# POSITIVE SOLUTIONS FOR SINGULAR $\alpha$ -ORDER $(2 \leq \alpha \leq 3)$ FRACTIONAL BOUNDARY VALUE PROBLEMS ON THE HALF-LINE

ABDELHAMID BENMEZAI, SOUAD CHENTOUT

ABSTRACT. This article deals with existence of positive solutions to the fractional boundary value problem

$$\begin{cases} D^{\alpha}u(t) + f(t, u(t)) = 0, \ 0 & 0 \leq t < \infty \\ u(0) = D^{\alpha-2}u(0) = \lim_{t \to \infty} D^{\alpha-1}u(t) = 0 \end{cases}$$

where $\alpha \in [2, 3]$, $D^{\alpha}$ is the standard Riemann-Liouville derivative and $f : (0, +\infty) \times (0, +\infty) \to \mathbb{R}^{+}$ is a continuous function and may exhibit singular at $u = 0$. The main existence result is obtained by means of Guo-Krasnoselskii's version of expansion and compression of a cone principal in a Banach space.

## 1. INTRODUCTION AND MAIN RESULTS

In the last few decades, fractional differential equations have gained a considerable interest and importance, since they arise from many physical applications. Physical experimentation showed that the integral and derivative operators of fractional order do share some of the characteristics exhibited by the processes associated with complex systems having long-memory in time and fractional calculus provide an excellent framework to describe the hereditary properties of various materials and processes. For recent developments in the theory fractional calculus and its applications, we refer to [2, 7, 8, 9, 12, 13, 14].

Often, for physical considerations, the positivity of the solution is required. This why existence of positive solutions for various classes of boundary value problems associated with fractional differential equations has been the subject many papers, see, [1, 3, 4, 6, 10, 11, 15, 16] and references therein. However, to the best of our knowledge, there are no works considering existence of positive solutions in the case where such boundary value problems are posed on infinite intervals and having a singular dependence on the variable space. Thus, the purpose of this paper is to fill in the gap in this area.

We are concerned in this paper with existence of positive solutions to the fractional boundary value problem (fbvp for short),

$$\begin{cases} D^\alpha u(t) + f(t, u(t)) = 0, \ t \in I \\ u(0) = D^{\alpha-2}u(0) = \lim_{t\to\infty} D^{\alpha-1}u(t) = 0 \end{cases} \tag{1.1}$$

where $I = (0, +\infty)$, $\alpha \in [2, 3]$, $D^\alpha$ is the standard Riemann-Liouville derivative and $f : I \times I \to \mathbb{R}^+$ is a continuous function.

Our approach in this work is based on a fixed point formulation of the fbvp (1.1) and the main existence result in this work is then proved by the Guo-Krasnoselskii's version of expansion and compression of a cone principal in a Banach space.

Set

$$\mathcal{Q}(\alpha) = \left\{ q \in C\left(I, \mathbb{R}^+\right) : \int_0^{+\infty} q(s)\left(1+s\right)^{\alpha-1} ds < \infty \right\}$$

and assume that the nonlinearity $f$ satisfies the following hypothesis:

$$\begin{cases} \text{for all } R > 0 \text{ there exists two functions } \omega_R, \Psi_R : I \to I \\ \text{such that } \Psi_R \text{ is nonincreasing,} \\ f(t, (1+t)^{\alpha-1} u) \le (1+t)^{\alpha-1} \omega_R(t) \Psi_R(u) \text{ for all } t > 0 \text{ and } u \in (0, R] \\ \text{and } \Phi_{r,R} \in \mathcal{Q}(\alpha) \text{ for all } r \in (0, R], \end{cases} \tag{1.2}$$

where

$$\begin{aligned} \Phi_{r,R}(t) &= \omega_R(t) \Psi_R\left(r\widetilde{\gamma}(t)\right), \\ \widetilde{\gamma}(t) &= \frac{\gamma(t)}{(1+t)^{\alpha-1}} \text{ and} \\ \gamma(t) &= \min\left(1, t^{\alpha-1}\right). \end{aligned}$$

The statement of the main existence result in this work needs to introduce the following additional notations. Set for $q \in \mathcal{Q}(\alpha)$, $\theta > 1$ and $\nu = 0, \infty$, $I_\theta = [1/\theta, \theta]$,

$$\begin{aligned} f^\nu(q) &= \limsup_{u\to\nu} \left( \max_{t\ge0} \frac{f(t, (1+t)^{\alpha-1} u)}{(1+t)^{\alpha-1} q(t)u} \right), \\ f_\nu(q, \theta) &= \liminf_{u\to\nu} \left( \min_{t\in I_\theta} \frac{f(t, (1+t)^{\alpha-1} u)}{(1+t)^{\alpha-1} q(t)u} \right), \\ \Delta(q) &= \sup_{t\ge0} \left( \frac{1}{(1+t)^{\alpha-1}} \int_0^t G(t,s)\left(1+s\right)^{\alpha-1} q(s)ds \right), \\ \Theta(q, \theta) &= \sup_{t\ge0} \left( \frac{1}{(1+t)^{\alpha-1}} \int_{1/\theta}^\theta G(t,s)\left(1+s\right)^{\alpha-1} p(s)\widetilde{\gamma}(s)ds \right). \end{aligned}$$

