Journal of Fractional Calculus and Applications
Vol. 12(3). No. 16, pp. 1-13
1st. Inter. E-Conf. in Math. Sciences and Fractional Calculus(ICMSFC Feb 2021).
ISSN: 2090-5858.
http://math-frac.org/Journals/JFCA/

CONTROLLABILITY OF NONLOCAL FRACTIONAL NON-INSTANTANEOUS IMPULSIVE SEMILINEAR DIFFERENTIAL INCLUSIONS WITHOUT COMPACTNESS

MARYAM IBRAHIM AL-SULTAN, AHMAD GAMAL IBRAHIM

ABSTRACT. In this paper, we are interested in studying the controllability for a system governed by a nonlocal fractional non-instantaneous impulsive semilinear differential inclusions, and the linear part is an infinitesimal generator of a non-compact semigroup of linear operators. We don't assume any condition in terms of the measure of non-compactness on the multi-valued function. We utilize a new version for weakly convergent sequence in the space of piecewise continuous functions.

1. INTRODUCTION

EL-Sayed et al.[5] initiated the existence of solutions for differential inclusions of fractional order. In recent years, many authors studied the controllability of problems governed by differential equations or inclusions of integer or fractional order with or without impulse and with local or non-local conditions. Liang et al.[8] established the controllability of simelinear differential equation assuming a compactness condition on both the nonlinear part and the controllability operator.

For more works in which the authors studied the controllability for systems of fractional order with assuming a compactness condition on the generating semigroup or the linear part we refer to [1,2,6,9,10,12,13].

It is worth mentioning that in the papers [1,2,6,8,10] there are not impulses effect on the system, while in [9,12,13] there are instantaneous impulses effect.

More recently, Wang et al.[14] investigated the controllability of fractional non-instantaneous impulsive differential inclusions without compactness condition.

In this paper, by means of weakly topology theory and avoiding any regularity conditions in terms of compactness, we study the controllability of the following

---

2010 *Mathematics Subject Classification.* 34A08, 26A33, 34A60.
Key words and phrases. Fractional differential inclusions, non-instantaneous impulses effect, non-local conditions, measure of non-compactness, weakly convergent sequence of functions.
Submitted March. 3, 2021.

nonlocal fractional non-instantaneous impulsive semilinear differentail inclusion of order $\alpha \in (0,1)$:

$$\begin{cases} {}^{c}D^{\alpha}_{s_i,t}x(t) \in Ax(t) + F(t,x(t)) + (Vz)(t), a.e.t \in (s_i,t_{i+1}], i = 0,1,..,m \\ x(t) = g_i(t,x(t_i^-)), t \in (t_i,s_i], i = 1,2,..,m \\ x(t_i^+) = g_i(t_i,x(t_i^-)), i = 1,2,..,m \\ x(0) = x_0 - q(x) \end{cases} \quad (1)$$

in a real separable Banach space E. Here, ${}^{c}D^{\alpha}_{s_i,t}$ is the Caputo's derivative of order α, A is the infinitesimal generator of a C_0-semigroup $\{T(t) : t \leq 0\}$ on a separable Banach space E, $F : [0,b] \times E \to 2^E - \{\phi\}$ is a multifunction, the fixed points t_i and s_i satisfy $0 = s_0 < t_1 \leq s_1 < t_2 \leq s_2 < t_3.... < t_m \leq s_m < t_{m+1} = b$ and $x(t_i^+)$, $x(t_i^-)$ represent the right and left limits of x at the point t_i respectively and $x_0 \in E$ is a fixed point. Moreover, $g_i : [t_i,s_i] \times E \to E$ is a continuous function for all $i = 1,2,...,m$ and $q : PC(J,E) \to E$ is a nonlocal function related to the nonlocal condition at the origin, the control function z is given in $L^p(J,X)$, $p > \frac{1}{\alpha}$, a Banach space of admissible control functions, where X is a real Banach space. $V : L^p(J,X) \to L^\infty(J,E)$ is a bounded linear operator and $PC(J,E)$ will be defined later.

We would like to refer that Wang et al.[14] studied the controllability of (1) when A=0.

The paper is organized as follows: in section 2, we put together some background materials,basic results multivalued analysis and fractional calculus which are needed later. In particular, we establish sufficient and necessary conditions to guarantee a sequence in piecewise continuous functions spaces which is weakly convergent. In section 3, we demonestrate the main controllability results for (1) under mild conditions via a fixed point theorem for weakly sequentially closed graph operator.

2. PRELIMINARIES AND NOTATIONS

Let $0 < \alpha < 1$, $J = [0,b]$, $C(J,E)$ be the space of E-valued continuous functions on J, $P_{cc}(E) = \{v \subseteq E : v$ is non-empty, convex and closed set$\}$, $P_{ck}(E) = \{v \subseteq E : v$ is non-empty, convex and compact set$\}$, $P_{cwk}(E) = \{v \subseteq E : v$ is non-empty, convex and weakly compact set$\}$, $co(v)$ and $\overline{co}(v)$ are the convex hull and closed convex hull respectively of a subset v in E. Let E_w be the space E endowed with the weak topology. If E is a normed space and $G : J \to P_{cl}(E)$, then the set $S^p_G = \{f \in L^p(J,E) : f(t) \in G(t), a.e.t \in J\}$ is the set of Lebesgue integrable selections of G where $L^p(J,E)$, $p \in [1,\infty)$ is the space of E-valued Bochner integrable functions on J with the norm $\|f\|_{L^p(J,E)} = \left(\int_0^b \|f(t)\|^p dt\right)^{\frac{1}{p}}$.

