_g(r))\geq(\lambda_k^{(l,n,t)L}(g)-\varepsilon)(\log^{[n]}r+\exp^{[t]}L(r))$$

$$i.e.,\ \log^{[n]}r+\exp^{[t]}L(r)\leq\frac{\log^{[l]}\mu_k^{-1}(\mu_g(r))}{(\lambda_k^{(l,n,t)L}(g)-\varepsilon)}. \quad (11)$$

Hence from (10) and (11), it follows for all sufficiently large values of r that

$$\log^{[p+1]}\mu_h^{-1}(\mu_{f\circ g}(r))\leq O(1)+\Big(\frac{\rho_g^L(m,n,t)+\varepsilon}{\lambda_k^{(l,n,t)L}(g)-\varepsilon}\Big)\cdot\log^{[l]}\mu_k^{-1}\mu_g(r)$$

$$+\log(\rho_h^{(p,q,t)L}(f)+\varepsilon)+\frac{1+\exp^{[t]}L(\mu_g(\beta r))+O(1)}{\log^{[q]}\mu_g(\beta r)}$$

$$i.e.,\ \frac{\log^{[p+1]}\mu_h^{-1}(\mu_{f\circ g}(r))}{\log^{[l]}\mu_k^{-1}(\mu_g(r))+\exp^{[t]}L(\mu_g(\beta r))}\leq\frac{O(1)}{\log^{[l]}\mu_k^{-1}(\mu_g(r))+\exp^{[t]}L(\mu_g(\beta r))}+$$
$$\Big(\frac{\rho_g^L(m,n,t)+\varepsilon}{\lambda_k^{(l,n,t)L}(g)-\varepsilon}\Big)\cdot\frac{\log^{[l]}\mu_k^{-1}\mu_g(r)}{\log^{[l]}\mu_k^{-1}(\mu_g(r))+\exp^{[t]}L(\mu_g(\beta r))}$$
$$+\frac{\log(\rho_h^{(p,q,t)L}(f)+\varepsilon)\log^{[q]}\mu_g(\beta r)+1+\exp^{[t]}L(\mu_g(\beta r))+O(1)}{(\log^{[l]}\mu_k^{-1}(\mu_g(r))+\exp^{[t]}L(\mu_g(\beta r)))\log^{[q]}\mu_g(\beta r)}$$

$$i.e.,\ \frac{\log^{[p+1]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[l]}\mu_k^{-1}(\mu_g(r))+\exp^{[t]}L(\mu_g(\beta r))}\leq\frac{\frac{O(1)}{\exp^{[t]}L(\mu_g(\beta r))}}{\frac{\log^{[l]}\mu_k^{-1}(\mu_g(r))}{\exp^{[t]}L(\mu_g(\beta r))}+1}+\frac{\Big(\frac{\rho_g^L(m,n,t)+\varepsilon}{\lambda_k^{(l,n,t)L}(g)-\varepsilon}\Big)}{1+\frac{\exp^{[t]}L(\mu_g(\beta r))}{\log^{[l]}\mu_k^{-1}(\mu_g(r))}}$$
$$+\frac{1+\frac{O(1)+1+\log^{[q]}\mu_g(\beta r)\cdot\log(\rho_h^{(p,q,t)L}(f)+\varepsilon)}{\exp^{[t]}L(\mu_g(\beta r))}}{\Big(1+\frac{\log^{[l]}\mu_k^{-1}(\mu_g(r))}{\exp^{[t]}L(\mu_g(\beta r))}\Big)\log^{[q]}\mu_g(\beta r)}. \quad (12)$$

Since $\exp^{[t]}L(\mu_g(\beta r))=o(\log^{[l]}\mu_k^{-1}(\mu_g(r)))$ as $r\to+\infty$ and $\varepsilon(>0)$ is arbitrary we obtain from (12) that

$$\limsup_{r\to+\infty}\frac{\log^{[p+1]}\mu_h^{-1}(\mu_{f\circ g}(r))}{\log^{[l]}\mu_k^{-1}(\mu_g(r))+\exp^{[t]}L(\mu_g(\beta r))}\leq\frac{\rho_g^L(m,n,t)}{\lambda_k^{(l,n,t)L}(g)}.$$

Thus from above, the first part of the theorem is established.