**Theorem 1.1.** *Assume that Hypothesis (1.2) holds and there exist two functions $p, q$ in $\mathcal{Q}(\alpha)$ such that one of the following Hypotheses (1.3) or (1.4) holds true;*

$$f^0(q)\Delta(q) < 1 < f_\infty(p, \theta)\Theta(p, \theta) \tag{1.3}$$

*or*

$$f^\infty(q)\Delta(q) < 1 < f_0(p, \theta)\Theta(p, \theta). \tag{1.4}$$

*Then the fbvp (1.1) admits at least one unbounded increasing positive solution.*

For the typical case of the fbvp (1.1) where $f(t, u) = (1+t)^{\alpha-1} p(t)u^\rho$ with $\rho < 0$ and $p \in C\left(\mathbb{R}^+, \mathbb{R}^+\right)$, we obtain from Theorem 1.1 the following corollary:

**Corollary 1.2.** *Assume that $f(t, u) = (1+t)^{\alpha-1} p(t)u^\rho$ where $\rho < 0$ and $p \in C\left(I, \mathbb{R}^+\right)$. If*

$$\int_0^1 t^{\rho(\alpha-1)}p(t)dt < \infty \text{ and } \int_1^{+\infty} (1+t)^{(1-\rho)(\alpha-1)} p(t)dt < \infty,$$

*then the fbvp (1.1) admits at least one unbounded increasing positive solution.*

*Proof.* We have for all $R > 0$

$$R^\rho \int_0^{+\infty} (1+t)^{\alpha-1} p(t)dt \leq \int_0^{+\infty} (1+t)^{\alpha-1} p(t) (R\widetilde{\gamma}(t))^\rho dt$$
$$\leq 2^{(1-\rho)(\alpha-1)} R^\rho \int_0^1 t^{\rho(\alpha-1)} p(t)dt + R^\rho \int_0^{+\infty} (1+t)^{(1-\rho)(\alpha-1)} p(t)dt < \infty.$$

This shows that $p \in \mathcal{Q}(\alpha)$ and Hypothesis (1.2) holds with $\omega_R(t) = p(t)$ and $\Psi_R(u) = u^\rho$. We have also,

$$f^0(p) = f_0(p,\theta) = +\infty \text{ and } f^\infty(p) = f_\infty(p,\theta) = 0 \text{ for all } \theta > 1,$$

proving that Condition (1.4) is satisfied. This ends the proof.          □

## 2. Abstract background

Let $(E, ||.||)$ be a real Banach space. A nonempty closed convex subset $C$ of $E$ is said to be a cone in $E$ if $C \cap (-C) = \{0_E\}$ and $tC \subset C$ for all $t \geq 0$.

Let $\Omega$ be a nonempty subset in $E$. A mapping $A : \Omega \to E$ is said to be compact if it is continuous and $A(\Omega)$ is relatively compact in $E$.

The main tool of this work is the following Guo-Krasnoselskii's version of expansion and compression of a cone principal in a Banach space.

**Theorem 2.1.** *Let $P$ be a cone in $E$ and let $\Omega_1, \Omega_2$ be bounded open subsets of $E$ with $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$. If $T : P \cap (\overline{\Omega_2} \backslash \Omega_1) \to P$ is a compact mapping such that either:*

(1) *$||Tu|| \leq ||u||$ for $u \in P \cap \partial\Omega_1$ and $||Tu|| \geq ||u||$ for $u \in P \cap \partial\Omega_2$, or*
(2) *$||Tu|| \geq ||u||$ for $u \in P \cap \partial\Omega_1$ and $||Tu|| \leq ||u||$ for $u \in P \cap \partial\Omega_2$,*

*Then $T$ has at least one fixed point in $P \cap (\overline{\Omega_2} \backslash \Omega_2)$.*

## 3. Riemann-Liouville fractional derivative

Now, let us recall some basic facts related to the theory of fractional differential equations. Let $\beta$ be a positive real number, the Riemann-Liouville fractional integral of order $\beta$ of a function $f : (0, +\infty) \to \mathbb{R}$ is defined by

$$I_{0+}^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s)ds,$$

where $\Gamma(\beta)$ is the gamma function, provided that the right side is pointwise defined on $(0, +\infty)$. For example, we have for any real $\sigma > -1$, $I_{0+}^\beta t^\sigma = \frac{\Gamma(\sigma+1)}{\Gamma(\sigma+\beta+1)} t^{\sigma+\beta}$.

