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INITIAL BOUNDS FOR A CLASS OF ANALYTIC UNIVALENT FUNCTIONS DEFINED BY Q-DIFFERENCE AL-OBOUDI OPERATOR

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Abstract. In this paper using a q-difference Al-Oboudi operator, we define a class of univalent functions and obtain initial bounds for functions in this class.

1. Introduction

Quantum calculus, occasionally named calculus without limits. It is known as $q-$calculus which has infuenced many scientific fields due to its importance. Geometric function theory is no exception in this regard and many authors have already made a substantial research in this field. The generalization of derivative and integral in $q-$calculus are known as $q-$derivative and $q-$integral, were introduced and studied by Jackson [20]. Recently, many authors used the $q-$derivative and $q-$integral to generalize many classes and many operators in geometric function theory see for example [3, 4, 5, 7, 9, 30, 31] also see Ibrahim et al. [15, 16, 17, 18, 19].

The class of univalent analytic functions of the form:

$$F(z) = z + \sum_{k=2}^{\infty} a_k z^k, \ z \in \mathcal{D} = \{z \in \mathbb{C} : \mid z \mid < 1\}, \tag{1}$$

is denoted by S. For $F \in S, \ 0 < q < 1$, the $q-$difference operator Δ_q is given by [20] (see also [2, 3,4,5], [7],[9],[13]) by:

$$\Delta_q F(z) = \begin{cases} \frac{F(z)-F(qz)}{(1-q)z} & , z \neq 0 \\ F^{'}(0) & , z = 0 \end{cases},$$

that is

$$\Delta_q F(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}, \tag{2}$$

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where

$$[j]_q = \frac{1-q^j}{1-q},\ [0]_q = 0. \tag{3}$$

As $q \to 1^-$, $[k]_q = k$ and $\Delta_q F(z) = F^{'}(z)$.

For $F \in S$, the $q-$ Al-Oboudi operator is defined by Aouf et al. [8] as:

$$D^m_{\delta,q}F(z) = z + \sum_{k=2}^{\infty}[1+\delta([k]_q - 1)]^m a_k z^k,\ \delta \geq 0,\ m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}. \tag{4}$$

Note that: $\lim_{q\to 1-} D^m_{\delta,q}F(z) = D^m_\delta F(z)$ (see Al-Oboudi [1], Aouf and Mostafa [6]; $D^m_{1,q}F(z) = D^m_q F(z)$ (see [14], [32] and [7]) and $\lim_{q\to 1-} D^m_q F(z) = D^m F(z)$ (see [28]).

Pommerenke [27] (see also [24]) defined the Hankel determinent for $\eta \geq 1$, $\gamma \geq 0$ as

$$H_\eta(\gamma) = \begin{vmatrix} a_\gamma & a_{\gamma+1} & a_{\gamma+\eta-1} \\ a_{\gamma+1} & a_{\gamma+2} & a_{\gamma+\eta} \\ & & \\ a_{\gamma+\eta-1} & a_{\gamma+\eta} & a_{\gamma+2\eta-2} \end{vmatrix} (a_1 = 1), \tag{5}$$

where $a_\gamma^{,}s$ are the coefficients of various power of z in $F(z)$ defined by (1).

This determinant has also been considered by several authors, for example $H_2(1) = a_3 - a_2^2$, is known as the Fekete-Szego functional (see Fekete-Szego [12] who generalized the estimate to $\left|a_3 - \mu a_2^2\right|$ where μ is real).

In the case $\eta = 2$ and $\gamma = 2$, the Hankel determinant $H_\eta(\gamma)$ is

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = \left|a_2 a_4 - a_3^2\right|. \tag{6}$$

For more studies of $H_\eta(\gamma)$ see [11, 21, 25].

Using (3) and (4), we define the following class.