Similarly, the second part of the theorem can be established from Lemma 3 and therefore their proofs are omitted.

Remark 2 Theorem 3 remains valid with "limit inferior " instead of " limit superior", if we take $\lambda_h^{(p,q,t)L}(f)<+\infty$ instead of $\rho_h^{(p,q,t)L}(f)<+\infty$ and the other conditions remain the same.

Remark 3 In Theorem 3, if we replace either "$\lambda_k^{(l,n,t)L}(g)$" by "$\rho_k^{(l,n,t)L}(g)$", or "$\rho_g^L(m,n,t)$" by "$\lambda_g^L(m,n,t)$", then Theorem 3 remains valid with "limit inferior" replaced by "limit superior".

Now we state the following theorem without its proof as it can be carried out in the line of Theorem 3:

Theorem 4 Let f, g and h be entire functions with $0 < \lambda_h^{(p,q,t)L}(f) \leq \rho_h^{(p,q,t)L}(f) < +\infty$ and $\rho_g^L(m,n,t) < +\infty$ where $q+1 \geq m$ and $q = n$. Then for any $\beta > 1$,
(i) if $\exp^{[t]} L(\mu_g(\beta r)) = o(\log^{[p]} \mu_h^{-1}(\mu_f(r)))$ as $r \to +\infty$, then

$$\limsup_{r\to+\infty} \frac{\log^{[p+1]} \mu_h^{-1}(\mu_{f\circ g}(r))}{\log^{[p]} \mu_h^{-1}(\mu_f(r)) + \exp^{[t]} L(\mu_g(\beta r))} \leq \frac{\rho_g^L(m,n,t)}{\lambda_h^{(p,q,t)L}(f)}$$

and
(ii) if $\exp^{[t]} L(M_g(r)) = o(\log^{[p]} M_h^{-1}(M_f(r)))$ as $r \to +\infty$, then

$$\limsup_{r\to+\infty} \frac{\log^{[p+1]} M_h^{-1} M_{f\circ g}(r)}{\log^{[p]} M_h^{-1}(M_f(r)) + \exp^{[t]} L(M_g(r))} \leq \frac{\rho_g^L(m,n,t)}{\lambda_h^{(p,q,t)L}(f)}.$$

Remark 4 Theorem 4 remains valid with "limit inferior " instead of "limit superior", if we take either "$0 < \lambda_h^{(p,q,t)L}(f) < +\infty$" or "$0 < \rho_h^{(p,q,t)L}(f) < +\infty$" instead of "$0 < \lambda_h^{(p,q,t)L}(f) \leq \rho_h^{(p,q,t)L}(f) < +\infty$" and the other conditions remain the same.

Remark 5 In Theorem 4, if we replace "$\rho_g^L(m,n,t)$" by "$\lambda_g^L(m,n,t)$", then Theorem 4 remains valid with "limit inferior" replaced by "limit superior".

Theorem 5 Let f, g and h be any functions such that $0 < \lambda_h^{(p,q,t)L}(f) \leq \rho_h^{(p,q,t)L}(f) < +\infty$, $\rho_g^L(m,n,t) > 0$ where $m > q \geq n$. Then for a real number x,

$$\limsup_{r\to+\infty} \frac{\log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r))}{(\log^{[p]} \mu_h^{-1}(\mu_f(r)))^{1+x}} = \infty$$

and

$$\limsup_{r\to+\infty} \frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(r))}{(\log^{[p]} M_h^{-1}(M_f(r)))^{1+x}} = \infty.$$