To give the conception of mild solution for system (1), we define the normed space of piecewise continuous functions :

$$\begin{aligned} PC(J,E) &= \{x : J \to E : x_{|_{J_i}} \in C(J_i,E), J_i := (t_i,t_{i+1}], \\ i &= 0,1,2,...,m, x(t_i^+) \text{ and } x(t_i^-) \text{ exist for each } i = 0,1,2,...,m\} \end{aligned}$$

which endowed with PC-norm: $\|x\|_{PC(J,E)} = \max\{\|x(t)\| : t \in J\}$.

For completion, we recall some results to be used later.

The following fixed point theorem is crucial in the proof of our main result.

Lemma 1. *([11]) Let X be metrizable locally convex linear topological space and let U be a weakly compact, convex subset of X. Suppose that $R : U \to P_{cl}(U)$ has weakly sequentially closed graph. Then R has a fixed point.*

The next result is familiar in weak topology theory. In fact, we denote $\rightharpoonup$ by weak convergence.

Definition 1. *([3]) A sequence $\{x_n\}$ of elements of Banach space X converges weakly to an element $x \in X$ and we write $x_n \rightharpoonup x$, if $\lim_{n\to\infty} \widetilde{T}(x_n) = \widetilde{T}(x)$ for each linear functional $\widetilde{T} \in X^*$.*

Lemma 2. *([3]) A sequence $\{x_n\} \subseteq C(J,X)$ converges weakly to $x \in C(J,X)$ if and only if there is a positive real number L in which, for every $n \in \mathbb{N}$ and $t \in J$, $\| x_n(t) \| \leq L$ and $x_n(t) \rightharpoonup x(t)$, $\forall t \in J$.*

Lemma 3. *([14]) Let E be a Banach space. A sequence $\{x_n\}$ in $PC(J,E)$ weakly converges to an element x in $PC(J,E)$ if and only if*

(i): *There exists a positive number $L > 0$ such that $\| x_n(t) \| \leq L$ for all $n \in \mathbb{N}$ and $\forall t \in J$.*
(ii): *For each $t \in J_i$, $i = 0,1,2,...,m$, $x_n(t) \rightharpoonup x(t)$.*
(iii): *For each $i = 0,1,2,...,m$, $x_n(t_i^+) \rightharpoonup x(t_i^+)$.*

Proof. (a) We show the sufficiency. Assume that (i), (ii) and (iii) hold. Let's show that $x_n \rightharpoonup x$ in $PC(J,E)$.

Let $T : PC(J,E) \to \mathbb{R}$ be a linear and bounded functional, for any $i = 0,1,2,..,m$, we define $T_i : C(\overline{J_i},E) \to \mathbb{R}$, $\overline{J_i} := [t_i, t_{i+1}]$ as follows : let $f \in C(\overline{J_i},E)$ and define $f_i : J \to E$ by

$$f_i(t) = \begin{cases} f(t), t \in J_i, i = 0,1,..,m, \\ 0, t \notin J_i. \end{cases}$$

Thus, we put $T_i(f) := T(f_i)$

Clearly, T_i is linear and bounded. Indeed for any $f,g \in C(\overline{J_i},E)$ and any $\alpha,\beta \in \mathbb{R}$, we have

$$T_i(\alpha f + \beta g) = T((\alpha f + \beta g)_i) = T(\alpha f_i + \beta g_i) = \alpha T(f_i) + \beta T(g_i) = \alpha T_i(f) + \beta T_i(g).$$

Next, for any $f \in C(\overline{J_i},E)$, we obtains

$$\|T_i(f)\| = \|T(f_i)\| \leq \|T\| \|f_i\|_{C(J,E)} = \|T\| \|f\|_{C(\overline{J_i},E)}$$

Now, for any $x \in PC(J,E)$, $x = \sum_{i=0}^{m} x_i$, where

$$x_i(t) = \begin{cases} x(t), t \in J_i, i = 0,1,..,m, \\ 0, t \notin J_i. \end{cases}$$

Since $x \in PC(J,E)$, then $x_{|\overline{J_i}} \in C(\overline{J_i},E)$ and $x(t_i^+)$ exists. Owing to the linearity of T, we get

$$T(x) = \sum_{i=0}^{m} T(x_i\ ) = \sum_{i=0}^{m} T\left((x_{|\overline{J_i}})^*\right) \tag{2}$$

where $(x_{|\overline{J_i}})^* : \overline{J_i} \to E$ is given by

$$(x_{|\overline{J_i}})^* = \begin{cases} x(t), t \in J_i, \\ x(\tau_i^+), t = \tau_i. \end{cases}$$

Since $(x_{|\overline{J_i}})^* \in C(\overline{J_i}, E)$ with $(x_{n|\overline{J_i}})^*(t) \rightharpoonup (x_{|\overline{J_i}})^*(t)$ and the sequence $(x_{n|\overline{J_i}})^*$is uniformly bounded, then from lemma 2, we obtain $(x_{n|\overline{J_i}})^* \rightharpoonup (x_{|\overline{J_i}})^*$ in $C(\overline{J_i}, E)$, which means

$$\lim_{n\to\infty} T_i\left((x_{n|\overline{J_i}})^*\right) = T_i\left((x_{|\overline{J_i}})^*\right), \ \forall T_i \in C^*(\overline{J_i}, E) \tag{3}$$

Therefor, linking (2) and (3), we have

$$\lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} \sum_{i=0}^{m} T_i\left((x_{n|\overline{J_i}})^*\right) = \sum_{i=0}^{m} T_i\left((x_{|\overline{J_i}})^*\right) = T(x)$$

Thus, $x_n \rightharpoonup x$ in $PC(J, E)$.