Definition 1 Let $F \in S$, $\zeta, \delta \geq 0$, $m \in \mathbb{N}_0$. Then $F \in \mathcal{M}^m_q(\delta,\zeta)$ if and only if

$$Re\{\frac{z\Delta_q(D^m_{\delta,q}F(z))}{D^m_{\delta,q}F(z)} + \zeta\frac{z^2\Delta_q(\Delta_q D^m_{\delta,q}F(z))}{D^m_{\delta,q}F(z)}\} > 0. \tag{7}$$

Note that:

(i) $\mathcal{M}^m_q(\delta,0) = \mathcal{M}^m_q(\delta) = \{F \in S : Re[\frac{z\Delta_q(D^m_{\delta,q}F(z))}{D^m_{\delta,q}F(z)}] > 0\}$;

(ii) $\mathcal{M}^0_q(\delta,0) = \mathcal{M}_q(0)$ (see Seoudy and Aouf [29]);

(iii) $\mathcal{M}^m_q(1,\zeta) = \mathcal{M}^m_q(\zeta) = \{F \in S : Re[\frac{z\Delta_q(D^m_q F(z))}{D^m_q F(z)} + \zeta\frac{z^2\Delta_q(\Delta_q D^m_q F(z))}{D^m_q F(z)}] > 0\}$;

(iv) $lim_{q\to 1-}\mathcal{M}^m_q(\delta,\zeta) = \mathcal{M}^m(\delta,\zeta) = \{F \in S : Re[\frac{z(D^m_\delta F(z))^{'}}{D^m_\delta F(z)} + \zeta\frac{z^2(D^m_\delta F(z))^{''}}{D^m_\delta F(z)}] > 0\}$.

So, see that (7) modifies the definition of Patil and Khairnar [26].

2. Main Results

Unless indicated, we assume that $0 < q < 1$, $m \in \mathbb{N}_0$, $z \in \mathcal{D}$, $F(z)$ given by (1) and $\delta, \zeta \geq 0$.

To prove our main results we shall need the following lemmas. Let P be the family of all functions p analytic in $\mathcal{D}$ for which $R\{p(z)\} > 0$ and

$$p(z) = 1 + c_1 z + c_2 z^2 + ... \tag{8}$$

Lemma 1 [10] Let $p \in P$, then $|c_k| \leq 2,\ k = 1, 2, ...$ and the inequality is sharp.

Lemma 2[22] Let $p \in P$, then

$$\begin{aligned} 2c_2 &= c_1^2 + x(4 - c_1^2) \\ 4c_3 &= c_1^3 + 2xc_1(4 - c_1^2) - x^2 c_1(4 - c_1^2) + 2y(1 - |x|^2)(4 - c_1^2) \end{aligned} \tag{9}$$

for some x and y such that $|x| \leq 1$, $|y| \leq 1$.
Lemma 3 [23] If $p \in P$ is of the form (8) and ν is a complex number, then

$$\left|c_2 - \nu c_1^2\right| \leq 2 \max\{1; |2\nu - 1|\}.$$

Theorem 1 Let $F(z) \in \mathcal{M}_q^m(\delta, \zeta)$, then

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{4}{(1 + ([3]_q - 1)\delta)^{2m}([2]_q[3]_q\zeta + ([3]_q - 1))^2}. \tag{10}$$

Proof. Let $F(z) \in \mathcal{M}_q^m(\delta, \zeta)$ then, there exist $p(z) \in P$ such that

$$z\Delta_q(D_{\delta,q}^m F(z)) + \zeta z^2 \Delta_q(\Delta_q D_{\delta,q}^m F(z)) = D_{\delta,q}^m F(z)p(z) \quad for\ some\ z \in \mathcal{D}. \tag{11}$$

Therefore,

$$\begin{aligned} z\Delta_q(D_{\delta,q}^m F(z)) + \zeta z^2 \Delta_q(\Delta_q D_{\delta,q}^m F(z)) &= z + (1 + q\delta)^m a_2 [2]_q (1 + \zeta) z^2 \\ &+ (1 + ([3]_q - 1)\delta)^m a_3 [3]_q (1 + [2]_q \zeta) z^3 \\ &+ (1 + ([4]_q - 1)\delta)^m a_4 [4]_q (1 + [3]_q \zeta) z^4 + ... \end{aligned} \tag{12}$$

and

$$\begin{aligned} D_{\delta,q}^m F(z)p(z) &= z + (c_1 + (1 + q\delta)^m a_2) z^2 \\ &+ (c_2 + c_1 a_2 (1 + q\delta)^m + (1 + ([3]_q - 1)\delta)^m a_3) z^3 \\ &+ (c_3 + c_2 a_2 (1 + q\delta)^m + c_1 a_3 (1 + ([3]_q - 1)\delta)^m + (1 + ([4]_q - 1)\delta)^m a_4) z^4 + ... \end{aligned} \tag{13}$$

