

## A NOTE ON THE LOCAL STABILITY THEORY FOR CAPUTO FRACTIONAL PLANAR SYSTEM

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ABSTRACT. In this manuscript a new approach into analyzing the local stability of equilibrium points of non-linear Caputo fractional planar systems is shown. It is shown that the equilibrium points of such systems can be a stable focus or unstable focus. In addition, it is proposed that previous results regarding the stability of equilibrium points have been incorrect, the results here attempt to correct such results. Lastly, it is proposed that a Caputo fractional planar system cannot undergo a Hopf bifurcation, contrary to previous results prior.

### 1. INTRODUCTION

Fractional Differential Equations (FDE) have been growing in popularity in the field of applied mathematics, in particular in the field of mathematical modeling, see [1, 5, 12, 13, 15, 16, 18]. The primary modelling approach in the references is via Dynamical Systems with  $\alpha \in (0, 1)$ , where  $\alpha$  is the order of the derivative. Such models are particularly popular in modelling disease spread of Predator-Prey interactions (ecosystems), see [2, 8, 17, 20]. Traditionally, the authors in those papers are interested in determining the qualitative behavior of the system near its equilibria points, by employing the classical theory of local stability analysis or bifurcation theory. Similarly, for the fractional case, authors attempt to do the same.

However, due to the complexity of the fractional derivative, the results obtained are not always as strong as the classical. non-fractional, case. That is to say, for the classical case the local stability theory is well developed and it is easy to justify the qualitative behavior of a system near its equilibria point, as well as provide a complete characterization of the solutions. However, for the fractional case this is not the case.

In [19] Corollary 2, it was shown that a Caputo autonomous fractional order system, with order  $\alpha \in (0, 1)$ , cannot have a non-constant smooth periodic solution. This result leads to Proposition 1. In which, it is concluded that a Caputo autonomous system cannot undergo a Hopf bifurcation. Contrary to the results

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mentioned in, for example, [18]. Additionally, it is shown that, under suitable conditions, that an equilibrium point of a Caputo fractional system can be a stable focus or unstable focus. The condition is given in terms of  $\alpha$ , indicating that the fractional order influences the qualitative behaviour.

Additionally, in Theorem 2 and its proof we show that the results obtained by the authors in [6] Theorem 4 (a), and (f) are incorrect, see remark 2

**In summary:**

- (1) It is shown that the local stability of hyperbolic equilibrium points of a Caputo fractional planar system can be analyzed by studying the asymptotic behavior of the Mittag-Leffler function defined in Definition 3.
- (2) It shown that, depending on  $\alpha \in (0, 1)$ , that the hyperbolic equilibrium point can be an unstable focus, stable focus or locally asymptotically stable, provided that the eigenvalues are complex with a positive real part. The results for unstable focus and stable focus are new.
- (3) It is concluded that a Caputo autonomous fractional order system of order  $\alpha \in (0, 1)$  cannot undergo a Hopf bifurcation.

## 2. PRELIMINARIES

**Definition 1** Let  $0 < \alpha < 1$ . The operator  $J_a^\alpha$ , defined on  $L^1[a, b]$  by

$$J_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1} f(x) dx \quad (1)$$

for  $a \leq t \leq b$ , is called the Riemann-Liouville fractional integral operator of order  $\alpha$ . Here, and in what follows  $\Gamma(\cdot)$  is the Gamma function.

**Definition 2** Let  $0 < \alpha < 1$ . Then, we define the Caputo fractional differential operator  ${}_c D_a^\alpha$  as

$${}_c D_a^\alpha f(t) := J_a^{1-\alpha} f'(t) \quad (2)$$

whenever  $f, f' \in L[a, b]$ .

**Definition 3** Let  $\alpha > 0, \beta > 0$ . The function  $E_\alpha$ , defined by

$$E_{\alpha, \beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}, \quad (3)$$

whenever the series converges is called the two parameter Mittag-Leffler function with parameters  $\alpha$  and  $\beta$ .

