Journal of Fractional Calculus and Applications
Vol. 12(3). No. 15, pp. 1-12
1st. Inter. E-Conf. in Math. Sciences and Fractional Calculus(ICMSFC Feb 2021).
ISSN: 2090-5858.
http://math-frac.org/Journals/JFCA/

A MODIFIED WAVELET MULTIGRID METHOD FOR THE NUMERICAL SOLUTION OF CONVECTION-DIFFUSION PROBLEM

M. H. KANTLI

Abstract. A modified wavelet multigrid method to solve convection-diffusion problem is proposed with new prolongation and restriction intergrid operators based on Daubechies high pass and low pass filter coefficients. The proposed method is the robust technique for faster convergence with low computational cost which is acceptable through operator complexity, grid complexity, rate of convergence, condition number and error analysis. Also, we discussed the different set of parameters for the nature of the solutions. We compare the modified method to standard methods namely multigrid and wavelet multigrid. It is concluded that the modified wavelet multigrid method easily outperforms over existing standard multigrid methods.

1. Introduction

Convection-diffusion problems arise in the modelling of many physical processes. Their typical solutions exhibit boundary and interior layers. In spite of the linear nature of the differential operator, these problems pose still-unanswered questions to the numerical analysis. Accurate modelling of the interaction between convective and diffusive processes is the most universal and challenging task in the numerical approximation of partial differential equations Morton's [1]. This paper is devoted to the application of the wavelet multigrid method to the numerical solution of the convection-diffusion equation

$$-\varepsilon(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2})+a\frac{\partial u}{\partial x}=f, \quad on \quad \Omega, \quad u=u_a \quad on \quad \partial\Omega. \tag{1}$$

where Ω is a bounded two-dimensional domain with a polygonal boundary $\partial\Omega$, f is a given outer source of the unknown scalar quantity $u, \varepsilon>0$ is the constant diffusivity, a is the flow velocity and u_a is a given condition. We are interested in the strongly convection-dominated cases in which the solution of (1) typically contains

2010 *Mathematics Subject Classification.* 34B60, 65F08, 65F10, 65M55, 65T60.
Key words and phrases. Wavelet multigrid, Convection-Diffusion problem, Operator complexity, Rate of convergence.
Submitted Nov. 13, 2020.

narrow inner and boundary layers. It is well known that the application of the classical methods is inappropriate in these cases since the discrete solution is usually globally polluted by spurious oscillations. Therefore, various stabilization strategies have been developed during the last three decades, recently reviewed paper by John and Knobloch [2]. Linear systems of algebraic equations are related with many problems, as well as with applications of mathematics. Direct methods are used to solve a linear system of N equations with N unknowns. Direct methods are theoretically produce the exact solution to the system in a finite number of steps. In practice, of course, the solution obtained will be polluted by the round-off error that is involved with the arithmetic being used. To minimize such round-off error iterative methods are infrequently used for solving linear systems. Since the time required for sufficient accuracy exceeds that required for direct methods. For large systems with a high amount of 0 entries, however, these methods are efficient in terms of both computer storage and computation cost. This type of system stands up frequently in numerical analysis of convection-diffusion problems. The multigrid method is largely applicable in increasing the efficiency of iterative methods used to solve large system of algebraic equations Burden and Faires [3].

The multigrid (MG) method is a well-founded numerical method for solving sparse linear system of equations approximating convection-diffusion equations. In the historical three decades the development of effective iterative solvers for systems of algebraic equations has been a significant research topic in numerical analysis and computational science and engineering. Nowadays it is recognized that multigrid iterative solvers are highly efficient for convection-diffusion problems and often have optimal complexity. For a detailed treatment of multigrid methods we refer Hackbusch [4]. An introduction of multigrid methods is found in Wesseling [5], Briggs [6] and Trottenberg et al. [7]. Authors, Griebel and Knapek [8] used matrix-dependent interpolations, where the coarse grid operator is determined to be a Schur complement using a Galerkin approach. However, when met by certain convection-diffusion problems, the standard multigrid procedure converges slowly with larger computational time. Whereas wavelet multigrid methods solves the system of equations in faster convergence with lesser computational cost refer article De Leon [9].

The mathematical theory of wavelets is more than two decades, yet already wavelets have become an important tool in many areas, such as image processing and time series analysis. In recent years, wavelet analysis is fast extensive kindness in the numerical solution of elliptic problems. The smooth orthogonal basis is obtained by the dilation and shift of a single special function, called "mother wavelet". Recently, many authors (De Leon [9], Kantli and Shiralashetti [10], Mundewadi et al. [11], Bujurke and Kantli [12-14] and Shiralashetti et al. [15-17]) have developed wavelet multigrid methods. These methods use a choice of the filter operators obtained from wavelets to define the prolongation and restriction operators. Avudainayagam and Vani [18] used wavelet-based interpolation and restriction operators for their multigrid approaches, and Vasilyev and Kevlahan [19] used a wavelet-collocation-based multigrid method. This paper outspreads the new approach of modified wavelet multigrid method (MWMG) to solve convection-diffusion problems. Thus, the proposed method can be applied to a wide range of science and engineering problems. The organization of this paper is as follows. Wavelet multigrid operators are given in section 2. Section 3, deals with the method of solution. Section 4, presents numerical experiments and results. Finally, conclusion of the MWMG is discussed in section 5.