Equating the coefficients of (12) and (13):

$$a_2 = \frac{c_1}{(1 + q\delta)^m([2]_q\zeta + q)}, \tag{14}$$

$$\begin{aligned} a_3 &= \frac{c_2}{(1 + ([3]_q - 1)\delta)^m([2]_q[3]_q\zeta + ([3]_q - 1))} \\ &+ \frac{c_1^2}{(1 + ([3]_q - 1)\delta)^m([2]_q\zeta + q)([2]_q[3]_q\zeta + ([3]_q - 1))}, \end{aligned} \tag{15}$$

and

$$\begin{aligned} a_4 &= \frac{c_1^3}{(1 + ([4]_q - 1)\delta)^m([3]_q[4]_q\zeta + ([4]_q - 1))([2]_q\zeta + q)([2]_q[3]_q\zeta + ([3]_q - 1))} \\ &+ \frac{c_1 c_2([2]_q\zeta(1 + [3]_q) + q + [3]_q - 1)}{(1 + ([4]_q - 1)\delta)^m([3]_q[4]_q\zeta + ([4]_q - 1))([2]_q\zeta + q)([2]_q[3]_q\zeta + ([3]_q - 1))} \\ &+ \frac{c_3}{(1 + ([4]_q - 1)\delta)^m([3]_q[4]_q\zeta + ([4]_q - 1))}. \end{aligned} \tag{16}$$

From (14), (15) and (16), we have

$$\left|a_2a_4-a_3^2\right| = \left|\begin{array}{c} \frac{c_1^4}{(1+([4]_q-1)\delta)^m(1+q\delta)^m([3]_q[4]_q\zeta+([4]_q-1))([2]_q\zeta+q)^2([2]_q[3]_q\zeta+([3]_q-1))} \\ +\frac{c_1^2c_2([2]_q\zeta(1+[3]_q)+q+[3]_q-1)}{(1+([4]_q-1)\delta)^m(1+q\delta)^m([3]_q[4]_q\zeta+([4]_q-1))([2]_q\zeta+q)^2([2]_q[3]_q\zeta+([3]_q-1))} \\ +\frac{c_1c_3}{(1+([4]_q-1)\delta)^m(1+q\delta)^m([3]_q[4]_q\zeta+([4]_q-1))([2]_q\zeta+q)} \\ -\{[\frac{c_2}{(1+([3]_q-1)\delta)^m([2]_q[3]_q\zeta+([3]_q-1))}+\frac{c_1^2}{(1+([3]_q-1)\delta)^m([2]_q[3]_q\zeta+([3]_q-1))([2]_q\zeta+q)}]^2\} \end{array}\right|. \tag{17}$$

By using Lemma 2,

$$\left|a_2a_4-a_3^2\right| = \left|\begin{array}{c} \frac{c_1^4}{(1+([4]_q-1)\delta)^m(1+q\delta)^m([3]_q[4]_q\zeta+([4]_q-1))([2]_q\zeta+q)^2([2]_q[3]_q\zeta+([3]_q-1))} \\ +\frac{c_1^2([2]_q\zeta(1+[3]_q)+q+[3]_q-1)[\frac{c_1^2+x(4-c_1^2)}{2}]}{(1+([4]_q-1)\delta)^m(1+q\delta)^m([3]_q[4]_q\zeta+([4]_q-1))([2]_q\zeta+q)^2([2]_q[3]_q\zeta+([3]_q-1))} \\ +\frac{c_1[\frac{c_1^3+2xc_1(4-c_1^2)-x^2c_1(4-c_1^2)+2y(1-|x|^2)(4-c_1^2)}{4}]}{(1+([4]_q-1)\delta)^m(1+q\delta)^m([3]_q[4]_q\zeta+([4]_q-1))([2]_q\zeta+q)} \\ -\frac{c_1^4}{(1+([3]_q-1)\delta)^{2m}([2]_q[3]_q\zeta+([3]_q-1))^2([2]_q\zeta+q)^2} \\ -\frac{[\frac{c_1^2+x(4-c_1^2)}{2}]^2}{(1+([3]_q-1)\delta)^{2m}([2]_q[3]_q\zeta+([3]_q-1))^2}-\frac{2c_1^2[\frac{c_1^2+x(4-c_1^2)}{2}]}{(1+([3]_q-1)\delta)^{2m}([2]_q[3]_q\zeta+([3]_q-1))^2([2]_q\zeta+q)} \end{array}\right|. \tag{18}$$