Proof If x is such that $1+x \leq 0$, then the theorem is obvious. So we suppose that $1+x > 0$. Now in view of Lemma 2 and Lemma 4, we have for all sufficiently large values of r that

$$\mu_{f\circ g}(r) \geq \mu_f\Big(\frac{1}{48}\mu_g\Big(\frac{r}{4}\Big)\Big).$$

Since $\mu_h^{-1}(r)$ is an increasing function, it follows from above for a sequence of values of r tending to infinity that

$$\log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r)) \geq \log^{[p]} \mu_h^{-1}\Big(\mu_f\Big(\frac{1}{48}\mu_g\Big(\frac{r}{4}\Big)\Big)\Big)$$

$$i.e.,\ \log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r)) \geq (\lambda_h^{(p,q,t)L}(f) - \varepsilon)\Big[\log^{[q]}\Big(\frac{1}{48}\mu_g\Big(\frac{r}{4}\Big)\Big) + \exp^{[t]} L\Big(\frac{1}{48}\mu_g\Big(\frac{r}{4}\Big)\Big)\Big]$$

$$i.e.,\ \log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r)) \geq (\lambda_h^{(p,q,t)L}(f) - \varepsilon)\Big[\log^{[q]}\Big(\frac{1}{48}\mu_g\Big(\frac{r}{4}\Big)\Big) + \exp^{[t]} L\Big(\mu_g\Big(\frac{r}{4}\Big)\Big)\Big]$$

$$i.e., \ \log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r)) \geq (\lambda_h^{(p,q,t)L}(f) - \varepsilon) \left[\log^{[m-1]}\left(\frac{1}{48}\mu_g\left(\frac{r}{4}\right)\right) + \exp^{[t]} L\mu_g\left(\frac{r}{4}\right)\right)\right]$$

$$i.e., \log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r)) \geq (\lambda_h^{(p,q,t)L}(f) - \varepsilon)\Big[\log^{[n-1]}\left(\frac{r}{4}\right)\Big) \exp^{[t+1]} L(r)\Big]^{\rho_g^L(m,n,t)-\varepsilon} + O(1) + \exp^{[t]} L\Big(\mu_g\Big(\frac{r}{4}\Big)\Big). \quad (13)$$

where we choose $0 < \varepsilon < \min\{\lambda_h^{(p,q,t)L}(f),\ \rho_g^L(m,n,t)\}$.

Also for all sufficiently large values of r we get that

$$\log^{[p]} \mu_h^{-1}(\mu_f(r)) \leq (\rho_h^{(p,q,t)L}(f) + \varepsilon)(\log^{[q]} r + \exp^{[t]} L(r))$$

$$i.e., (\log^{[p]} \mu_h^{-1}(\mu_f(r)))^{1+x} \leq (\rho_h^{(p,q,t)L}(f) + \varepsilon)^{1+x} \cdot [\log^{[q]} r + \exp^{[t]} L(r)]^{1+x}. \quad (14)$$

Therefore from (13) and (14), it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r))}{(\log^{[p]} \mu_h^{-1}(\mu_f(r)))^{1+x}} \geq$$

$$\frac{O(1) + (\lambda_h^{(p,q,t)L}(f) - \varepsilon)\Big(\Big(\log^{[n-1]}\left(\frac{r}{4}\right)\Big) \exp^{[t+1]} L(r)\Big)^{\rho_g^L(m,n,t)-\varepsilon} + \exp^{[t]} L\Big(\mu_g\left(\frac{r}{4}\right)\Big)}{(\rho_h^{(p,q,t)L}(f) + \varepsilon)^{1+x} \cdot (\log^{[q]} r + \exp^{[t]} L(r))^{1+x}},$$

thus the first part of the theorem follows from above.

Since $M_h^{-1}(r)$ is an increasing function of r, by similar reasoning as above the second part of the theorem follows from Lemma 3 and therefore the proof is omitted.