2. Wavelet multigrid operators

The scaling functions, which are both compactly supported and continuous, were first constructed by Daubechies [20, 21] that created great excitement among mathematicians and scientists performing research in the area of wavelets. Daubechies low pass and high pass filter coefficients are described in [11, 15-17] and are used in the wavelet intergrid operators (prolongation and restriction).
The matrix formulation of the discrete signals and discrete wavelet transforms (DWT), which play an important part in the wavelet method. This is highly expedient and informative, particularly for the numerical computations. As we already know about the DWT matrix and its applications in the wavelet method and is given in [15-17] as,

$$W_1(2^J \times 2^J) = \begin{bmatrix} h_0 & h_1 & h_2 & h_3 & 0 & 0 & ... & 0 & 0 \\ g_0 & g_1 & g_2 & g_3 & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & h_0 & h_1 & h_2 & h_3 & ... & 0 & 0 \\ 0 & 0 & g_0 & g_1 & g_2 & g_3 & ... & 0 & 0 \\ .. & .. & .. & .. & .. & .. & ... & .. & .. \\ .. & .. & .. & .. & .. & .. & ... & .. & .. \\ h_2 & h_3 & 0 & 0 & 0 & 0 & ... & h_1 & h_2 \\ g_2 & g_3 & 0 & 0 & 0 & 0 & ... & g_1 & g_2 \end{bmatrix} \quad (2)$$

Using this matrix authors used restriction and prolongation operators W and W^T respectively given in section 3.2, alike to multigrid operators.
Here, we developed modified DWT matrix similar to DWT matrix in which we have added rows and columns consecutively with diagonal element as 1, which is built as,

$$W_2(2^J \times 2^J) = \begin{bmatrix} h_0 & 0 & h_1 & 0 & h_2 & 0 & h_3 & 0 & ... & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & ... & 0 \\ g_0 & 0 & g_1 & 0 & g_2 & 0 & g_3 & 0 & ... & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & ... & 0 \\ .. & .. & .. & .. & .. & .. & .. & .. & ... & .. \\ .. & .. & .. & .. & .. & .. & .. & .. & ... & .. \\ g_2 & 0 & g_3 & 0 & ... & 0 & g_0 & 0 & g_1 & 0 \\ 0 & 0 & 0 & .. & .. & ... & .. & 0 & 0 & 1 \end{bmatrix} \quad (3)$$

where $h_0 = \frac{1+\sqrt{3}}{4\sqrt{2}}, h_1 = \frac{3+\sqrt{3}}{4\sqrt{2}}, h_2 = \frac{3-\sqrt{3}}{4\sqrt{2}}, h_3 = \frac{1-\sqrt{3}}{4\sqrt{2}}$ are low pass filter coefficients and $g_0 = \frac{1-\sqrt{3}}{4\sqrt{2}}, g_1 = \frac{3-\sqrt{3}}{4\sqrt{2}}, g_2 = \frac{3+\sqrt{3}}{4\sqrt{2}}, g_3 = \frac{1+\sqrt{3}}{4\sqrt{2}}$ are the high pass filter coefficients. Using W_2 matrix, we developed restriction and prolongation operators WP and WP^T respectively alike to wavelet multigrid operators given in section 3.3.

3. Method of solutions

Consider the differential equation, after discretizing the differential equation through the finite difference method (FDM), we get system of algebraic equations. Through this system we can write the system as

$$Au = f. \quad (4)$$

where A is $2^J \times 2^J$ coefficient matrix, b is $2^J \times 1$ matrix and u is $2^J \times 1$ matrix to be determined. Where J is the maximum level of resolution. Solve the equation (4) through the iterative method, we get the approximate solution v of u. i.e. $u = e + v \Rightarrow v = u - e$, where e is ($2^J \times 1$ matrix) error to be determined. In the computation of numerical analysis, approximate solution containing some error. There are many approaches known to minimize the error. Some of them are Multigrid (MG), Wavelet multigrid (WMG) and modified wavelet multigrid (MWMG) Methods etc. Now we are deliberating about the method of solution of these mentioned methods as below.