Substituting for c_2 and c_3 from (9) and since $|c_1| \leq 2$ by Lemma 1, let $c_1 = c$ and assuming without restriction that $c \in [0, 2]$ we obtain, by triangle inequality,

$$\begin{aligned} \left|a_2a_4-a_3^2\right| \leq\ & \frac{c^4}{(1+([4]_q-1)\delta)^m(1+q\delta)^m([3]_q[4]_q\zeta+([4]_q-1))([2]_q\zeta+q)^2([2]_q[3]_q\zeta+([3]_q-1))} \\ & + \frac{c^4([2]_q\zeta(1+[3]_q)+q+[3]_q-1)}{2(1+([4]_q-1)\delta)^m(1+q\delta)^m([3]_q[4]_q\zeta+([4]_q-1))([2]_q\zeta+q)^2([2]_q[3]_q\zeta+([3]_q-1))} \\ & + \frac{pc^2(4-c^2)([2]_q\zeta(1+[3]_q)+q+[3]_q-1)}{2(1+([4]_q-1)\delta)^m(1+q\delta)^m([3]_q[4]_q\zeta+([4]_q-1))([2]_q\zeta+q)^2([2]_q[3]_q\zeta+([3]_q-1))} \\ & + \frac{c^4+2pc^2(4-c^2)-p^2c^2(4-c^2)+2(4-c^2)(1-p^2)}{4(1+([4]_q-1)\delta)^m(1+q\delta)^m([3]_q[4]_q\zeta+([4]_q-1))([2]_q\zeta+q)} \\ & + \frac{c^4}{(1+([3]_q-1)\delta)^{2m}([2]_q[3]_q\zeta+([3]_q-1))^2([2]_q\zeta+q)^2} \\ & + \frac{c^2(c^2+p(4-c^2)}{(1+([3]_q-1)\delta)^{2m}([2]_q[3]_q\zeta+([3]_q-1))^2([2]_q\zeta+q)} \\ & + \frac{c^4+2pc^2(4-c^2)+p^2(4-c^2)^2}{4(1+([3]_q-1)\delta)^{2m}([2]_q[3]_q\zeta+([3]_q-1))^2} \\ \leq\ & G(p), \end{aligned} \tag{19}$$

with $p = |x| \leq 1$. Furthermore,

$$\begin{aligned} G^{'}(p) &\leq \frac{c^2(4-c^2)([2]_q\zeta(1+[3]_q)+q+[3]_q-1)}{2(1+([4]_q-1)\delta)^m(1+q\delta)^m([3]_q[4]_q\zeta+([4]_q-1)) \\ ([2]_q\zeta+q)^2([2]_q[3]_q\zeta+([3]_q-1))} \\ &+ \frac{2c^2(4-c^2)-2pc^2(4-c^2)-4(4-c^2)p}{4(1+([4]_q-1)\delta)^m(1+q\delta)^m([3]_q[4]_q\zeta+([4]_q-1)) \\ ([2]_q\zeta+q)} \\ &+ \frac{c^2(4-c^2)}{(1+([3]_q-1)\delta)^{2m}([2]_q[3]_q\zeta+([3]_q-1))^2([2]_q\zeta+q)} \\ &+ \frac{2c^2(4-c^2)+2p(4-c^2)^2}{4(1+([3]_q-1)\delta)^{2m}([2]_q[3]_q\zeta+([3]_q-1))^2}. \end{aligned} \tag{20}$$

By elementary calculations, we can show that $G^{'}(p) \geq 0$ for $p > 0$, which implies that G is an increasing function and thus the upper bound for (17) corresponds to $p = 1$ & $c = 0$, we have (10).