(b) Here, we reveal the necessity. Assume that $x_n \rightharpoonup x$ in $PC(J, E)$. For any $t \in J$, consider the following two functions

$$\delta_t : PC(J, E) \to \mathbb{R}, \ \delta_t(f) = x(f(t))$$

and

$$\rho_t : PC(J, E) \to \mathbb{R}, \ \rho_t(f) = x(f(t^+))$$

It's obvious that δ_t and ρ_t are linear and bounded. Since $x_n \rightharpoonup x$ in $PC(J, E)$, then $\delta_t(x_n) \to \delta_t(x)$ and $\rho_t(x_n) \to \rho_t(x)$. Hence, we get (ii) and (iii) through Definition 1. Moreover, it's well known that any weakly convergent sequence is bounded. Hence the property (i) is satisfied. □

In the following lemma, we recall Krein-Simulian theorem which is another well-known result.

Lemma 4. *([4]) The convex hull of a weakly compact set in a Banach space is weakly compact.*

Lemma 5. *([4])(Mazure's lemma) Any weakly convergent sequence $\{x_n\}$ in a Banach space has a sequence $\{\widetilde{x_n}\}$ of convex combination of its members which strongly converges to the same limit.*

Remark 1. *([14])*

(a): *Every closed (open) set in E_w is closed (open) in E. If the set F is closed and convex in E, then F is closed in E_w. As a matter of fact, let $x_n \rightharpoonup x$, $x_n \in F$. From Mazure's lemma, there is a sequence of convex combination of x_n, denoted by $\widetilde{x_n}$, where $\widetilde{x_n} \to x$ in E. Since F is convex, $\widetilde{x_n} \in F$. From the closedness of F we get $x \in F$.*

(b): *If $x_n \rightharpoonup x$, and F is weakly open and $x \in F$, then there is a natural number N such that $x_n \in F$, $\forall n \geq N$. Since F is weakly open and $x \in F$, thus from the definition of the weak topology there is $\varepsilon > 0$ and finite number of linear continuous functionals $f_1, f_2, ..., f_m$ such that $x + \{c :\mid f_k(c) \mid < \varepsilon, \forall k = 1, 2, ..., m\} \subseteq F$. From the weak convergence of x_n towards x, there is a natrural number N such that$\mid f_k(x_n - x) \mid < \varepsilon, \forall k = 1, 2, .., m$, $\forall n \geq N$, which implies $x_n - x \in \{c :\mid f_k(c) \mid < \varepsilon, \forall k = 1, 2, ..., m\}$ and hence, $x_n \in F, \forall n \geq N$.*

The PC-mild solution of (1) is introduced as follows:

Definition 2. *([14]) A function $x \in PC(J,E)$ is said to be a mild solution for (1) if there is an integrable selection $f \in S^1_{F(.,x(.))}$, such that for each $t \in J$,*

$$x(t) = \begin{cases} k_1(t)(x_0 - q(x)) + \int_0^t (t-s)^{\alpha-1} k_2(t-s)[f(s) + (Vz)(s)]ds, t \in [0, t_1], \\ g_i(t, x(t_i^-)), t \in (t_i, s_i], i = 1, 2, .., m, \\ k_1(t - s_i) g_i(s_i, x(t_i^-)) \\ + \int_{s_i}^t (t-s)^{\alpha-1} k_2(t-s)[f(s) + (Vz)(s)]ds, t \in (s_i, t_{i+1}], i = 1, 2, .., m, \end{cases}$$

where $k_1(t) = \int_0^\infty \xi_\alpha(\theta) T(t^\alpha \theta) d\theta$, $k_2(t) = \alpha \int_0^\infty \theta \xi_\alpha(\theta) T(t^\alpha \theta) d\theta$, $\xi_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}} \omega_\alpha(\theta^{-\frac{1}{\alpha}}) \geq 0$, $\omega_\alpha(\theta) = \sum\limits_{n=1}^{\infty} \frac{1}{\pi}(-1)^{n-1} \theta^{-\alpha n - 1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha)$, $\theta \in (0, \infty)$ and ξ_α is a probability density function defined on $(0, \infty)$, that $\int_0^\infty \xi_\alpha(\theta) d\theta = 1$.

In the following we recall the properties of $k_1(.)$ and $k_2(.)$.

Lemma 6. *([14])*

- **(i):** *For any fixed point $t \geq 0$, $k_1(t)$ and $k_2(t)$ are linear bounded operators.*
- **(ii):** *For $\lambda \in [0,1]$, $\int_0^\infty \theta^\lambda \xi_\alpha(\theta) d\theta = \frac{\Gamma(1+\lambda)}{\Gamma(1+\alpha\lambda)}$.*
- **(iii):** *If $\|T(t)\| \leq M, t \geq 0$, then for any $x \in E$, $\|k_1(t)x\| \leq M\|x\|$ and $\|k_2(t)x\| \leq \frac{M}{\Gamma(\alpha)}\|x\|$.*
- **(iv):** *$\{k_1(t) : t \geq 0\}$ and $\{k_2(t) : t \geq 0\}$ are strongly continuous.*
- **(v):** *If for any $t > 0$, $T(t)$ is compact, then $k_1(t)$ and $k_2(t)$, $t > 0$are compact.*

Definition 3. *([14]) The system (1) is said to be controllable on J if for every $x_0, x_1 \in E$, there exists a control function $z \in L^p(J, X)$ such that a mild solution of (1) satisfies $x(0) = x_0 - q(x)$ and $x(b) = x_1 - q(x)$.*

3. MAIN CONTROLLABILITY RESULT FOR (1)

In this section, we discuss the controllability of the system (1).