3.1. **Multigrid (MG) method.** From equation (4), we get the approximate solution v for u. Now we find the residual as

$$r_{2^{J-\eta}\times 1} = [f]_{2^{J-\eta}\times 1} - [A]_{2^{J-\eta}\times 2^{J-\eta}}[v]_{2^{J-\eta}\times 1}. \tag{5}$$

where $\eta = 0, 1, 2, ..., J$
We reduce the matrices in the finer (J^{th}) level to coarsest level using Restriction operator, i.e.

$$R(2^{J-1-\eta} \times 2^{J-\eta}) = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & ... & 0 & 0 \\ .. & .. & .. & .. & .. & ... & .. & .. \\ 0 & 0 & 0 & 0 & 0 & ... & 1 & 2 \end{bmatrix} \tag{6}$$

and then construct the matrices back to finer level from the coarsest level using Prolongation operator, i.e.

$$P(2^{J-\eta} \times 2^{J-1-\eta}) = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & ... & 0 & 0 \\ 2 & 0 & 0 & ... & 0 & 0 \\ 1 & 1 & 0 & ... & 0 & 0 \\ 0 & 2 & 0 & ... & 0 & 0 \\ 0 & 1 & 1 & ... & 0 & 0 \\ .. & .. & .. & ... & .. & .. \\ 0 & 0 & 0 & ... & 0 & 1 \\ 0 & 0 & 0 & ... & 0 & 2 \end{bmatrix} \tag{7}$$

From (5),

$$r_{2^{J-1}\times 1} = [R]_{2^{J-1}\times 2^J}[r]_{2^J\times 1}. \tag{8}$$

and $[A]_{2^{J-1}\times 2^{J-1}} = [R]_{2^{J-1}\times 2^J}[A]_{2^J\times 2^J}[P]_{2^J\times 2^{J-1}}$. Residual equation becomes, $[A]_{2^{J-1}\times 2^{J-1}}[e]_{2^{J-1}\times 1} = [r]_{2^{J-1}\times 1}$ where $e_{2^{J-1}\times 1}$ is to be determined. Solve $e_{2^{J-1}\times 1}$ with initial guess '0'.
From (8),

$$r_{2^{J-2}\times 1} = [R]_{2^{J-2}\times 2^{J-1}}[r]_{2^{J-1}\times 1}. \tag{9}$$

and $[A]_{2^{J-2}\times 2^{J-2}} = [R]_{2^{J-2}\times 2^{J-1}}[A]_{2^{J-1}\times 2^{J-1}}[P]_{2^{J-1}\times 2^{J-2}}$. Then residual equation becomes, $[A]_{2^{J-2}\times 2^{J-2}}[e]_{2^{J-2}\times 1} = [r]_{2^{J-2}\times 1}$. Solve $e_{2^{J-2}\times 1}$ with initial guess '0'.
The procedure is continuing up to the coarsest ($\eta = J$) level, we have,

$$r_{1\times 1} = [R]_{1\times 2}[r]_{2\times 1}. \tag{10}$$

and $[A]_{1\times 1} = [R]_{1\times 2}[A]_{2\times 2}[P]_{2\times 1}$. Residual equation is, $[A]_{1\times 1}[e]_{1\times 1} = [r]_{1\times 1}$. Solve $e_{1\times 1}$ exactly.
Now correct the solution

$$u_{2\times 1} = [e]_{2\times 1} + [P]_{2\times 1}[e]_{1\times 1}.$$

Solve $[A]_{2\times2}[u]_{2\times1} = [r]_{2\times1}$ with initial guess $u_{2\times1}$.
Similarly, correct the solution

$$u_{4\times1} = [e]_{4\times1} + [P]_{4\times2}[u]_{2\times1}.$$

Solve $[A]_{4\times4}[u]_{4\times1} = [r]_{4\times1}$ with initial guess $u_{4\times1}$. Continue the procedure up to the finer ($\eta = 0$) level, Finally, correct the solution

$$u_{2^J\times1} = [e]_{2^J\times1} + [P]_{2^J\times2^{J-1}}[u]_{2^{J-1}\times1}.$$

Solve $[A]_{2^J\times2^J}[u]_{2^J\times1} = [f]_{2^J\times1}$ with initial guess $u_{2^J\times1}$.
$u_{2^J\times1}$ is the required solution of system (4).

3.2. **Wavelet multigrid (WMG) method.** The same procedure is applied as explained in the MG method (section 3.1). Instead of using R and P matrices, we used as

$$W(2^{J-1}\times 2^J) = \begin{bmatrix} h_0 & h_1 & h_2 & h_3 & 0 & 0 & ... & 0 & 0 \\ g_0 & g_1 & g_2 & g_3 & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & h_0 & h_1 & h_2 & h_3 & ... & 0 & 0 \\ 0 & 0 & g_0 & g_1 & g_2 & g_3 & ... & 0 & 0 \\ .. & .. & .. & .. & .. & .. & ... & .. & .. \\ 0 & ... & g_0 & g_1 & g_2 & g_3 & ... & 0 & 0 \end{bmatrix} \tag{11}$$

and W^T respectively.

3.3. **Modified wavelet multigrid (MWMG) method.** Here also the same procedure is applied as explained in the above methods (section 3.1). Instead of using Instead of using R and P matrices, we used as

$$WP(2^{J-1}\times 2^J) = \begin{bmatrix} h_0 & 0 & h_1 & 0 & h_2 & 0 & h_3 & 0 & ... & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & ... & 0 \\ g_0 & 0 & g_1 & 0 & g_2 & 0 & g_3 & 0 & ... & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & ... & 0 \\ .. & .. & .. & .. & .. & .. & ... & .. & .. & .. \\ 0 & ... & g_0 & 0 & g_1 & 0 & g_2 & 0 & g_3 & ... \end{bmatrix} \tag{12}$$

and WP^T respectively.