Theorem 2 Let $F(z) \in \mathcal{M}_q^m(\delta, \zeta)$, then

$$\begin{aligned} \left|a_3 - \mu a_2^2\right| &\leq \frac{2}{(1+([3]_q-1)\delta)^m([2]_q[3]_q\zeta+([3]_q-1))} \max \\ &\{1; \left|1 + \frac{2}{([2]_q\zeta+q)}(1 - \frac{(1+([3]_q-1)\delta)^m([2]_q[3]_q\zeta+([3]_q-1))}{(1+q\delta)^{2m}([2]_q\zeta+q)}\mu)\right|\}. \end{aligned} \tag{21}$$

Proof. Since if $F(z) \in \mathcal{M}_q^m(\delta, \zeta)$, then a_2 and a_3 are given by (14) and (15), we have

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{c_2}{(1+([3]_q-1)\delta)^m([2]_q[3]_q\zeta+([3]_q-1))} \\ &+ \frac{c_1^2}{(1+([3]_q-1)\delta)^m([2]_q\zeta+q)([2]_q[3]_q\zeta+([3]_q-1))} \\ &- \mu\frac{c_1^2}{(1+q\delta)^{2m}([2]_q\zeta+q)^2}. \end{aligned} \tag{22}$$

Therefore,

$$\left|a_3 - \mu a_2^2\right| = \left|\frac{1}{(1+([3]_q-1)\delta)^m([2]_q[3]_q\zeta+([3]_q-1))}\{c_2 - \nu c_1^2\}\right|, \tag{23}$$

where

$$\nu = \frac{1}{([2]_q\zeta+q)}[\frac{(1+([3]_q-1)\delta)^m([2]_q[3]_q\zeta+([3]_q-1))}{(1+q\delta)^{2m}([2]_q\zeta+q)}\mu - 1]. \tag{24}$$

Our result now follows by an application of Lemma 3. This completes the proof of Theorem 2.

Remarks (i) Letting $q \to 1-$ in Theorem 1, we have the results obtained by [26];

(ii) For different values δ and ζ, we obtain results for the classes mentioned in the introduction.

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References

[1] F. M. Al-Oboudi, On univalent functions defined by a generalized Sălăgean operator, Int. J. Math. Math. Sci., 27 (2004), 1429–1436.

[2] M. H. Annby and Z. S. Mansour, $q-$Fractional Calculus Equations, Lecture Notes in Mathematics, Vol. 2056. Springer, Berlin , 2012.

[3] M. K. Aouf, H. E. Darwish and G. S. Sălăgean, On a generalization of starlike functions with negative coefficients, Math. Tome, 43 66 (2001), no. 1, 3–10.

[4] M. K. Aouf and A. O. Mostafa, Subordination results for analytic functions associated with fractional $q-$calculus operators with complex order, Afr. Mat., 31 (2020), 1387–1396.

[5] M. K. Aouf and A. O. Mostafa, Some subordinating results for classes of functions defined by Sălăgean type $q-$derivative operator, Filomat, 34 (2020), no. 7, 2283–2292.

[6] M. K. Aouf and A. O. Mostafa, Some properties of a subclass of uniformly convex functions with negative coefficients, Demonstratio Math., 41 (2008), no. 2, 353-370.

[7] M. K. Aouf, A. O. Mostafa and F. Y. AL-Quhali, Properties for class of $\beta-$ uniformly univalent functions defined by Sălăgean type $q-$difference operator, Int. J. Open Probl. Complex Anal. 11 (2019), no. 2, 1–16.

[8] M. K. Aouf, A. O. Mostafa and R. E. Elmorsy, Certain subclasses of analytic functions with varying arguments associated with $q-$difference operator, Afrika Math., (To apear).

[9] A. Aral, V. Gupta and R. P. Agarwal, Applications of $q-$Calculus in Operator Theory, Springer, New York, 2013.

[10] P. L. Duren, Univalent Functions. Grundlehren der Mathematishen Wissenschaften, New York: Springer, 1983.

[11] R. Ehrenborg, The Hankel determinant of exponantial polynomials, American Mathematical Monthly, 107 (2000), 557–560.