The Riemann-Liouville fractional derivative of order $\beta \geq 0$, of a continuous function
$f : (0, +\infty) \to \mathbb{R}$ is given by

$$D_{0+}^\beta f(t) = \frac{1}{\Gamma(n-\beta)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t-s)^{\beta-n+1}} ds,$$

where $n = [\beta] + 1$, $[\beta]$ denotes the integer part of the number $\beta$. See that $D^0 f = f$.

As a basic example, we quote for $\sigma > \beta - 1$, $D_{0+}^\beta t^\sigma = \frac{\Gamma(\sigma+1)}{\Gamma(\sigma-\beta+1)} t^{\sigma-\beta}$. Thus, if $u \in C(0, +\infty) \cap \mathbb{L}^1(0, +\infty)$, then the fractional differential equation $D_{0+}^\beta u(t) = 0$

has $u(t) = \sum_{i=1}^{i=[\beta]+1} c_i t^{\beta-i}$, $c_i \in \mathbb{R}$, as unique solution and if $u$ has a fractional derivative of order $\beta$ in $C(0, +\infty) \cap \mathbb{L}^1(0, +\infty)$, then

$$I_{0+}^{\beta} D_{0+}^{\beta} u(t) = u(t) + \sum_{i=1}^{i=[\beta]+1} c_i t^{\beta-i}, \; c_i \in \mathbb{R}. \qquad (3.1)$$

For a detailed presentation on fractional differential calculs, see [8] or [13].

## 4. FIXED POINT FORMULATION

Firstly, we introduce the necessary framework for the fixed point formulation of the fbvp (1.1). Throughout, we let $E$ be the linear space defined by

$$E = \left\{ u \in C(\mathbb{R}^+, \mathbb{R}) : \lim_{t \to \infty} \frac{u(t)}{t^{\alpha-1}} = 0 \in \mathbb{R} \right\},$$

Equipped with the norm $\|\cdot\|_E$ where for all $u \in E$, $\|u\|_E = \sup_{t>0} \frac{|u(t)|}{(1+t)^{\alpha-1}}$, $E$ becomes a Banach space.

In all what follows $E^+$ denote the cone of nonnegative functions in $E$ and the subset $P$ of $E$ defined by

$$P = \{ u \in E : u(t) \geq \gamma(t) \|u\|_E \text{ for all } t \geq 0 \}$$

is a cone in $E$.

Let $G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ the function given by

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1} - (t-s)^{\alpha-1} & 0 \leq s \leq t < \infty \\ t^{\alpha-1} & 0 \leq t \leq s < \infty. \end{cases}$$

**Lemma 4.1.** *(Lemma 1, [3])The function $G$ is continuous and has the following properties:*

$$G(0,s) = 0 \text{ for all } s \geq 0, \qquad (4.1)$$

$$0 < G(t,s) \leq \frac{t^{\alpha-1}}{\Gamma(\alpha)} \text{ for all } t, s \geq 0, \qquad (4.2)$$

$$\lim_{t \to 0} \frac{G(t,s)}{t^{\alpha-1}} = \frac{1}{\Gamma(\alpha)}, \quad \lim_{t \to +\infty} \frac{G(t,s)}{t^{\alpha-1}} = 0 \text{ for all } s \geq 0, \qquad (4.3)$$

$$G(t,s) \geq \gamma(t) \frac{G(\tau,s)}{(1+\tau)^{\alpha-1}} \text{ for all } t, \tau, s \geq 0. \qquad (4.4)$$

The following lemma is an adapted version for the case of the space $E$ of Corduneanu's compactness criterion ([5], p. 62). It will be used in this work to prove that some operator is completely continuous.

**Lemma 4.2.** *A nonempty subset $M$ of $E$ is relatively compact if the following conditions hold:*

(a) *$M$ is bounded in $E$,*

(b) *the functions belonging to $\left\{ u : u(t) = \dfrac{x(t)}{(1+t)^{\alpha-1}}, \; x \in M \right\}$ are locally equicontinuous on $[0, +\infty)$, that is, equicontinuous on every compact interval of $\mathbb{R}^+$ and*

(c) *the functions belonging to* $\left\{ u : u(t) = \dfrac{x(t)}{(1+t)^{\alpha-1}}, \ x \in M \right\}$ *are equiconver-*
*gent at* $+\infty$, *that is, given* $\epsilon > 0$, *there corresponds* $T(\epsilon) > 0$ *such that* $|x(t) - x(+\infty)| < \epsilon$ *for any* $t \geq T(\epsilon)$ *and* $x \in M$.