4. Numerical experiments

In this section, we present convection-diffusion problem which show the efficiency of MWMG based new prolongation and restriction operators in multigrid methods in the place of conventional prolongation and restriction operators. The error will be considered by $L_\infty = max|u_e - u_a|$, where u_e and u_a are exact and approximate solution respectively. The grid complexity (G_c) and operator complexity (O_c) of the methods are given as follows,

$$G_c = \frac{T_g}{F_g} \quad and \quad O_c = \frac{T_n}{F_n}. \tag{13}$$

where, T_g- Total number of all grid points, F_g- Number of grid points on the finest grid, T_n- Total number of non-zero entries in all matrices (A^k) and F_n- Number of non-zero entries in the finest grid matrix ($A^0 = A$).
Consider the Convection-Diffusion problem on the unit square

$$\Omega = (x, y) : 0 < x < 1, 0 < y < 1$$

$$-\varepsilon(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2})+a\frac{\partial u}{\partial x}=f(x,y), \quad on \;\; \Omega, \quad with \;\; u=0 \;\; on \;\; \partial\Omega. \tag{14}$$

where, $f(x,y) = \delta sin(l\pi y)(C_2x^2 + C_1x + C_0)$, $\varepsilon > 0$, $\delta \in R$, $a \in R$, l is an integer, $C_2 = -\varepsilon l^2\pi^2$, $C_1 = \varepsilon l^2\pi^2 - 2a$ and $C_0 = a + 2\varepsilon$. It has the exact solution $u(x,y) = \delta x(l-x)sin(l\pi y)$. After discretizing the Eqn. (14) through the finite difference method, we get system of equations, for $\varepsilon = 0.1, a = 0.1, l = 1, \delta = 0.1$ and $J = 4$ of the form

$$[A]_{16\times16}[u]_{16\times1} = [f]_{16\times1}. \tag{15}$$

Solve (15) through the iterative method, we get the approximate solution v of u. i.e. $u = e + v \Rightarrow v = u - e$, where e is (16×1 matrix) error to be determined. The implementation of the problem is given as the MWMG method discussed in section 3.3, as follows,
From equation (15), we find the residual as

$$r_{16\times1} = [f]_{16\times1} - [A]_{16\times16}[v]_{16\times1}. \tag{16}$$

We reduce the matrices in the finer level to coarsest level using WP Restriction operator and then construct the matrices back to finer level from the coarsest level using Prolongation operator WP^T.
From (16),

$$r_{8\times1} = [WP]_{8\times16}[r]_{16\times1}. \tag{17}$$

and $[A]_{8\times8} = [WP]_{8\times16}[A]_{16\times16}[WP^T]_{16\times8}$. Residual equation becomes, $[A]_{8\times8}[e]_{8\times1} = [r]_{8\times1}$. where $e_{8\times1}$ to be determine. Solve $e_{8\times1}$ with initial guess '0'.
From (17),

$$r_{4\times1} = [WP]_{4\times8}[r]_{8\times1}. \tag{18}$$

and $[A]_{4\times4} = [WP]_{4\times8}[A]_{8\times8}[WP^T]_{8\times4}$. Then residual equation becomes, $[A]_{4\times4}[e]_{4\times1} = [r]_{4\times1}$. where $e_{4\times1}$ to be determine. Solve $e_{4\times1}$ with initial guess '0'.
From (18),

$$r_{2\times1} = [WP]_{2\times4}[r]_{4\times1}. \tag{19}$$

and $[A]_{2\times2} = [WP]_{2\times4}[A]_{4\times4}[WP^T]_{4\times2}$. The residual equation becomes, $[A]_{2\times2}[e]_{2\times1} = [r]_{2\times1}$. Solve $e_{2\times1}$ with initial guess '0'.
From (19),

$$r_{1\times1} = [WP]_{1\times2}[r]_{2\times1}. \tag{20}$$

and $[A]_{1\times1} = [WP]_{1\times2}[A]_{2\times2}[WP^T]_{2\times1}$. Finally, residual equation is, $[A]_{1\times1}[e]_{1\times1} = [r]_{1\times1}$. Solve $e_{1\times1}$ exactly.
From $e_{1\times1}$, now correct the solution

$$u_{2\times1} = [e]_{2\times1} + [WP^T]_{2\times1}[e]_{1\times1}.$$

Solve $[A]_{2\times2}[u]_{2\times1} = [r]_{2\times1}$ with initial guess $u_{2\times1}$.
Similarly, correct the solution from $u_{2\times1}$,

$$u_{4\times1} = [e]_{4\times1} + [WP^T]_{4\times2}[u]_{2\times1}.$$

Solve $[A]_{4\times4}[u]_{4\times1} = [r]_{4\times1}$ with initial guess $u_{4\times1}$.
From $u_{4\times1}$, correct the solution,

$$u_{8\times1} = [v]_{8\times1} + [WP^T]_{8\times4}[u]_{4\times1}.$$

Solve $[A]_{8\times8}[u]_{8\times1} = [r]_{8\times1}$ with initial guess $u_{8\times1}$.
From $u_{8\times1}$, correct the solution,