Theorem 1. *Let $F : J \times E \to P_{cwk}(E)$ be a multifunction, $g_i : [t_i, s_i] \times E \to E$ $(i = 1, 2, ..., m)$, $q : PC(J,E) \to E$ be functions and x_0, x_1 be two fixed points in E. Suppose that the following conditions are satisfied:*

- **(HA):** *The operator A is the infinitesimal generator of a C_0-semigroup $\{T(t) : t \geq 0\}$, and there is a constant $M \geq 1$ such that*

$$M = \sup_{t \in [0,\infty)} \|T(t)\| < \infty.$$

- **(HF)(i):** *For every $x \in E$, $t \to F(t,x)$ has a measurable selection and for $a.e. t \in J$, $x \to F(t,x)$ is upper semicontinuous from E_w to E_w.*
- **(ii):** *For any natural number n there exists a function $\varphi_n \in L^p(J, \mathbb{R}^+)$, $p > 1$, such that*

$$\sup_{\|x\| \leq n} \|F(t,x)\| < \varphi_n(t), a.e. t \in J,$$

and $\lim\limits_{n\to\infty} \inf \frac{\|\varphi_n\|_{L^p(J,\mathbb{R}^+)}}{n} = 0.$

- **(Hq):** *If $x_n \rightharpoonup x$ in $PC(J,E)$, then $q(x_n) \rightharpoonup q(x)$ and there are two positive constants a, d such that*

$$\|q(x)\| \leq a\|x\|_{PC(J,E)} + d.$$

(Hg): *For every $i = 1, 2, ..., m$, $g_i(t, .)$ is continuous from E_w to E_w and there exists $h_i > 0$ such that*

$$\|g_i(t, x)\| \leq h_i \|x\|, \forall x \in E, \forall t \in [t_i, s_i].$$

(HD): *The linear operator $D : L^p(J, X) \to E$ defined by*

$$D(z) = \int_{s_m}^{b} (b-s)^{\alpha-1} k_2(b-s)(Vz)(s)ds. \tag{4}$$

has an invertible operator $D^{-1} : E \to L^p(J, X)/Ker(D)$, and there is $N > 0$ such that $\|D^{-1}\| \leq N$ and $\|V\| \leq N$.

Then, system (1) is controllable on J provided that

$$M(h + a) + \frac{MN^2 b^\alpha}{\alpha\Gamma(\alpha)}(a + Mh) < 1, \tag{5}$$

where $h = \sum_{i=0}^{m} h_i$.

Proof. Notice that the operator D is well defined . Indeed, from (iii) of lemma 6, it yields for any $z \in L^p(J, X)$,

$$\begin{aligned}
\|D(z)\| &\leq \frac{M}{\Gamma(\alpha)} \int_{s_m}^{b} (b-s)^{\alpha-1} \|(Vz)(s)\| ds \\
&\leq \frac{M}{\Gamma(\alpha)} \|(Vz)\|_{L^\infty(J,E)} \int_{s_m}^{b} (b-s)^{\alpha-1} ds \\
&\leq \frac{M}{\Gamma(\alpha)} \|V\| \|z\|_{L^p(J,X)} \frac{b^\alpha}{\alpha} \\
&\leq \frac{MNb^\alpha}{\alpha\Gamma(\alpha)} \|z\|_{L^p(J,X)}
\end{aligned}$$

Moreover, thanks to $(HF)(i)$ for every $x \in PC(J, E)$, the multifunction $t \to F(t, x)$ has a measurable selection which, by $(HF)(ii)$, belongs to $L^p(J, E)$. Then, for any $x \in PC(J, E)$, the set $S^1_{F(.,x(.))}$ is not empty. This allows us to define a multifunction $R : PC(J, E) \to 2^{PC(J,E)}$ as follows:

For $x \in PC(J, E)$, $R(x)$ is the set of all functions $y \in PC(J, E)$, $y \in R(x)$ such that

$$y(t) = \begin{cases} k_1(t)(x_0 - q(x)) \\ + \int_0^t (t-s)^{\alpha-1} k_2(t-s)[f(s) + (Vz_{x,f})(s)]ds, t \in [0, t_1], \\ g_i(t, x(t_i^-)), t \in (t_i, s_i], i = 1, 2, .., m, \\ k_1(t - s_i) g_i(s_i, x(t_i^-)) \\ + \int_{s_i}^t (t-s)^{\alpha-1} k_2(t-s)[f(s) + (Vz_{x,f})(s)]ds, t \in (s_i, t_{i+1}], i = 1, 2, .., m, \end{cases}$$

where $f \in S^1_{F(.,x(.))}$ and

$$z_{x,f} = D^{-1}[x_1 - q(x) - k_1(b - s_m) g_m(s_m, x(t_m^-)) \quad - \int_{s_m}^{b} (b-s)^{\alpha-1} k_2(b-s) f(s) ds] \tag{6}$$

In order to show that system (1) is controllable, it sufficies to demonstrate that R has a fixed point. Let x be such fixed point. Then

$$
\begin{aligned}
x(b) &= k_1(b-s_m)g_m(s_m, x(t_m^-)) \\
&\quad + \int_{s_m}^{b}(b-s)^{\alpha-1}k_2(b-s)[f(s)+(Vz_{x,f})(s)ds] \\
&= k_1(b-s_m)g_m(s_m, x(t_m^-)) \\
&\quad + \int_{s_m}^{b}(b-s)^{\alpha-1}k_2(b-s)f(s)ds + D(z) \\
&= k_1(b-s_m)g_m(s_m, x(t_m^-)) \\
&\quad + \int_{s_m}^{b}(b-s)^{\alpha-1}k_2(b-s)f(s)ds \\
&\quad + x_1 - q(x) - k_1(b-s_m)g_m(s_m, x(t_m^-)) \\
&\quad - \int_{s_m}^{b}(b-s)^{\alpha-1}k_2(b-s)f(s)ds \\
&= x_1 - q(x).
\end{aligned}
$$

Now, we turn to prove that R has a fixed point by utilizing lemma 1.