**Lemma 4.3.** *Assume that Hypothesis (1.2) holds, then there exists a continuous operator* $T : K\{0\} \to K$ *such that for all* $r, R$ *with* $0 < r < R$, $T(K \cap (B(0, R)B(0, r)))$ *is relatively compact and fixed points of* $T$ *are positive solutions to the fbvp (1.1).*

*Proof.* Let $u \in K\{0\}$ and let $\Phi_r$ be the function given by Hypothesis (1.2) for $r = \|u\|$. For all $t > 0$, we have

$$\int_0^{+\infty} G(t,s)f(s,u(s))ds = \int_0^{+\infty} G(t,s)f(s,(1+s)^{\alpha-1}\frac{u(s)}{(1+s)^{\alpha-1}})ds$$
$$\leq \int_0^{+\infty} G(t,s)(1+s)^{\alpha-1}\Phi_r(s)ds$$
$$\leq \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^{+\infty}(1+s)^{\alpha-1}\Phi_r(s)ds < \infty.$$

Set

$$v(t) = \int_0^{+\infty} G(t,s)f(s,u(s))ds.$$

Clearly, $v$ is continuous and for all $t \geq 0$, we have from (4.2)

$$\frac{v(t)}{(1+t)^{\alpha-1}} = \int_0^{+\infty} G(t,s)f(s,u(s))ds \leq \frac{1}{\Gamma(\alpha)}\frac{t^{\alpha-1}}{(1+t)^{\alpha-1}}\int_0^{+\infty}(1+s)^{\alpha-1}\Phi_r(s)ds$$
$$\leq \frac{1}{\Gamma(\alpha)}\int_0^{+\infty}(1+s)^{\alpha-1}\Phi_r(s)ds < \infty.$$

Moreover, it follows from (4.4) that for all $t, \tau \geq 0$

$$v(t) = \int_0^{+\infty} G(t,s)f(s,u(s))ds \geq \frac{\gamma(t)}{(1+\tau)^{\alpha-1}}\int_0^{+\infty} G(\tau,s)f(s,u(s))ds$$
$$= \gamma(t)\frac{v(\tau)}{(1+\tau)^{\alpha-1}}.$$

Passing to the supremum on $\tau$, we obtain $v(t) \geq \gamma(t)\|v\|$ for all $t \geq 0$, that is $v \in K$.

Thus, we have proved that the oerator $T : K\{0\} \longrightarrow K$, where for $u \in K\{0\}$ and $t \geq 0$

$$Tu(t) = \int_0^{+\infty} G(t,s)f(s,u(s))ds,$$

is well defined.

Now, let $r, R$ with $0 < r < R$, $\Omega = K \cap (B(0,R)B(0,r))$ and $\Phi_{r,R}$ be the function given by Hypothesis (1.2). For a sequence $(u_n) \subset \Omega$ such that $\lim u_n = u \in K\{0\}$, we have

$$\|Tu_n - Tu\|_E \leq \sup_{t \geq 0}\int_0^{+\infty}\frac{G(t,s)}{(1+t)^{\alpha-1}}|f(s,u_n(s)) - f(s,u(s))|\,ds$$
$$\leq \frac{1}{\Gamma(\alpha)}\sup_{t \geq 0}\frac{t^{\alpha-1}}{(1+t)^{\alpha-1}}\int_0^{+\infty}|f(s,u_n(s)) - f(s,u(s))|\,ds$$
$$\leq \int_0^{+\infty}|f(s,u_n(s)) - f(s,u(s))|\,ds,$$

$$|f(s,u_n(s)) - f(s,u(s))| \leq 2(1+s)^{\alpha-1}\Phi_{r,R}(s),$$
$$\int_0^{+\infty}(1+s)^{\alpha-1}\Phi_{r,R}(s)ds < \infty \text{ and}$$
$$\lim|g(s,u_n(s)) - g(s,u(s))| = 0 \text{ for all } s \geq 0.$$

Thus, we conclude by means of Lebesgue dominated convergence theorem that $\lim\|Tu_n - Tu\|_E = 0$. Proving the continuity of $T$ on $\Omega$.

For all $u \in \Omega$, we have then

$$\frac{|Tu(t)|}{(1+t)^{\alpha-1}} \leq \int_0^{+\infty} \frac{G(t,s)}{(1+t)^{\alpha-1}} f(s, (1+s)^{\alpha-1} \frac{u(s)}{(1+s)^{\alpha-1}}) ds$$
$$\leq \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} (1+s)^{\alpha-1} \Phi_{r,R}(s) ds < \infty.$$

This shows that $T\Omega$ is bounded.