**Lemma 1** If  $0 < \alpha < 1, \beta \in \mathbb{C}$  and  $\mu \in \mathbb{R}$  such that

$$\alpha \frac{\pi}{2} < \mu < \alpha \pi,$$

then for a arbitrary integer  $p \geq 1$  the following expansion holds:

$$E_\alpha(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} e^{z^{1/\alpha}} - \sum_{k=1}^p \frac{z^{-k}}{\Gamma(\beta - \alpha k)} + O\left(|z|^{-1-p}\right) \quad |z| \rightarrow \infty, |arg(z)| \leq \mu \quad (4)$$

or

$$E_\alpha(z) = - \sum_{k=1}^p \frac{z^{-k}}{\Gamma(\beta - \alpha k)} + O\left(|z|^{-1-p}\right) \quad |z| \rightarrow \infty, \mu \leq |\arg(z)| \leq \pi \quad (5)$$

**Remark 1** Note, that the terms

$$\sum_{k=1}^p \frac{z^{-k}}{\Gamma(\beta - \alpha k)} + O\left(|z|^{-1-p}\right)$$

become arbitrary small as  $|z| \rightarrow \infty$ . Fix  $\beta = 1$ , then

$$E_\alpha(z) = \frac{1}{\alpha} e^{z^{1/\alpha}} \quad |z| \rightarrow \infty, |\arg(z)| \leq \mu$$

### 3. LOCAL STABILITY THEORY OF PLANAR FRACTIONAL SYSTEM

In this section we provide the stability theory that we will use in the paper for Planar Fractional Systems. Specifically, we build on the the classical theory and extend the results to the fractional case. We only consider the case when  $\alpha \in (0, 1)$ .

$$\begin{cases} cD_0^\alpha x(t) = f(x, y), \\ cD_0^\alpha y(t) = g(x, y), \end{cases} \quad (6)$$

subject to the initial condition:

$$(x(0), y(0)) = (x_0, y_0),$$

where  $x, (t)$ , and  $y(t)$  are assumed to be in  $AC(0, T]$  for every  $T > 0$ , and  $f, g \in C^1(\mathbb{R}^2)$ .

Since,  $f, g \in C^1(\mathbb{R}^2)$ , it is well known that for any  $(x_0, y_0) \in \mathbb{R}^2$  the initial value problem (6) has a unique solution, see[14].

We denote by  $A(x, y)$  the Jacobian matrix of  $f$  and  $g$  at  $(x, y)$ , that is,

$$A(x, y) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}, \quad (7)$$

and by  $|A(x, y)|$  and  $\text{tr}(A(x, y))$  the determinant and trace of  $A(x, y)$ , respectively.

Below we provide the definitions for a cycle or periodic orbit, and Hopf bifurcation.

**Definition 3** A point  $(x^*, y^*) \in \mathbb{R}^2$  is called an equilibrium point of (6) if  $f(x^*, y^*) = 0$  and  $g(x^*, y^*) = 0$ .

Below we define the linearized system of (6) about the equilibrium point  $(x^*, y^*)$ .

**Definition 4** Let  $A$  be the matrix defined in (7) is evaluated at the equilibrium point  $(x^*, y^*)$ . Then,

$$cD_0^\alpha X = AX, \quad (8)$$

where  $X = (x, y)^T$ , is the linearization of system (6) at the equilibrium point  $(x^*, y^*)$ .

**Definition 5** A cycle or periodic orbit of (6) is any closed trajectory curve of (6) which is not an equilibrium point of (6). A limit cycle  $\Phi$  of a planar system is a cycle of (6) which belongs to the  $\alpha$  or  $\omega$  limit set of some trajectory of (6) other than  $\Phi$ . If a cycle  $\Phi$  is the  $\omega$  limit set of every trajectory in some neighborhood of  $\Phi$ , then  $\Phi$  is a stable limit cycle.

**Definition 6** Hopf bifurcation is a local bifurcation in which a steady state of a dynamical system changes its stability, so that the appearance or disappearance of a periodic orbit occurs.

The following Lemma can be found in [19] as Corollary 2.

**Lemma 2** The Caputo fractional order system defined in (6), where  $\alpha \in (0, 1)$ , cannot have any non-constant smooth periodic solutions.

As a direct consequence of Lemma 2 above, we obtain the following result.

**Proposition 1** (1) The Caputo fractional order system defined in (6), where  $\alpha \in (0, 1)$ , cannot have a limit cycle.

(2) The Caputo fractional order system defined in (6), where  $\alpha \in (0, 1)$ , cannot undergo a Hopf bifurcation.

The following Lemma is a special case ( $n = 2$ ) of Lemma 3.2 in [18].

**Lemma 3** Let  $(x^*, y^*)$  be an equilibrium point of (6) and  $A$  be defined as in (7). Let  $\lambda_1$  and  $\lambda_2$  be the eigenvalues of  $A$ . Then, the following assertions hold.