Step1. In this step we claim that there is a natural number n_0 such that $R(B_{n_0}) \subseteq B_{n_0}$, where $B_{n_0} = \{x \in PC(J,E) : \|x\|_{PC(J,E)} \leq n_0\}$. Suppose that for any natural number n, there are x_n, $y_n \in PC(J,E)$ with $y_n \in R(x_n)$, $\|x_n\|_{PC(J,E)} \leq n$ and $\|y_n\|_{PC(J,E)} > n$. Then, according to the definition of R, there is a sequence of integrable functions $\{f_n\}_{n\geq 1} \in S^1_{F(.,x(.))}$ such that

$$
y_n(t) = \begin{cases}
k_1(t)(x_0 - q(x_n)) \\
+\int_0^t (t-s)^{\alpha-1}k_2(t-s)[f_n(s)+(Vz_{x_n,f_n})(s)]ds, t \in [0,t_1], \\
g_i(t, x_n(t_i^-)), t \in (t_i, s_i], i = 1,2,..,m, \\
k_1(t-s_i)g_i(s_i, x_n(t_i^-)) \\
+\int_{s_i}^t (t-s)^{\alpha-1}k_2(t-s)[f_n(s)+(Vz_{x_n,f_n})(s)]ds, t \in (s_i, t_{i+1}], i = 1,2,..,m.
\end{cases}
$$

Let $t \in [0,t_1]$, then

$$
\begin{aligned}
\|y_n(t)\| \leq\ & M\ (\|x_0\| + \|q(x_n)\|) + \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f_n(s)\|ds \\
& + \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|(Vz_{x_n,f_n})(s)\|ds \\
& \leq M\ (\|x_0\| + an + d) \\
& + \frac{M}{\Gamma(\alpha)}\|\varphi_n\|_{L^p(J,+)}\left(\int_0^t \mid (t-s)^{\alpha-1}\mid^{\frac{p}{p-1}} ds\right)^{\frac{p-1}{p}} \\
& + \frac{M}{\Gamma(\alpha)}\|Vz_{x_n,f_n}\|_{L^\infty(J,E)}\int_0^t (t-s)^{\alpha-1}ds. \qquad (7)
\end{aligned}
$$

Notice that

$$\begin{aligned}\left(\int_0^t \mid (t-s)^{\alpha-1} \mid^{\frac{p}{p-1}} ds\right)^{\frac{p-1}{p}} &= \left(\frac{(t)^{\frac{(\alpha-1)p}{p-1}+1}}{\frac{(\alpha-1)p}{p-1}+1}\right)^{\frac{p-1}{p}} = \left(\frac{(t)^{\frac{\alpha p-1}{p-1}}}{\frac{\alpha p-1}{p-1}}\right)^{\frac{p-1}{p}} \\ &= \left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}} t^{\alpha-\frac{1}{p}} \leq \left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}} b^{\alpha-\frac{1}{p}}.\end{aligned}$$

For simplification, put $\eta = \left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}} b^{\alpha-\frac{1}{p}}$.

Observe that

$$\begin{aligned}\|(Vz_{x_n,f_n})(s)\| &\leq \|Vz_{x_n,f_n}\|_{L^\infty(J,E)}, \ a.e. \\ &\leq \|V\|\|z_{x_n,f_n}\|_{L^p(J,X)}, \ a.e..\end{aligned}$$

Moreover, the relation (6) tell us

$$\begin{aligned}\|z_{x_n,f_n}\|_{L^p(J,X)} &\leq \|D^{-1}\|(\|x_1 - q(x_n) - k_1(b-s_m)g_m(s_m, x_n(t_m^-)) \\ &- \int_{s_m}^b (b-s)^{\alpha-1}k_2(b-s)f_n(s)ds\|) \\ &\leq N\ (\|x_1\| + a\|x_n\| + d + Mhn \\ &+ \frac{M}{\Gamma(\alpha)}\|\varphi_n\|_{L^p(J,+)}\left(\int_{s_m}^b \mid (b-s)^{\alpha-1} \mid^{\frac{p}{p-1}} ds\right)^{\frac{p-1}{p}}) \\ &\leq N\ (\|x_1\| + an + d + Mhn \\ &+ \frac{M}{\Gamma(\alpha)}\eta\|\varphi_n\|_{L^p(J,+)}).\end{aligned} \tag{8}$$

Relation (8) and (7) give us

$$\begin{aligned}\|y_n(t)\| &\leq M\ (\|x_0\| + an + d) + \frac{M\eta}{\Gamma(\alpha)}\|\varphi_n\|_{L^p(J,^+)} \\ &+ \frac{MN^2b^\alpha}{\alpha\Gamma(\alpha)}(\|x_1\| + an + d \\ &+ Mhn + \frac{M\eta}{\Gamma(\alpha)}\|\varphi_n\|_{L^p(J,^+)}).\end{aligned} \tag{9}$$

Let $t \in (t_i, s_i], i = 1, 2, .., m$. Then

$$\|y_n(t)\| = \|g_i(t, x_n(t_i^-))\| \leq h_i\|x_n\| < hn. \tag{10}$$

For $t \in (s_i, t_{i+1}], i = 1, 2, .., m$ and by using the same steps used on (7), we obtain

$$\begin{aligned}\|y_n(t)\| &\leq Mhn + \frac{M\eta}{\Gamma(\alpha)}\|\varphi_n\|_{L^p(J,^+)} \\ &+ \frac{MN^2b^\alpha}{\alpha\Gamma(\alpha)}\left(\|x_1\| + an + d + Mhn + \frac{M\eta}{\Gamma(\alpha)}\|\varphi_n\|_{L^p(J,^+)}\right).\end{aligned} \tag{11}$$