We have then for all $u \in \Omega$This shows that the operator $T$ is bounded on $\Omega$.

Let $[\xi, \eta]$ be an interval of $\mathbb{R}^+$. For all $u \in \Omega$ and all $t_1, t_2 \in [\xi, \eta]$ with $0 < t_2 - t_1 < 1$, We have

$$\left| \frac{Tu(t_2)}{(1+t_2)^{\alpha-1}} - \frac{Tu(t_1)}{(1+t_1)^{\alpha-1}} \right| \leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left| \left( \frac{t_2-s}{1+t_2} \right)^{\alpha-1} - \left( \frac{t_1-s}{1+t_1} \right)^{\alpha-1} \right| \Phi_{r,R}(s) (1+s)^{\alpha-1} ds$$
$$+ \left( \frac{t_2-t_1}{1+t_2} \right)^{\alpha-1} |\Phi_R|_\alpha + \left| \left( \frac{t_2}{1+t_2} \right)^{\alpha-1} - \left( \frac{t_1}{1+t_1} \right)^{\alpha-1} \right| |\Phi_{r,R}|_\alpha,$$

where $|\Phi_{r,R}|_\alpha = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} (1+s)^{\alpha-1} \Phi_{r,R}(s) ds$.

We have by the mean value theorem:

$$\left| \left( \frac{t_2-s}{1+t_2} \right)^{\alpha-1} - \left( \frac{t_1-s}{1+t_1} \right)^{\alpha-1} \right| \leq (\alpha-1) \left( \frac{\eta}{1+\eta} \right)^{\alpha-2} \left| \frac{t_2-s}{1+t_2} - \frac{t_1-s}{1+t_1} \right|$$
$$\leq (\alpha-1) \left( \frac{\eta}{1+\eta} \right)^{\alpha-2} \frac{(t_2-t_1)(1+s)}{(1+t_2)(1+t_1)} \qquad (4.5)$$
$$\leq (\alpha-1) \left( \frac{\eta}{1+\eta} \right)^{\alpha-2} (t_2-t_1)$$

and

$$\left| \left( \frac{t_2}{1+t_2} \right)^{\alpha-1} - \left( \frac{t_1}{1+t_1} \right)^{\alpha-1} \right| \leq (\alpha-1) \left( \frac{\eta}{1+\eta} \right)^{\alpha-2} \left( \frac{t_2}{1+t_2} - \frac{t_1}{1+t_1} \right)$$
$$\leq (\alpha-1) \left( \frac{\eta}{1+\eta} \right)^{\alpha-2} (t_2-t_1).$$

The above calculations lead to

$$\left| \frac{Tu(t_2)}{(1+t_2)^{\alpha-1}} - \frac{Tu(t_1)}{(1+t_1)^{\alpha-1}} \right| \leq 2 \frac{(\alpha-1)}{\Gamma(\alpha)} \left( \frac{\eta}{1+\eta} \right)^{\alpha-2} |\Phi_{r,R}|_\alpha (t_2-t_1) + \left( \frac{t_2-t_1}{1+t_2} \right)^{\alpha-1} |\Phi_{r,R}|_\alpha$$
$$\leq \frac{1}{\Gamma(\alpha)} \left( 2(\alpha-1) \left( \frac{\eta}{1+\eta} \right)^{\alpha-2} + 1 \right) |\Phi_{r,R}|_\alpha (t_2-t_1)$$

Proving that $T(\Omega)$ is equicontinuous on compact intervals.

We have for any $u$ in $\Omega$ and $t \geq 0$

$$\left| \frac{Tu(t)}{(1+t)^{\alpha-1}} \right| \leq \int_0^{+\infty} \frac{G(t,s)}{(1+t)^{\alpha-1}} |g(s, u(s))| ds$$
$$\leq \int_0^{+\infty} \frac{G(t,s)}{(1+t)^{\alpha-1}} (1+s)^{\alpha-1} \Phi_{r,R}(s) ds = H(t).$$

Since $\Phi_{r,R} \in \mathcal{Q}(\alpha)$, the Property (4.3) of the function $G$ and the dominated convergence theorem lead to $\lim_{t \to \infty} H(t) = 0$, proving the equiconvergence $T\Omega$.

In view of Lemma 4.2 $T\Omega$ is relatively compact in $E$ and since $r, R$ are arbitrary, the operator $T$ is continuous.