(1) The equilibrium point  $(x^*, y^*)$  is locally asymptotically stable if and only if  $|\arg(\lambda_{1,2})| > \frac{\alpha\pi}{2}$ .

(2) The equilibrium point  $(x^*, y^*)$  is stable if and only if  $|\arg(\lambda_{1,2})| \geq \frac{\alpha\pi}{2}$  and the eigenvalues with  $|\arg(\lambda_{1,2})| = \frac{\alpha\pi}{2}$  have the same geometric multiplicity and algebraic multiplicity.

(3) The equilibrium point  $(x^*, y^*)$  is unstable if and only if  $|\arg(\lambda_{1,2})| < \frac{\alpha\pi}{2}$ .

**Lemma 4** If the origin  $(0, 0)$  is a hyperbolic equilibrium point of (6), then vector field  $(f(x, y), g(x, y))$  is topologically equivalent with its linearization vector field given by the linear system  $cD_0^\alpha X = AX$  in the neighborhood of the origin  $(0, 0)$ .

The following Theorem follows from Lemma 3, where the conditions are expressed in terms of  $\text{tr}(A(x^*, y^*))$ , and  $|A(x^*, y^*)|$ .

**Theorem 1** If  $(x^*, y^*)$  is a equilibrium point of (6), then the following assertions hold.

- (i) If  $|A(x^*, y^*)| < 0$ , then  $(x^*, y^*)$  is unstable. (6).
- (ii) If  $|A(x^*, y^*)| > 0$ ,  $\text{tr}(A(x^*, y^*)) > 0$  and  $(\text{tr}(A(x^*, y^*)))^2 - 4|A(x^*, y^*)| \geq 0$ , then  $(x^*, y^*)$  is unstable.
- (iii) If  $|A(x^*, y^*)| > 0$ ,  $\text{tr}(A(x^*, y^*)) < 0$ , then  $(x^*, y^*)$  is Locally Asymptotically stable.

The Theorem below is new.

**Theorem 2** Let  $(x^*, y^*)$  be an equilibrium point of (6). Let  $\lambda_1$  and  $\lambda_2$  be eigenvalues of the matrix  $A$  defined in (7) and suppose that  $\lambda_1 = a + ib$  and  $\lambda_2 = a - ib$ , with  $b \neq 0$ .

- (i) If  $|A(x^*, y^*)| > 0$ ,  $\text{tr}(A(x^*, y^*)) > 0$  and  $(\text{tr}(A(x^*, y^*)))^2 - 4|A(x^*, y^*)| < 0$ , then  $(x^*, y^*)$  is an unstable focus of (6) for  $\alpha \in (\alpha^*, 1)$ ; stable focus of (6) for  $\alpha \in (0, \alpha^*)$ .
- (ii) If  $|A(x^*, y^*)| > 0$ ,  $\text{tr}(A(x^*, y^*)) < 0$  and  $(\text{tr}(A(x^*, y^*)))^2 - 4|A(x^*, y^*)| < 0$ , then  $(x^*, y^*)$  is a stable focus of (6) for  $\alpha \in (\frac{1}{2}, 1)$ ; locally asymptotically stable for  $\alpha \in (0, \frac{1}{2}]$ .

*Proof.* By Lemma 4 we can study (8) to determine the qualitative behavior of the equilibrium point  $(x^*, y^*)$ . If  $(x^*, y^*) \neq (0, 0)$ , then we can use the following substitution

$$x_1 = x - x^*, \quad y_1 = y - y^*,$$

to translate the equilibrium point  $(x^*, y^*)$  to the origin.

Define  $|\arg(\lambda_{1,2})|$  to be the argument of the eigenvalues  $\lambda_1$ , and  $\lambda_2$ , which are equal since the eigenvalues are complex conjugates. Additionally, since  $b \neq 0$ , then there exists a  $\mu \in (\frac{\alpha\pi}{2}, \alpha\pi)$  such that  $|\arg(\lambda_{1,2})| \leq \mu$ . Moreover,  $\lambda_1 \neq \lambda_2$ , thus the general solution,  $X$ , to (8) can be expressed as follows

$$X(t) = c_1 u_1 E_\alpha(\lambda_1 t^\alpha) + c_2 u_2 E_\alpha(\lambda_2 t^\alpha), \quad (9)$$

where  $X(t) = (x(t), y(t))$ ,  $c_1$  and  $c_2$  are constants and  $u_1$  and  $u_2$  are eigenvectors corresponding to the eigenvalues  $\lambda_1$  and  $\lambda_2$ , respectively. Since, we are interested in the qualitative behaviour solutions about the equilibrium points, instead of transient dynamics, we can consider the behaviour of the solution

$$X(t) = c_1 u_1 E_\alpha(\lambda_1 t^\alpha) + c_2 u_2 E_\alpha(\lambda_2 t^\alpha), \quad \text{as } t \rightarrow \infty.$$