Combining (9), (10) and (11), and noting that $hn < Mhn$, to get for $t \in J$,

$$\begin{aligned} n \quad < \quad & \|y_n(t)\|_{PC(J,E)} \leq Mhn + \frac{M\eta}{\Gamma(\alpha)}\|\varphi_n\|_{L^p(J,\mathbb{R}^+)} + M\left(\|x_0\| + an + d\right) \\ & + \frac{MN^2b^\alpha}{\alpha\Gamma(\alpha)}\left(\|x_1\| + an + d + Mhn + \frac{M\eta}{\Gamma(\alpha)}\|\varphi_n\|_{L^p(J,\mathbb{R}^+)}\right). \end{aligned}$$

By dividing both side in the last inequality by n and then taking the limit as $n \to \infty$, we obtain

$$1 \leq Mh + Ma + \frac{MN^2b^\alpha}{\alpha\Gamma(\alpha)}(a + Mh),$$

which contradicts (5) and our claim is proved. Then there exists n_0 with $R(B_{n_0}) \subseteq B_{n_0}$.

Step2. In this step we demonestrate that the graph of $R\mid_{B_{n_0}}$ is weakly sequentially closed. For this purpose let $x_n \in B_{n_0}$ with $x_n \rightharpoonup x$ in $PC(J,E)$ and $y_n \in R(x_n)$ with $y_n \rightharpoonup y$ in $PC(J,E)$. We have to show that $y \in R(x)$. According to the previous step, there are $\{f_n\}_{n\geq 1} \in S^1_{F(.,x(.))}$ such that

$$y_n(t) = \begin{cases} k_1(t)(x_0 - q(x_n)) \\ + \int_0^t (t-s)^{\alpha-1}k_2(t-s)[f_n(s) + (Vz_{x_n,f_n})(s)]ds, t \in [0,t_1], \\ g_i(t, x_n(t_i^-)), t \in (t_i, s_i], i = 1,2,..,m, \\ k_1(t-s_i)g_i(s_i, x_n(t_i^-)) \\ + \int_{s_i}^t (t-s)^{\alpha-1}k_2(t-s)[f_n(s) + (Vz_{x_n,f_n})(s)]ds, t \in (s_i, t_{i+1}], i = 1,2,..,m. \end{cases} \tag{12}$$

Notice that $\|x_n(t)\| \leq n_0, \forall t \in J$ and $\forall n \geq 1$.

Then by $(HF)(ii)$ there is $\varphi_{n_0} \in L^p(J,\mathbb{R}^+)$ such that

$$\|f_n(t)\| \leq \|F(t, x_n(t))\| < \varphi_{n_0}(t), \ a.e. \tag{13}$$

This implies that the sequence $(f_n)_{n\geq 1}$ is bounded in $L^p(J,E)$, and hence by the reflexivity of $L^p(J,E)$, $(f_n)_{n\geq 1}$ has a supsequence which denoted by $(f_n)_{n\geq 1}$ again such that $f_n \rightharpoonup f \in L^p(J,E)$.

Now , for any $i = 1,2,...,m$ we define operators as follows:

$T_i : L^p([s_i,b],E) \to C([s_i,b],E)$,

$$T_i(h)(t) = \int_{s_i}^t (t-s)^{\alpha-1}k_2(t-s)h(s)ds, \tag{14}$$

and $O_i : E \to C([s_i,b],E)$,

$$O_i(z)(t) = \int_{s_i}^t (t-s)^{\alpha-1}k_2(t-s)\left(V(D^{-1}z)\right)(s)ds. \tag{15}$$

Obviously from the linearity of the integral operator and of the operators k_2, V and D^{-1} , the operators T_i and O_i, $i = 0,1,2,...,m$, are linear. Furthermore by lemma 6, we get

$$\|T_i(h)(t)\| \leq \frac{M\eta}{\Gamma(\alpha)}\|h\|_{L^p([s_i,b],E)},$$

then

$$\|T_ih\|_{C([s_i,b],E)} \leq \frac{M\eta}{\Gamma(\alpha)}\|h\|_{L^p([s_i,b],E)}$$

which means that T_i is bounded for $i = 0,1,2,...,m$, and $\|T_i\| \leq \frac{M\eta}{\Gamma(\alpha)}$.

As known, any linear bounded operator maps weakly convergent sequence to weakly convergent sequence (see[7]). Hence, $f_n \rightharpoonup f$ in $L^p(J,E)$ yeilds to $(T_i f_n)(t) \rightharpoonup (T_i f)(t)$in E $(i = 0,1,2,...,m)$. Thus, for $i = 0,1,2,...,m$ and $t \in [s_i, b]$ we have

$$\int_{s_i}^{t}(t-s)^{\alpha-1}k_2(t-s)f_n(s)ds \rightharpoonup \int_{s_i}^{t}(t-s)^{\alpha-1}k_2(t-s)f(s)ds. \tag{16}$$

Similarly, for $i = 0,1,2,...,m$,

$$\|O_i(z)(t)\| \le \int_{s_i}^{t}(t-s)^{\alpha-1}\|k_2(t-s)\|\|\left(V(D^{-1}z)\right)(s)\|ds.$$

To simplify the notations, let

$$g = D^{-1}(z)$$

We have $g \in L^p(J,X)$ and $Vg \in L^\infty(J,E)$. Then, for $a.e.s \in J$,

$$\begin{aligned}
\|Vg(s)\| &\le \|Vg\|_{L^\infty(J,E)}\\
&\le \|V\|\|g\|_{L^p(J,X)}\\
&\le N\|D^{-1}z\|_{L^p(J,X)}\\
&\le N\|D^{-1}\|\|z\|_E.
\end{aligned}$$

Therefore, for $t \in [s_i, b]$,

$$\begin{aligned}
\|O_i(z)(t)\| &\le \frac{MN}{\Gamma(\alpha)}\|D^{-1}\|\|z\|_E\int_{s_i}^{t}(t-s)^{\alpha-1}ds\\
&\le \frac{MN^2b^\alpha}{\alpha\Gamma(\alpha)}\|z\|_E,
\end{aligned}$$

this means that O_i is bounded for $i = 0,1,2,..,m$, and $\|O_i\| \le \frac{MN^2b^\alpha}{\alpha\Gamma(\alpha)}$.