$$u_{16\times1} = [v]_{16\times1} + [WP^T]_{16\times8}[u]_{8\times1}.$$

TABLE 1. Comparison with numerical solutions of Eq. (14) and exact solution for $\varepsilon = 0.1, a = 0.1, l = 1, \delta = 0.1, J = 4$

x	y	$MWMG$	$Exact$
0.2	0.2	0.00954	0.0094
0.4	0.2	0.01544	0.01522
0.6	0.2	0.01544	0.01522
0.8	0.2	0.00954	0.0094
0.2	0.4	0.01434	0.01411
0.4	0.4	0.0232	0.02283
0.6	0.4	0.0232	0.02283
0.8	0.4	0.01434	0.01411
0.2	0.6	0.01435	0.01411
0.4	0.6	0.02322	0.02283
0.6	0.6	0.02322	0.02283
0.8	0.6	0.01435	0.01411
0.2	0.8	0.00957	0.0094
0.4	0.8	0.01548	0.01522
0.6	0.8	0.01548	0.01522
0.8	0.8	0.00957	0.0094

Solve $[A]_{16\times16}[u]_{16\times1} = [f]_{16\times1}$ with initial guess $u_{16\times1}$. where, $u_{16\times1}$ is the required solution of system (14).
The numerical solutions of the given equation is obtained through the methods as explained in section 3 and are presented in comparison with the exact solution, in the tables 1 and 2 for $\varepsilon = 0.1, a = 0.1, l = 1, \delta = 0.1, J = 4$ and $J = 8$ respectively. In the figure 1 for $\varepsilon = 0.1, a = 0.1, l = 1, \delta = 0.1, J = 10$. Similarly we presented numerical solutions with exact solution for different set of parameters for $\varepsilon = 0.01, a = 1, l = 3, \delta = 0.2, J = 10$ in figure 2 and for $\varepsilon = 0.01, a = 10, l = 16, \delta = 0.3, J = 10$ in figure 3. The grid complexity and operator complexity of example as shown in the table 3, maximum error with the computational time given in the table 4 and advantage of the wavelet matrices give the well-conditioned system through the condition number of the matrices, which is defined by $K(A) = \parallel A \parallel \parallel A^{-1} \parallel$. We have presented the experimental rate of convergence $R_c(2^J)$ which is defined as,

$$R_c(2^J) = \frac{log[max(E(2^{J-2}))]/max(E(2^J))}{log2}. \tag{21}$$

The condition number of A and rate of convergence for different grid points are expressed in table 5.

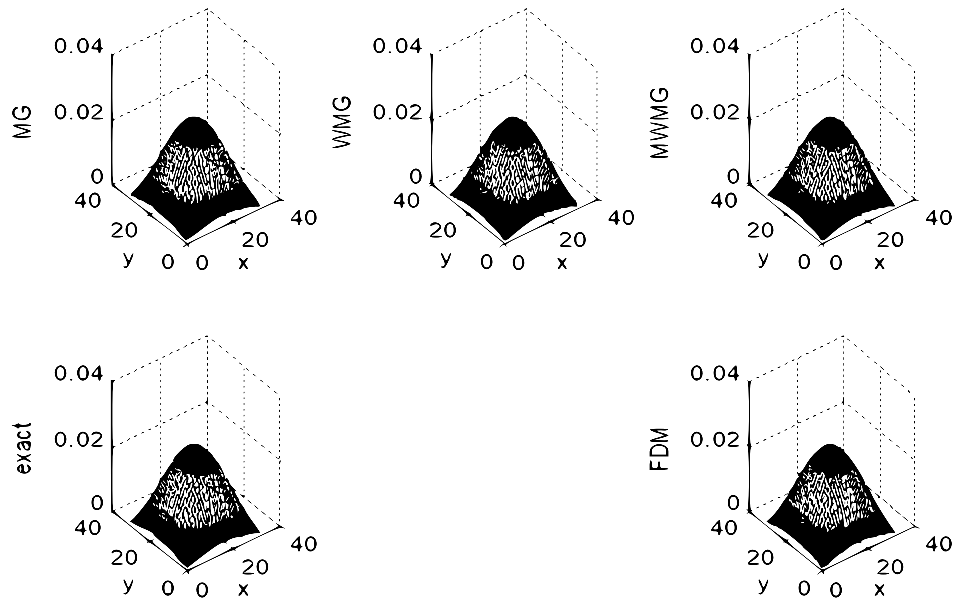

Fig. 1. Comparison with numerical solutions of Eq. (14) and exact solution for $\varepsilon = 0.1, a = 0.1, l = 1, \delta = 0.1, J = 10$.