Since there exists a  $\mu \in (\frac{\alpha\pi}{2}, \alpha\pi)$  such that  $|\arg(\lambda_{1,2})| \leq \mu$ , by Lemma 1 we have,

$$E_\alpha(\lambda_1 t^\alpha) = (1/\alpha) e^{(\lambda_1 t^\alpha)^{1/\alpha}} \quad \text{as } t \rightarrow \infty \text{ and } |\arg(\lambda_{1,2})| \leq \mu. \quad (10)$$

Define  $\alpha_1 := \frac{1}{\alpha}$ , then  $\alpha_1 \in \mathbb{Z}_+$ .

$$\begin{aligned} (1/\alpha) e^{(\lambda_1 t^\alpha)^{1/\alpha}} &= \alpha_1 e^{(a+ib)^{\alpha_1} t} \\ &= \alpha_1 e^{\left( \sum_{j=0}^{\alpha_1-1} \binom{\alpha_1}{j} a^{\alpha_1-j} (ib)^{j-1} \right) t} \\ &= \alpha_1 e^{\left( \binom{\alpha_1}{0} a^{\alpha_1} + \binom{\alpha_1}{1} a^{\alpha_1-1} (ib) + \dots + (ib)^{\alpha_1} \right) t} \\ &= \alpha_1 e^{\left( \binom{\alpha_1}{0} a^{\alpha_1} + \binom{\alpha_1}{1} a^{\alpha_1-1} (ib) + \dots + (ib)^{\alpha_1} \right) t} \\ &= \alpha_1 e^{(\omega+i\beta)t} = \alpha_1 e^{\omega t} \left( \cos(\beta t) + i \sin(\beta t) \right), \end{aligned} \quad (11)$$

where

$$\binom{\alpha_1}{0} a^{\alpha_1} + \binom{\alpha_1}{1} a^{\alpha_1-1} (ib) + \dots + (ib)^{\alpha_1} = \omega + i\beta \quad \text{for some } \omega, \beta \in \mathbb{R}.$$

Note that, the qualitative behaviour of (10) is the same the qualitative behaviour of (11), which depend only on the eigenvalues and  $\alpha$ . See Lemma 3.

Similarly,

$$(1/\alpha) e^{(\lambda_2 t^\alpha)^{1/\alpha}} = \alpha_1 e^{\omega t} \left( \cos(\beta t) - i \sin(\beta t) \right)$$

Thus, (9) can be expressed as

$$\begin{aligned} X(t) &= c_1 u_1 \left[ \alpha_1 e^{\omega t} (\cos(\beta t) + i \sin(\beta t)) \right] \\ &\quad + c_2 u_2 \left[ \alpha_1 e^{\omega t} (\cos(\beta t) - i \sin(\beta t)) \right] \end{aligned} \quad (12)$$

as  $t \rightarrow \infty$  and  $|\arg(\lambda_{1,2})| \leq \mu$ .

Thus, from (12) we can see that the solutions, can be expressed strictly as real valued solutions, similarly as the classical case. In addition, the solutions behave in an oscillatory manner, similarly as the classical case, qualitatively.

Next, recall that the eigenvalues for a planar system can be represented as

$$\lambda_1 = \frac{\operatorname{tr}(A(x, y)) + \sqrt{\operatorname{tr}(A(x, y))^2 - 4 \det(A(x, y))}}{2},$$

$$\lambda_2 = \frac{\operatorname{tr}(A(x, y)) - \sqrt{\operatorname{tr}(A(x, y))^2 - 4 \det(A(x, y))}}{2}.$$

(1) If  $\operatorname{tr}(A(x, y)) > 0$ , and  $\operatorname{tr}(A(x, y))^2 - 4 \det(A(x, y)) < 0$ , then the eigenvalues are complex conjugates. Note that the eigenvalues are independent of  $\alpha$ , so  $|\arg(\lambda_{1,2})|$  is fixed. Specifically, the term  $\alpha^* = \frac{2}{\pi} |\arg(\lambda_{1,2})|$  is fixed.