[12] M. Fekete and G. Szego, Eine Bemerkung uber ungerade schlichte Funktionen, J. London Math. Soc., 8 (1933), 85–89.

[13] B. A. Frasin and G. Murugusundaramoorthy, A subordination results for a class of analytic functions defined by $q-$differential operator, Ann. Univ. Paedagog. Crac. Stud. Math., 19 (2020), 53-64.

[14] M. Govindaraj and S. Sivasubramanian, On a class of analytic function related to conic domains involving $q-$calculus, Anal. Math., 43 (2017), no. 2, 475–487.

[15] R. W. Ibrahim, Classes of quantum integral operator in a complex domain, J. Fractional Calculus and Appl., 12(2021), no. 1101-109.

[16] R. W. Ibrahim and D. Baleanu, On quantum hybrid fractional conformable differential and integral operators in a complex domain, Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales.Serie A. Matematicas, 115(2021), no. 1, 1-13.

[17] R. W. Ibrahim, R. M. Elobaid and S. J. Obaiys, A class of quantum Briot–Bouquet differential equations with complex coefficients, Mathematics, 8(2020), no. 5, 1-13.

[18] R. W. Ibrahim, R. M. Elobaid, and S. J. Obaiys, On subclasses of analytic functions based on a quantum symmetric conformable differential operator with application, Advances in Difference Equations, (2020), no. 1, 1-14.

[19] R. W. Ibrahim, S. B. Hadid, and S. Momani, Generalized Briot–Bouquet differential equation by a quantum difference operator in a complex domain, Internat. J. Dynamics and Control (2020), 1-10.

[20] F. H. Jackson, On $q-$functions and a certain difference operator, Trans. R. Soc. Edinb., 46 (1908), 253–281.

[21] J. W. Layman, The Hankel transform and some of its properties, J. Integer Sequences, 4 (2001), 1–11.

[22] R. J. Libera and E. J. Zlotkiewicz, Coefficient bounds for the inverse of a function with derivative in P, Proc. Amer. Math. Soc., 87 (1983), no. 2, 251- 257.

[23] W. C. Ma and D. Minda, A unified treatment of some special classes of univalent functions, in Proceedings of the Conference on Complex Analysis, Internat. Press, Cambridge, Ma (Tianjin, 1992), 157–169.

[24] J. W. Noonan and D. K. Thomas, On the second Hankel determinant of a really mean $p-$valent functions, Trans. Amer. Math. Soc., 223 (1976), no.2, 337–346.

[25] K. I. Noor, Hankel determinant problem for the class of functions with bounded boundary rotation, Rev. Roum. Math. Pures Et Appl., 28 (1983), no. 8, 731–739.

[26] S. M. Patile and S. M. Khairnar, Second Hankel determinant for certain classes of analytic functions, Internat. J. Comput. Appl. Math., 12 (2017), no. 3, 675–680.

[27] Ch. Pommerenke, Univalent Function, Vandenhoek and Ruprecht, Gottingen, 1975.

[28] G. Sălăgean, Subclasses of univalent functions, Lecture note in Math., Springer-Verlag, 1013(1983), 362-372.

[29] T. M. Seoudy and M. K. Aouf, Coefficient estimates of new classes of $q-$starlike and $q-$convex functions of complex order, J. Math. Inequal., 10 (2016), no. 1, 135–145.

[30] H. M. Srivastava, Operators of basic (or $q-$) calculus and fractional $q-$calculus and their applications in geometric function theory of complex analysis, Iran. J. Sci. Technol. Trans. Sci., 44(2020), 327–344.

[31] H. M. Srivastava, A. O. Mostafa, M. K. Aouf and H. M. Zayed, Basic and fractional $q-$calculus and associated Fekete–Szego problem for $p-$valently $q-$starlike functions and $p-$valently $q-$convex functions of complex order, Miskolc Math. Notes, 20 (2019), no. 1, 489–509.

[32] K. Vijaya, M. Kasthuri and G. Murugusundaramoorthy, Coefficient bounds for subclasses of bi-univalent functions defined by the Sălăgean derivative operator, Boletin de la Asociaciton, Matematica Venezolana, 21(2014), no. 2, 1-9.

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