Now, let $u \in K\{0\}$ be a fixed point of $T$. Therefore, we have

$$u(t) = \int_0^{+\infty} G(t,s) f(s, u(s)) ds$$
$$= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s)) ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^\infty f(s, u(s)) ds$$
$$= -I_{0^+}^\alpha f(t, u(t)) + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^\infty f(s, u(s)) ds.$$

$$D^{\alpha-2}u(t) = D^{\alpha-2}u(t) = -\int_0^t (t-s) f(s,u(s))ds + t\int_0^{+\infty} f(s,u(s))ds.$$
$$D^{\alpha-1}u(t) = -\int_0^t f(s,u(s))ds + \int_0^{+\infty} f(s,u(s))ds = \int_t^{+\infty} f(s,u(s))ds.$$
$$D^{\alpha}u(t) = -f(t,u(t)),$$
$$D^{\alpha-2}u(0) = \lim_{t\to+\infty} D^{\alpha-1}u(t) = 0$$

and we obtain from (4.1), $u(0) = \int_0^{+\infty} G(0,s)f(s,u(s))ds = 0$.

These show that $u$ is a positive solution to the fbvp (1.1), ending the proof. $\square$

## 5. Proof of Theorem 1.1

**Step 1. Existence in the case where (1.3) holds**

Let $\epsilon > 0$ be such that $(f^0(q)+\epsilon)\Delta(q) < 1$. For such a $\epsilon$, there exists $R_1 > 0$ such that $f(t,(1+t)^{\alpha-1}w) \le (f^0(q)+\epsilon)(1+t)^{\alpha-1}q(t)w$ for all $w \in (0,R_1)$. Thus, for all $u \in K \cap \partial\Omega_1$, where $\Omega_1 = \{u \in E, \|u\| < R_1\}$, the following estimates hold.

$$
\begin{aligned}
\|Tu\| &= \sup_{t\ge 0}\left(\frac{1}{(1+t)^{\alpha-1}}\int_0^{+\infty} G(t,s)f(s,u(s))ds\right) \\
&\le \sup_{t\ge 0}\left(\frac{1}{(1+t)^{\alpha-1}}\int_0^{+\infty} G(t,s)f(s,(1+s)^{\alpha-1}\frac{u(s)}{(1+s)^{\alpha-1}})ds\right) \\
&\le \sup_{t\ge 0}\left(\frac{1}{(1+t)^{\alpha-1}}\int_0^t G(t,s)(f^0(q)+\epsilon)(1+s)^{\alpha-1}q(s)\frac{u(s)}{(1+s)^{\alpha-1}}ds\right) \\
&\le (f^0(q)+\epsilon)\sup_{t\ge 0}\left(\frac{1}{(1+t)^{\alpha-1}}\int_0^t G(t,s)(1+s)^{\alpha-1}q(s)ds\right)\|u\| \\
&\le (f^0(q)+\epsilon)\Delta(q)\|u\| \le \|u\|.
\end{aligned}
$$

Now, suppose that $f_\infty(p,\theta) > \Theta(p,\theta)$ and let $\varepsilon > 0$ be such that $(f_\infty(p,\theta)-\varepsilon) > \Theta(p,\theta)$. There exists $R_2 > R_1$ such that $f(t,(1+t)^{\alpha-1}w) \ge (f_\infty(\beta,,\theta)-\varepsilon)(1+t)^{\alpha-1}p(t)w$ for all $t \in I_\theta$ and $w \ge 0$ with $w \ge R_2$. Let $\Omega_2 = \{u \in E : \|u\| < R_2/\gamma_*\}$, where $\gamma_* = \inf_{t\in I_\theta}\widetilde{\gamma}(t)$. For all $u \in K \cap \partial\Omega_2$, we have

$$
\begin{aligned}
\|Tu\| &\ge \sup_{t\ge 0}\left(\frac{1}{(1+t)^{\alpha-1}}\int_{1/\theta}^\theta G(t,s)f(s,u(s))ds\right) \\
&\ge \sup_{t\ge 0}\left(\frac{1}{(1+t)^{\alpha-1}}\int_{1/\theta}^\theta G(t,s)f(s,(1+s)^{\alpha-1}\frac{u(s)}{(1+s)^{\alpha-1}})ds\right) \\
&\ge (f_\infty(p,\theta)-\varepsilon)\sup_{t\ge 0}\left(\frac{1}{(1+t)^{\alpha-1}}\int_{1/\theta}^\theta G(t,s)(1+s)^{\alpha-1}p(s)\frac{u(s)}{(1+s)^{\alpha-1}}ds\right) \\
&\ge (f_\infty(p,\theta)-\varepsilon)\sup_{t\ge 0}\left(\frac{1}{(1+t)^{\alpha-1}}\int_{1/\theta}^\theta G(t,s)(1+s)^{\alpha-1}p(s)\widetilde{\gamma}(s)ds\right)\|u\| \\
&= (f_\infty(p,\theta)-\varepsilon)\Theta(p,,\theta)\|u\| \ge \|u\|.
\end{aligned}
$$

Therefore, we deduce from Theorem 2.1, that $T$ admits a fixed point $u \in K$ with $R_1 \le \|u\| \le R_2/\gamma_*$ which is, by Lemma 4.3 a positive solution to the bvp (1.1).