(i) Suppose that  $\alpha \in (\alpha^*, 1)$ , then  $|\arg(\lambda_{1,2})| < \frac{\alpha\pi}{2} \leq \mu < \alpha\pi$ . From (12) the solution has a oscillatory behaviour. This together with Lemma 3 allows us to conclude that  $(x^*, y^*)$  is an unstable focus.

(ii) Suppose that  $\alpha \in (0, \alpha^*)$ , then there exists a  $\mu \in (\frac{\alpha\pi}{2}, \alpha\pi)$  such that  $\frac{\alpha\pi}{2} < |\arg(\lambda_{1,2})| \leq \mu < \alpha\pi$ . From (12) the solution has a oscillatory behavior. This together with Lemma 3 allows us to conclude that  $(x^*, y^*)$  is a stable focus.

(2) If  $\operatorname{tr}(A(x, y)) < 0$ , and  $\operatorname{tr}(A(x, y))^2 - 4 \det(A(x, y)) < 0$ , then the eigenvalues are complex conjugates, and  $|\arg(\lambda_{1,2})| > \frac{\pi}{2}$ . Let  $\alpha \in (\frac{1}{2}, 1)$  and  $\mu \in (\frac{\alpha\pi}{2}, \alpha\pi)$ . Then, there exists a  $\mu \in (\frac{\alpha\pi}{2}, \alpha\pi)$  such that  $\frac{\alpha\pi}{2} < \frac{\pi}{2} < |\arg(\lambda_{1,2})| < \mu < \alpha\pi$ . From (12) the solution has a oscillatory behaviour. This together with Lemma 3 allows us to conclude that  $(x^*, y^*)$  is a stable focus.

□

**Remark 2** In Theorem 2 we provide a new method for determining the qualitative behavior of solutions near equilibrium point  $(x^*, y^*)$ . In particular, we use the asymptotic expansion properties of the Mittag-Leffler functions to achieve the results. Furthermore, it is noted that under the condition  $\alpha = \alpha^*$ , we cannot conclude that the equilibrium point  $(x^*, y^*)$  undergoes a Hopf bifurcation, under suitable conditions, see proposition 1.

In fact, the condition  $\alpha = \alpha^*$ , has been misrepresented in the literature, see [18, 11, 6]. The authors in [18, 11] claimed that equilibrium point  $(x^*, y^*)$  of (6) is a Hopf bifurcation if  $\alpha = \alpha^*$ . However, this is not correct, if the claim is to be true, then it is to follow that (6) undergoes a limit cycle, which from Proposition 1 (1) cannot be the case. Thus, it is not possible to treat  $\alpha$  as bifurcation parameter in this case. Nor, is it correct to claim that this leads to (6) undergoing a Hopf bifurcation. It only follows that the equilibrium point  $(x^*, y^*)$  is stable under this case. The authors in [6] claimed that the equilibrium point  $(x^*, y^*)$  is a stable node, if  $\alpha = \alpha^*$ . This claim is also not correct, in fact this would require an additional constraint on the equilibrium point. namely, that it is locally asymptotically stable, from Lemma 3 this is not the case. Additionally, Theorem (4) (f) in [6] is not correct. In fact, the author states that if all the eigenvalues are complex and satisfy  $|\arg(\lambda_{1,2})| > \frac{\alpha\pi}{2}$ , then the equilibrium point  $(x^*, y^*)$  is a stable focus of (6). However, this can only be concluded in its entirety for the linear system (8),

provided that the equilibrium point is hyperbolic ((8) has no zero eigenvalues). Indeed, consider the case when the complex eigenvalues have a zero real part, then  $|\arg(\lambda_{1,2})| = \frac{\pi}{2} \geq \frac{\alpha\pi}{2}$ . However, since the eigenvalues zero real parts, then this equilibrium point is a non hyperbolic equilibrium point, and the linearization Lemma 4 does not apply.

Lastly, due to the conditions associated with the asymptotic expansion of the Mittag-Leffler function, we cannot conclude that under (ii) the equilibrium point  $(x^*, y^*)$  is a stable focus for  $\alpha \in (0, \frac{1}{2}]$ .

#### 4. CONCLUSION

In this manuscript a new method for analyzing the local stability of hyperbolic equilibrium points for a Caputo fractional planar system was presented, see the proof of Theorem 2. It was shown that the equilibrium point could be an unstable focus, stable focus or locally asymptotically stable, under suitable conditions. In addition, it is concluded in Proposition 1 that (6) can not undergo a Hopf bifurcation. The future work in this area would be to determine the stability of non-hyperbolic equilibria points of (6).

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