Next, let

$$\begin{aligned}
\kappa_n &= x_1 - q(x_n) - k_1(b-s_m)g_m(s_m, x_n(t_m^-))\\
&\quad - \int_{s_m}^{b}(b-s)^{\alpha-1}k_2(b-s)f_n(s)ds,
\end{aligned}$$

and

$$\begin{aligned}
\kappa &= x_1 - q(x) - k_1(b-s_m)g_m(s_m, x(t_m^-))\\
&\quad - \int_{s_m}^{b}(b-s)^{\alpha-1}k_2(b-s)f(s)ds.
\end{aligned}$$

Notice that

$$x_n \rightharpoonup x \Longrightarrow x_n(t_m^-) \rightharpoonup x(t_m^-),$$

and from (Hg),

$$g_m(s_m, x_n(t_m^-)) \rightharpoonup g_m(s_m, x(t_m^-)).$$

Moreover, the linearity and boundedness of k_1 implies to

$$k_1(b-s_m)g_m(s_m, x_n(t_m^-)) \rightharpoonup k_1(b-s_m)g_m(s_m, x(t_m^-)).$$

From the above discussion and (16), we conclude that $\kappa_n \rightharpoonup \kappa$ in E.

Now because of the linearity and boundedness of the operators O_i for $i = 0,1,...,m$, we can easily see that

$$O_i(\kappa_n)(t) \rightharpoonup O_i(\kappa)(t) \text{ in } E$$

This means, for $i = 0, 1, ..., m$, $t \in [s_i, b]$,

$$\begin{aligned} &\int_{s_i}^{t} (t-s)^{\alpha-1} k_2(t-s) V(D^{-1}(\kappa_n))(s) ds \\ \rightharpoonup \ &\int_{s_i}^{t} (t-s)^{\alpha-1} k_2(t-s) V(D^{-1}(\kappa))(s) ds, \end{aligned}$$

which implies to

$$\begin{aligned} &\int_{s_i}^{t} (t-s)^{\alpha-1} k_2(t-s) V(z_{x_n,f_n})(s) ds \\ &\rightharpoonup \int_{s_i}^{t} (t-s)^{\alpha-1} k_2(t-s) V(z_{x,f})(s) ds. \end{aligned} \tag{17}$$

As above, we have for $i = 0, 1, .., m$

$$k_1(t-s_i) g_i(s_i, x_n(t_i^-)) \rightharpoonup k_1(t-s_i) g_i(s_i, x(t_i^-)) \tag{18}$$

From (16), (17) and (18) we obtain $y_n(t) \rightharpoonup w(t)$ in E and $y_n(t_i^+) \rightharpoonup w(t_i^+), i = 1, 2, .., m$ where

$$w(t) = \begin{cases} k_1(t)(x_0 - q(x)) \\ + \int_0^t (t-s)^{\alpha-1} k_2(t-s)[f(s) + (V z_{x,f})(s)] ds, t \in [0, t_1], \\ g_i(t, x(t_i^-)), t \in (t_i, s_i], i = 1, 2, .., m, \\ k_1(t-s_i) g_i(s_i, x(t_i^-)) \\ + \int_{s_i}^t (t-s)^{\alpha-1} k_2(t-s)[f(s) + (V z_{x,f})(s)] ds, t \in (s_i, t_{i+1}], i = 1, 2, .., m. \end{cases} \tag{19}$$

Notice that $\|y_n\| \leq n_0, \forall n \geq 1$. Then, from lemma 3, $y_n \rightharpoonup w$ in $PC(J, E)$. From the uniqueness of the weak limit we get $y(t) = w(t), t \in J$.

In order to complete the proof of our claim in this step , we have to show that $f(t) \in F(t, x(t))$, for $a.e.t \in J$. Since $f_n \rightharpoonup f$ then, in view of Mazure's lemma, we can find a sequence $\left(\widetilde{f_n}\right)_{n \geq 1}$ of convex combinations of f_n and $\widetilde{f_n}$ converges strongly to f, and hence there is a subsequence of $\left(\widetilde{f_n}\right)_{n \geq 1}$, denoted again by $\left(\widetilde{f_n}\right)_{n \geq 1}$ and $\left(\widetilde{f_n}\right)_{n \geq 1}(t) \to f(t)$, for $a.e.t \in J$. Let G be the set of points of J such that for any $t \in J - G$, $\left(\widetilde{f_n}\right)(t) \to f(t)$ and $F(t, .)$ is upper semi-continuous multifunction from E_w to E_w. Obviously the lebesgue measure of G is equal to zero. We will show that $f(t) \in F(t, x(t))$, for every $t \in J - G$. Assume that there is $t_0 \in J - G$ such that $f(t_0) \notin F(t_0, x(t_0))$. From the fact that $F(t_0, x(t_0))$ is closed and convex, we can find, by Hahn-Banach theorem, a weakly open convex subset V and $F(t_0, x(t_0)) \subseteq V$ but $f(t_0) \notin \overline{V}$.

From the upper semicontinuity of $F(t_0, .)$ from E_w to E_w at $x(t_0)$ there is a weak neighborhood U for $x(t_0)$ such that if $z \in U$, then $F(t_0, z) \subseteq V$, and hence by remark 1(b) we can find a natural number N such that $x_n(t_0) \in U, \forall n \geq N$. Then $f_n(t_0) \in F(t_0, x_n(t_0)) \subseteq V, \forall n \geq N$. Due to the convexity of V, $\widetilde{f_n}(t_0) \in V, \forall n \geq N$ and so $f(t_0) \in \overline{V}$. Consequentially $f(t) \in F(t, x(t))$, for $a.e.t \in J$.