**Step 2. Existence in the case where (1.4) holds**

Let $\varepsilon > 0$ be such that $(f_0(p,\theta)-\varepsilon)\Theta(p,\theta) > 1$. There exists $\widetilde{R}_1 > 0$ such that $f(t,(1+t)^{\alpha-1}w) > (f_0(p,\theta)-\varepsilon)(1+t)^{\alpha-1}p(t)w$ for all $w \in [0,\widetilde{R}_1]$, and all $t \in I_\theta$. Thus, for all $u \in K \cap \partial\Omega_1$, where $\Omega_1 = \left\{u \in E : \|u\| < \widetilde{R}_1/\gamma_*\right\}$, we have

$$
\begin{aligned}
\|Tu\| &\ge \sup_{t\ge 0}\left(\frac{1}{(1+t)^{\alpha-1}}\int_{1/\theta}^\theta G(t,s)f(s,u(s))ds\right) \\
&\ge \sup_{t\ge 0}\left(\frac{1}{(1+t)^{\alpha-1}}\int_{1/\theta}^\theta G(t,s)f(s,(1+s)^{\alpha-1}\frac{u(s)}{(1+s)^{\alpha-1}})ds\right) \\
&\ge (f_\infty(p,\theta)-\varepsilon)\sup_{t\ge 0}\left(\frac{1}{(1+t)^{\alpha-1}}\int_{1/\theta}^\theta G(t,s)(1+s)^{\alpha-1}p(s)\frac{u(s)}{(1+s)^{\alpha-1}}ds\right) \\
&\ge (f_\infty(p,\theta)-\varepsilon)\sup_{t\ge 0}\left(\frac{1}{(1+t)^{\alpha-1}}\int_{1/\theta}^\theta G(t,s)(1+s)^{\alpha-1}p(s)\widetilde{\gamma}(s)ds\right)\|u\| \\
&= (f_\infty(p,\theta)-\varepsilon)\Theta(p,,\theta)\|u\| \ge \|u\|.
\end{aligned}
$$

Let $\epsilon > 0$ be such that $(f^\infty(q) + \epsilon)\Delta(q) < 1$. Then there exists $R_\epsilon > 0$ such that

$$f(t, (1+t)^{\alpha-1} w) \quad \leq (f^\infty(q) + \epsilon)q(t)w + (1+t)^{\alpha-1}\omega_\epsilon(t)\Psi_\epsilon(w), \text{ for all } t, w > 0,$$

where $\omega_\epsilon$ and $\Psi_\epsilon$ are the functions given by Hypothesis (1.2) for $R = R_\epsilon$.

Let $\Phi_\epsilon(t) = \Phi_{R_\epsilon, R_\epsilon}(t)$ and

$$R_2 = \frac{\overline{\Phi_\epsilon}}{1 - \Delta(q)(f^\infty(q) + \epsilon)}, \text{ where}$$
$$\overline{\Phi_\epsilon} = \sup_{t\geq 0}\left(\frac{1}{(1+t)^{\alpha-1}}\int_0^{+\infty} G(t,s)(1+s)^{\alpha-1}\Phi_\epsilon(s)\,ds\right).$$

For all $R > R_2$ we have $\Delta(q)(f^\infty(\alpha) + \epsilon)R + \overline{\Phi_\epsilon} \leq R$. Let $\widetilde{R}_2 > \max(\widetilde{R}_1/\gamma_*, R_2, R_\epsilon)$. Thus for all $u \in K \cap \partial\Omega_2$, where $\Omega_2 = \left\{u \in E : \|u\| < \widetilde{R}_2\right\}$, we have

$$
\begin{aligned}
\|Tu\| \quad &= \sup_{t\geq 0}\left(\frac{1}{(1+t)^{\alpha-1}}\int_0^{+\infty} G(t,s)f(s, (1+s)^{\alpha-1}\frac{u(s)}{(1+s)^{\alpha-1}})ds\right)\\
&\leq \sup_{t\geq 0}\left(\frac{1}{(1+t)^{\alpha-1}}\int_0^t G(t,s)((f^\infty(q) + \epsilon)(1+s)^{\alpha-1}q(s)\frac{u(s)}{(1+s)^{\alpha-1}}\right.\\
&+ (1+s)^{\alpha-1}\omega_\epsilon(s)\Psi_\epsilon\left(\frac{u(s)}{(1+s)^{\alpha-1}}\right)ds\right)\\
&\leq (f^\infty(q) + \epsilon)\sup_{t\geq 0}\left(\frac{1}{(1+t)^{\alpha-1}}\int_0^t G(t,s)(1+s)^{\alpha-1}q(s)ds\right)\|u\| + \overline{\Phi_\epsilon}\\
&(f^\infty(q) + \epsilon)\Delta(q)\|u\| + \overline{\Phi_\epsilon} \leq \|u\|.
\end{aligned}
$$