Remark 6 The conclusion of Theorem 5 can also be drawn if we take $0 < \rho_h^{(p,q,t)L}(f) < +\infty$ instead of $0 < \lambda_h^{(p,q,t)L}(f) \leq \rho_h^{(p,q,t)L}(f) < +\infty$ and $\lambda_g^L(m,n,t) > 0 > 0$ instead of $\rho_g^L(m,n,t) > 0$.

Remark 7 The conclusion of Theorem 5 can also be drawn if we take $0 < \lambda_h^{(p,q,t)L}(f) < +\infty$ instead of $0 < \lambda_h^{(p,q,t)L}(f) \leq \rho_h^{(p,q,t)L}(f) < +\infty$ and $\lambda_g^L(m,n,t) > 0 > 0$ instead of $\rho_g^L(m,n,t) > 0$.

Using the same technique of Theorem 5 one may easily verify the following theorem:

Theorem 6 Let f, g, h and k be any four entire functions such that $\lambda_h^{(p,q,t)L}(f) > 0$, $\rho_k^{(l,n,t)L}(g) < +\infty$, $\rho_g^L(m,n,t) > 0$ where $m > q$. Then for a real number x,

$$\limsup_{r\to+\infty} \frac{\log^{[p]} \mu_h^{-1}(\mu_{f\circ g}(r))}{(\log^{[l]} \mu_k^{-1}(\mu_g(r)))^{1+x}} = +\infty$$

and

$$\limsup_{r\to+\infty} \frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(r))}{(\log^{[l]} M_k^{-1}(M_g(r)))^{1+x}} = +\infty.$$

Remark 8 The conclusion of Theorem 6 can also be drawn if we take "$\rho_h^{(p,q,t)L}(f) > 0$, $\rho_k^{(l,n,t)L}(g) < +\infty$" instead of "$\lambda_h^{(p,q,t)L}(f) > 0$, $\rho_k^{(l,n,t)L}(g) < +\infty$" and $\lambda_g^L(m,n,t) > 0$ instead of $\rho_g^L(m,n,t) > 0$.

Remark 9 The conclusion of Theorem 6 can also be drawn if we take "$\lambda_h^{(p,q,t)L}(f) > 0$, $\lambda_k^{(l,n,t)L}(g) < +\infty$" instead of "$\lambda_h^{(p,q,t)L}(f) > 0$, $\rho_k^{(l,n,t)L}(g) < +\infty$" and $\lambda_g^L(m,n,t) > 0$ instead of $\rho_g^L(m,n,t) > 0$.

Remark 10 Theorem 5 and Theorem 6 remain valid with "limit" instead of " limit superior", if we take $\lambda_g^L(m,n,t) > 0$ instead of $\rho_g^L(m,n,t) > 0$ and the other conditions remain the same.

4. Conclusion

The main purpose of this paper is to extend and to modify the notion of (m,n)-φ order and (m,n)-φ lower order to relative $(p,q,t)L$ -th order and relative $(p,q,t)L$ -th lower order of higher dimentions in case of entire functions. Actually we are trying to generalize some growth properties of entire functions using the concept of relative $(p,q,t)L$ -th order and relative $(p,q,t)L$ -th lower order.

The results of this paper in connection with Nevanlinna's Value Distribution theory of entire functions on the basis of relative $(p,q,t)L$ -th order and relative $(p,q,t)L$ -th lower order may have a wide range of applications in Complex Dynamics, Factorization Theory of entire functions of single complex variable, the solution of complex differential equations etc. In fact, Complex Dynamics is a thrust area in modern function theory and it is solely based on the study of fixed points of entire functions as well as the normality of them. Factorization theory of entire functions is another branch of applications of Nevanlinna's theory which actually deals how a given entire function can be factorized into other simpler entire functions in the sense of composition.

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Tanmay Biswas
Rajbari, Rabindrapalli, R. N. Tagore Road, P.O.- Krishnagar, Dist-Nadia, PIN- 741101, West Bengal, India
E-mail address: tanmaybiswas_math@rediffmail.com