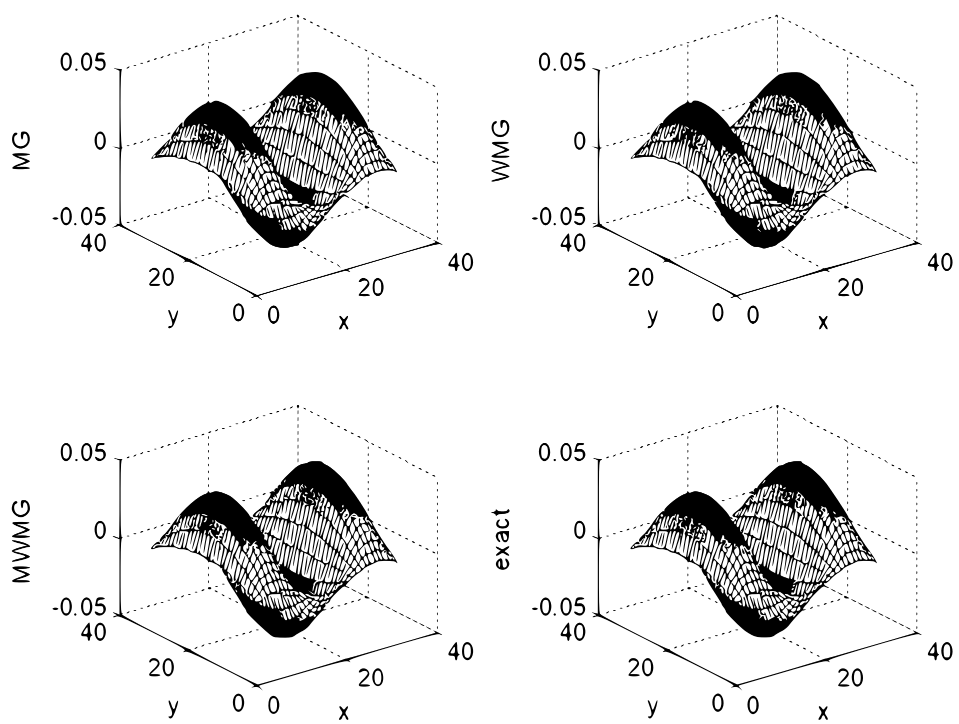

Fig. 2. Comparison with numerical solutions of Eq. (14) and exact solution for $\varepsilon = 0.01, a = 1, l = 3, \delta = 0.2, J = 10$.

Table 2. Comparison with numerical solutions of Eq. (14) and exact solution for $\varepsilon = 0.1, a = 0.1, l = 1, \delta = 0.1, J = 8$

x	y	$MWMG$	$Exact$	x	y	$MWMG$	$Exact$
0.1111	0.1111	0.00339	0.00338	0.5555	0.1111	0.00849	0.00844
0.1111	0.2222	0.00637	0.00635	0.5555	0.2222	0.01595	0.01587
0.1111	0.3333	0.00859	0.00855	0.5555	0.3333	0.0215	0.02138
0.1111	0.4444	0.00977	0.00973	0.5555	0.4444	0.02444	0.02432
0.1111	0.5555	0.00977	0.00973	0.5555	0.5555	0.02444	0.02432
0.1111	0.6666	0.00859	0.00855	0.5555	0.6666	0.02150	0.02138
0.1111	0.7777	0.00637	0.00635	0.5555	0.7777	0.01595	0.01587
0.1111	0.8888	0.00339	0.00338	0.5555	0.8888	0.00849	0.00844
0.2222	0.1111	0.00594	0.00591	0.6666	0.1111	0.00764	0.00760
0.2222	0.2222	0.01116	0.01111	0.6666	0.2222	0.01436	0.01428
0.2222	0.3333	0.01504	0.01497	0.6666	0.3333	0.01935	0.01925
0.2222	0.4444	0.01710	0.01702	0.6666	0.4444	0.02200	0.02188
0.2222	0.5555	0.01710	0.01702	0.6666	0.5555	0.02200	0.02188
0.2222	0.6666	0.01504	0.01497	0.6666	0.6666	0.01935	0.01925
0.2222	0.7777	0.01116	0.01111	0.6666	0.7777	0.01436	0.01428
0.2222	0.8888	0.00594	0.00591	0.6666	0.8888	0.00764	0.00760
0.2222	0.1111	0.00764	0.00760	0.7777	0.1111	0.00594	0.00591
0.3333	0.2222	0.01435	0.01428	0.7777	0.2222	0.01117	0.01111
0.3333	0.3333	0.01934	0.01925	0.7777	0.3333	0.01505	0.01497
0.3333	0.4444	0.02199	0.02188	0.7777	0.4444	0.01711	0.01702
0.3333	0.5555	0.02199	0.02188	0.7777	0.5555	0.01711	0.01702
0.3333	0.6666	0.01934	0.01925	0.7777	0.6666	0.01505	0.01497
0.3333	0.7777	0.01435	0.01428	0.7777	0.7777	0.01117	0.01111
0.3333	0.8888	0.00764	0.00760	0.7777	0.8888	0.00594	0.00591
0.4444	0.1111	0.00849	0.00844	0.8888	0.1111	0.00340	0.00338
0.4444	0.2222	0.01595	0.01587	0.8888	0.2222	0.00638	0.00635
0.4444	0.3333	0.02149	0.02138	0.8888	0.3333	0.00860	0.00855
0.4444	0.4444	0.02444	0.02432	0.8888	0.4444	0.00978	0.00973
0.4444	0.5555	0.02444	0.02432	0.8888	0.5555	0.00978	0.00973
0.4444	0.6666	0.02149	0.02138	0.8888	0.6666	0.00860	0.00855
0.4444	0.7777	0.01595	0.01587	0.8888	0.7777	0.00638	0.00635
0.4444	0.8888	0.00849	0.00844	0.8888	0.8888	0.00340	0.00338