We deduce from ii) of Theorem 2.1 that $T$ admits a fixed point $u \in K$ with $\widetilde{R}_1/\gamma_* \leq \|u\| \leq \widetilde{R}_2$ which is, by Lemma 4.3, a positive solution to the bvp (1.1).

**Step 3. Unboundedness of the obtained positive solution**

Let $u$ be the positive solution obtained in Step1 or in Step 2. Then, for all $t \geq 0$, we have

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s, u(s))ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^\infty f(s, u(s))ds \text{ and}$$
$$u'(t) = -\frac{\alpha-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-2}f(s, u(s))ds + \frac{(\alpha-1)t^{\alpha-2}}{\Gamma(\alpha)}\int_0^\infty f(s, u(s))ds.$$

The above shows that $u'(0) = 0$ and $u'$ is increasing on $I$. Therefore, we have that $u'(t) > 0$ and $\lim_{t\to+\infty} u'(t) = l > 0$ and so, the solution $u$ is increasing and $\lim_{t\to+\infty} u(t) = +\infty$.

The proof of the main theorem is complete.

## REFERENCES

[1] Y. Arioua and M. Titraoui, New class of boundary value problem for nonlinear fractional differential equations involving Erdélyi-Kober derivative, Communications in Mathematics 27 (2019) 113-141.

[2] D. Baleanu, K. Diethelm, E. Scalas, J.J. Trujillo, Fractional calculus models and numerical methods, in: Series on Complexity, Nonlinearity and Chaos, World Scientific, Boston, 2012.

[3] A. Benmezaï and S. Chentout, Eigenvalue criteria for existence and nonexistence of positive solutions for $\alpha$-order $(2 < \alpha < 3)$ fractional differential equations on the half-line, Diff. Equ. Appl., 11 (2019), 463-480.

[4] T. S. Cerdik, F. Y. Deren and N. A. Hamal, Unbounded solutions for boundary value problems of Riemann Liouville Fractional diffrential equations on the half-line, Fixed Point Theory, 19(2018), 93-106.

[5] C. Corduneanu, Integral Equations and Stability of Feedback Systems, Academic Press, New York, 1973.

[6] M. Jia, H. Zhang and Q. Chen, Existence of positive solutions for fractional differential equation with integral boundary conditions on the half-line, Boundary Value Problems (2016) 2016: 104.

[7] K. Diethelm, The Analysis of Fractional Differential Equations, Springer-Verlag, Berlin, 2010.

[8] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, **204**, Elsevier Science B. V., Amsterdam, 2006.

[9] V. Lakshmikantham, S. Leela, J.V. Devi, Theory of Fractional Dynamic Systems, Cambridge Scientific Publishers, Cambridge, 2009.

[10] X. Li, X. Liu, M. Jia and L. Zhang, The positive solutions of infinite-point boundary value problem of fractional differential equations on the infinite interval, Advances in Difference Equations (2017) 2017: 126.

[11] S. Liang and J. Zhang, Existence of multiple positive solutions for m-point fractional boundary value problems on an infinite interval, Math. Comput. Modelling 54 (2011) 1334–1346.

[12] K. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York, 1993.

[13] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.

[14] S. Samko, A. Kilbas, O. Marichev, Fractional Integral and Derivatives, in: Theory and Applications, Gordon and Breach, Switzerland, 1993.

[15] G. Wang, S. Liu and L. Zhang, Iterative approximation of the minimal and maximal positive solutions for multipoint fractional boundary value problem on an unbounded domain, J. Function Spaces, Volume 2014, Article ID 469509, 5 pages.

[16] J. Xiao, J. J. Nieto and Z. Luo, Multiple positive solutions of the singular boundary value problem for second-order impulsive differential equations on the half-line, Boundary Value Problems, Volume 2010, Article ID 281908, 13 pages.

A. Benmezai

Faculty of Mathematics, USTHB, Algeirs, Algeria
  *E-mail address*: aehbenmezai@gmail.com

S. Chentout

Faculty of Mathematics, USTHB, Algeirs, Algeria
  *E-mail address*: chentout@yahoo.fr