Therefore the values of R are weakly closed

Step3. Showing that $R \mid_{B_{n_0}}$ is weakly compact. Obviously, it suffices to show that $R(B_{n_0})$ is relatively weakly compact. Let $\{x_n\}_{n \geq 1}$ in B_{n_0} and $y_n \in R(x_n)$.

Then, there is $f_n \in S^1_{F(.,x_n(.))}$ such that

$$y_n(t) = \begin{cases} k_1(t)(x_0 - q(x_n)) \\ + \int_0^t (t-s)^{\alpha-1} k_2(t-s)[f_n(s) + (Vz_{x_n,f_n})(s)]ds, t \in [0,t_1] \\ g_i(t, x_n(t_i^-)), t \in (t_i, s_i], i = 1,2,..,m \\ k_1(t-s_i) g_i(s_i, x_n(t_i^-)) \\ + \int_{s_i}^t (t-s)^{\alpha-1} k_2(t-s)[f_n(s) + (Vz_{x_n,f_n})(s)]ds, t \in (s_i, t_{i+1}], i = 1,2,..,m \end{cases}$$

By following the argument in (Step2), there is a subsequence of $(f_n)_{n\geq 1}$, denoted again by $(f_n)_{n\geq 1}$ such that $(f_n) \to f \in L^p(J,E)$, and $y_n \rightharpoonup w$, where w is given by (19). Then $R(B_{n_0})$ is relatively weakly compact.

Now since the values of F are convex, it is easy to see that $R(x)$ is convex for each $x \in B_{n_0}$. Then the restriction of R on B_{n_0} is convex.

Step 4. Set $W_{n_0} = \overline{co}(\overline{R(B_{n_0})}^w)$. From step3, $\overline{R(B_{n_0})}^w$ is weakly compact and , by lemma 4, W_{n_0} is a weakly compact and convex set. Moreover, from the fact that B_{n_0} is closed and convex then, by remark 1(a), we deduce that $\overline{B_{n_0}}^w = B_{n_0}$. Then it follows, by step1, that

$$R(W_{n_0}) = R(\overline{co}(\overline{R(B_{n_0})}^w)) \subseteq R\left(\overline{co}((\overline{B_{n_0}})^w)\right) = R\left(\overline{co}(B_{n_0})\right) = R(B_{n_0}) \subseteq W_{n_0}$$

Hence, by step2, $R : W_{n_0} \to P_{cl}(W_{n_0})$ has a weakly sequentially closed graph. Now, by applying lemma 1, R has a fixed point and the proof is complete. $\square$

References

[1] Aissani, K. & Benchohra, M. (2014). Controllability of Impulsive Fractional Differential Equations with Infinite Delay. LIBERTAS MATHEMATICA,34, 1-18.

[2] Balachandran, K. &Park, J. (2009). Controllability of fractional integrodifferential systems in Banach spaces. Nonlinear analysis: Hybrid Systems, 3, 363-367.

[3] Bochner, S. & Taylor, A. E. (1938). Linear Functionals on Certain Spaces of Abstractly-Valued Functions. The Annals of Mathematics, 39, 913-944.

[4] Dunford, N. & Schwartz, J. H. (1976). Linear operators, New York: John Wiley and Sons,Inc.

[5] El-Sayed, A.M.A. & Ibrahim, A.G. (1995). Multivalued differential inclusions. Appl.Math.C, 120, 355-374.

[6] Hernǎndez, E. & O'Regan, D. (2009). Controllability of Volterra-Fredholm type systems in Banach spaces. Journal of the Franklin Institute,346, 95-101.

[7] Kreyszig, E. (1978). Introductory Functional Analysis With Applications. New York, Chichester, Brisbane, Toronto, Singapore: John Wiley and Sons. Inc.

[8] Liang, J. & Yang, H. (2015). Controllabibilty of fractional integro-differentail evolution equations with nonlocal conditions. Applied Mathematics and Computation, 254, 20-29.

[9] Machado, J. A., Ravichandran, C., Rivero, M. & Trujillo, J. J. (2013). Controllability results for impulsive mixed-type functional integro-differentail evolution equations with nonlocal conditions. Fixed Point Theory and Applications, 66.

[10] Obukhovski, V. & Zecca, P. (2009). Controllability for systems governed by semilinear differential inclusions in a Banach space with a noncompact semigroup. Nonlinear Analysis, 70, 3424-3436.

[11] D'ORegan (2000). Fixed point theorems for weakly sequentiallyclosed maps, Archivum Mathematicum, 36, 61-70.

[12] Ravichandran, C & Trujillo, J. J. (2013). Controllability of Impulsive Fractional Functional Integro-differential Equations in Banach Spaces. Journal of Function Spaces and Applications, 1-8.

[13] Wang, J. R. &Ibrahim, A. G. (2013). Existence and Controllability Results for Nonlocal Fractional Impulsive Differential Inclusions in Banach Spaces. Journal of Function Spaces and Applications, 1-16.

[14] Wang, J., Ibrahim, A. G.,Fečkan, M. & Zhou, Y. (2017). Controllability of fractional non-instantaneous impulsive differential inclusions without compactness. IMA Journal of Mathematical Control and Information.

Maryam Ibrahim Al-Sultan
Al-Ahssa, Kingdom of Saudi Arabia
Email address: mariam831991@gmail.com

Ahmad Gamal Ibrahim
Department of Mathematics and Statistics, King Faisal University
Al-Ahssa, Kingdom of Saudi Arabia
Email address: agamal@kfu.edu.sa