Table 3. Operator and Grid complexity of linear systems for $\varepsilon = 0.1, a = 0.1, l = 1, \delta = 0.1, J = 10$

$Methods$	$Operator\ \ complexity$	$Grid\ \ complexity$
MG	2.74	1.94
WMG	2.67	1.94
MWMG	2.41	1.94

Table 4. Linear systems ($J = 10$) for ($\varepsilon = 0.1, a = 0.1, l = 1, \delta = 0.1$), ($\varepsilon = 0.01, a = 1, l = 3, \delta = 0.2$) and ($\varepsilon = 0.01, a = 10, l = 16, \delta = 0.3$) respectively. The comparison of Error (L_∞) and CPU time of the methods

Methods	*Error*(L_∞)	*Setup time*	*Running time*	*Total time*
FDM	1.38E-05	0.07	4.80	4.87
MG	9.89E-06	0.06	0.31	0.37
WMG	9.75E-06	0.07	0.16	0.23
MWMG	9.61E-06	0.01	0.09	0.10
FDM	1.98E-04	0.07	3.51	3.58
MG	1.76E-04	0.01	0.30	0.31
WMG	1.71E-04	0.01	0.13	0.14
MWMG	1.60E-04	0.01	0.12	0.13
FDM	1.67E-02	0.08	3.80	3.88
MG	1.65E-02	0.01	0.46	0.47
WMG	1.65E-02	0.01	0.22	0.23
MWMG	1.62E-02	0.01	0.10	0.11

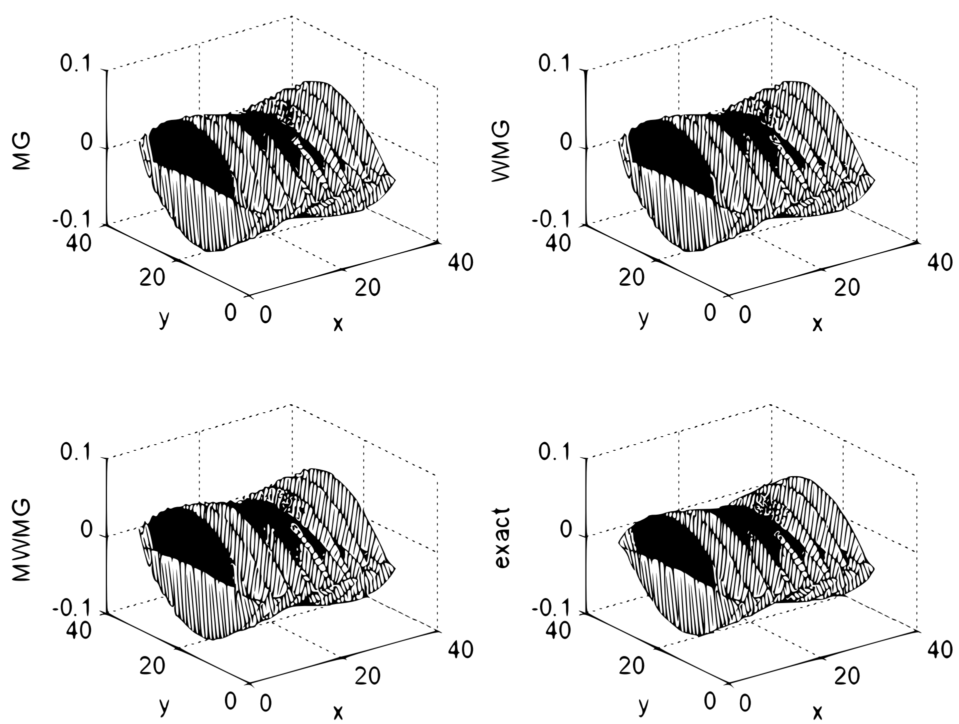

Fig. 3. Comparison with numerical solutions of Eq. (14) and exact solution for $\varepsilon = 0.01, a = 10, l = 16, \delta = 0.3, J = 10$.

Conclusions

In this paper, we introduced the modified wavelet multigrid method for the numerical solution of convection-diffusion problem using new intergrid operators of

TABLE 5. Linear systems for ($\varepsilon = 0.1, a = 0.1, l = 1, \delta = 0.1$). Condition number $K(A)$ and rate of convergence for different grids points

J	FDM	WMG	$MWMG$	$MWMG$ $Rate$ of $convergence$ $R_c(2^J)$
4	9.43E+00	1	1	-
6	3.20E+01	1	1	1.63
8	1.16E+02	1	1	1.82
10	4.38E+02	1	1	1.98

prolongation and restrictions based on Daubechies filters coefficients. In many of those problems where the standard multigrid and other wavelet multigrid methods are converges slowly with larger computational time whereas the modified wavelet multigrid method does ensure such slower convergence with lesser computational cost. This method is efficiently applied in such cases, the results are presented in this paper have demonstrated that it is worthwhile to further explore the effectiveness of improved wavelet multigrid method.

Acknowledgement: Author acknowledges the financial support from VGST-RGS/F, GRD. No. 898, Karnataka, India and also thank to BGMIT, Mudhol for their encouragement and support.

REFERENCES

[1] K. W. Morton, Numerical Solution of Convection-Diffusion Problems. Chapman and Hall, London, 1996.
[2] V. John, P. Knobloch, A comparison of spurious oscillations at layers diminishing (SOLD) methods for convection-diffusion equations: Part I, Preprint Nr. 156, FR 6.1- Mathematik, Universitat des Saarlandes, Saarbrucken, 2005.
[3] R. L. Burden, J. D. Faires, Numerical Analysis, 9th Edition, Brooks/Cole, Cengage Learning, 2011.
[4] W. Hackbusch, Multigrid Methods and Applications, vol. 4 of Springer Series in Computational Mathematics, Springer, Berlin, Heidelberg, 1985.
[5] P. Wesseling, An introduction to Multigrid Methods, Wiley, Chichester, 1992.
[6] W. L. Briggs, V. E. Henson, S.F. McCormick, A Multigrid Tutorial (2nd ed.), SIAM, Philadelphia, 2000.
[7] U. Trottenberg, C.W. Oosterlee, A. Schuler, Multigrid, Academic Press, London, 2001.
[8] M. Griebel, S. Knapek, A multigrid homogenization method, Modeling and Computation in Environmental Sciences, Stuttgart, 1995, Notes on Numerical Fluid Mechanics, vol. 59, Vieweg, Braunschweig, 1995, pp. 187-202.
[9] D. De Leon, A new wavelet multigrid method, J. of Comp. and Appl. Math. 220 (2008) 674-685.
[10] M. H. Kantli, S. C. Shiralashetti, Finite difference Wavelet–Galerkin method for the numerical solution of elastohydrodynamic lubrication problems, J. Analysis, 26:2 (2018) 285-295.
[11] R. A. Mundewadi, B. A. Mundewadi, M.H. Kantli, Iterative Scheme of Integral and Integro-differential Equations using Daubechies Wavelets New Transform Method, Int. J. Appl. Comput. Math., 6:135, (2020), https://doi.org/10.1007/s40819-020-00879-2.
[12] N. M. Bujurke, M. H. Kantli, Jacobian-free Newton-Krylov subspace method with wavelet-based preconditioner for analysis of transient elastohydrodynamic lubrication problems with surface asperities, Applied Mathematics and Mechanics, 41, 881-898, (2020).
[13] N. M. Bujurke, M. H. Kantli, B. M. Shetter, Wavelet preconditioned Newton-Krylov method for elastohydrodynamic lubrication of line contact problems, Appl. Math. Model, 46 (2017) 285-298.

[14] N. M. Bujurke, M. H. Kantli, B. M. Shetter, Jacobian free Newton-GMRES method for the solution of elastohydrodynamic grease lubrication in line contact using wavelet based preconditioners", Proce. Nati. Acad. Sci. India Sect. A: Phy. Sci. 88:2 (2018) 247-265.
[15] S. C. Shiralashetti, M. H. Kantli, A. B. Deshi, New wavelet based full-approximation scheme for the numerical solution of nonlinear elliptic partial differential equations, Alex. Eng. J., 55 (2016) 2797-2804.
[16] S. C. Shiralashetti, M. H. Kantli, A. B. Deshi, A new wavelet multigrid method for the numerical solution of elliptic type differential equations, Alex. Eng. J., 57 (2018) 203-209.
[17] S. C. Shiralashetti, M. H. Kantli, Wavelet based decoupled method for the investigation of surface roughness effects in elastohydrodynamic lubrication problems using couple stress fluid", Ain Shams Eng. J. 9 (2018) 757-766. https://doi.org/10.1016/ j.asej.2016.04.024.
[18] A. Avudainayagam, C. Vani, Wavelet based multigrid methods for linear and nonlinear elliptic partial differential equations, Appl. Math. Comput. 148 (2004) 307-320.
[19] O. V. Vasilyev, N. K. R. Kevlahan, An adaptive multilevel wavelet collocation method for elliptic problems, J. Comput. Phys. 206 (2005) 212-431.
[20] I. Daubechies, Ten Lectures on Wavelets, MA, SIAM, Philadelphia, 1992.
[21] I. Daubechies, Orthonormal bases of compactly supported wavelets, Commun. Pure Appl. Math., 41 (1988) 909-996.

Mounesha H. Kantli
Department of Mathematics, Biluru Gurubasava Mahaswamiji Institute of Technology, Mudhol-587313, India
Email address: mkantli@gmail.com