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ANALYTICAL SOLUTION OF RAYLEIGH-STOKES PROBLEM WITH KATUGAMPOLA FRACTIONAL DERIVATIVE

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Abstract. In this paper, the Katugampola Fourier sine transform and Katugampola Laplace transform were used to solve the Stokes first problem and the Rayleigh-Stokes problem for generalized second grade fluid with Katugampola fractional derivative. Exact solutions for both the velocity and temperature have been achieved. The solutions of the classical problem for both Stokes first problem and Rayleigh-Stokes problem have been obtained as limiting cases.

1. Introduction

Fractional partial differential equations which are the generalization of differential equations are successful models of real events and have many applications in various fields in science and engineering [4, 7, 11, 12]. These applications appears in gravitations elastic membrane, electrostatics, fluid flow, steady state, heat conduction and many other topics in both pure and applied mathematics. Typical example of fractional partial differential equations of time fractional advection-dispersion equation as in [6, 8], fractional diffusion equations as in [5, 9, 10, 16, 17], fractional wave equations as in [15]. The Rayleigh-Stokes fractional equations [1]. In this paper we consider Stokes first fractional equations for the flat plate and the Rayleigh-Stokes fractional equation. Exact solution of these equations will be investigated. The Katugampola Fourier sine and Laplace transforms are used for getting exact solutions for these equations. The fractional terms in Stokes and Rayleigh-Stokes equations are considered as Katugampola fractional derivatives.
Most of the fractional derivatives are defined via fractional integral .Two of which are the most popular ones.

(i) Riemann $-$ Liouville definition [1, 12]. For $\alpha \in [n-1, n)\,,\ n \in N$ the derivative $of\ f\ is$

$$^{RL}D_a^{\alpha}\left(f\right)\left(t\right) = \tfrac{1}{\Gamma(n-\alpha)}\tfrac{d^n}{dt^n}\int_a^t f\left(x\right)\left(t-x\right)^{n-\alpha-1}dx.$$

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(ii) Caputo definition [12] . For $\alpha \in [n-1, n)$, $n \in N$ the derivative of f *is*

$$ {}^{C}D_{a}^{\alpha}\left(f\right)\left(t\right)=\frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}f^{(n)}\left(x\right)\left(t-x\right)^{n-\alpha-1}dx. $$

In [3] U.N. Katugampola in 2014 introduced and studies a new fractional derivative called "Katugampola fractional derivative" as follows

Definition 1.1 Let $f : [0, \infty) \to R$ and $t>0$. Then the" Katugampola fractional derivative" of f of order α is defined by

$$ D_{t}{}^{\alpha}(f)(t)=\lim_{\varepsilon\to 0}\frac{f(t e^{\varepsilon t^{-\alpha}})-f(t)}{\varepsilon}, \tag{1} $$

for $t>0$ *and* $\alpha \in (0, 1]$. *If* f is $\alpha-$*differentiable in some*$(0, a)$, $a>0$ *and* $\lim_{t\to 0^{+}} D^{\alpha}(f)(t)$ *exists, then* $D_t{}^{\alpha}(f)(0) = \lim_{t\to 0^{+}} D_t{}^{\alpha}(f)(t)$.
If $\alpha \in (n,\ n+1]$, *for some* $n \in N$ *and* f *be an* $n-$*differentiable at* $t > 0$. *Then the* $\alpha-$*fractional derivative of* f *is defined by*

$$ D^{\alpha}f(t)=\lim_{\varepsilon\to 0}\frac{f^{(n)}\left(t.e^{\varepsilon.t^{n-\alpha}}\right)-f^{(n)}\left(t\right)}{\varepsilon} \tag{2} $$

if the limit exists.

Note that Katugampola derivative satisfies product rule, quotient rule, chain rule,. . . etc. and it is consistent in its properties with the classical calculus of integer order. In addition
If $\alpha \in (n,\ n+1]$, for some $n \in N$ and f an $(n+1)-$differentiable at $t > 0$. Then

$$ D^{\alpha}f(t)=\ t^{n+1-\alpha}f^{(n+1)}\left(t\right), \tag{3} $$

and for $\alpha \in (0,\ 1]$, $t > 0$, we have $D^{\alpha}\mathrm{f}(t) =\ t^{1-\alpha}\frac{df}{dt}(t)$.

Definition 1.2 [13] **(Katugampola Fourier cosine and sine transforms)**
Let $\alpha \in (n, n+1]$, $n \in \mathrm{N}$ *and* $f(x)$*be a real valued function. The infinite Katugampola Fourier cosine transform of a function* $f : [0, \infty) \to R$ *which denoted by* $F_c^{\alpha}\{f(x)\}$ *is defined as*

$$ F_{c}^{\alpha}\{f(x)\}=F_{c}^{\alpha}\left(\kappa\right)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}f(x)\cos\left(\kappa\frac{x^{\alpha-n}}{\alpha-n}\right)x^{\alpha-n-1}\,dx, \tag{4} $$

and the infinite Katugampola Fourier sine transform of a function $f : [0, \infty) \to R$ *which denoted by* $F_s^{\alpha}\{f(x)\}$ *is defined as*

$$ F_{s}^{\alpha}\{f(x)\}=F_{s}^{\alpha}\left(\kappa\right)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}f(x)\sin\left(\kappa\frac{x^{\alpha-n}}{\alpha-n}\right)x^{\alpha-n-1}\,dx. \tag{5} $$

The following Lemma is the relation between the Katugampola Fourier sine transform and the usual Fourier sine transform.
Lemma 1.3 [13]
Let $\alpha \in (n, n+1]$ *and* $f : [0, \infty) \to R$, $n \in N$ *be a function. Then*

$$ \Im_{s}\{f(x)\}(\kappa)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}f(x)\sin\left(\kappa x\right)dx $$

, where $F_s^{\alpha}\{f(x),\kappa\} = \Im_s\left\{f(((\alpha-n)\,x)^{\frac{1}{\alpha-n}})\right\}(\kappa)$

Lemma 1.4 [13]
Let $\alpha \in (n, n+1]$, $n \in N$ and $F_s^{\alpha}\{f(x),\kappa\}$ be Katugampola Fourier sine transform of a function $f(x)$. Then, the inversion formula for Katugampola Fourier sine transform of $F_s^{\alpha}\{f(x)\}(\kappa)$ is

$$(F_s^{\alpha})^{-1}\{F_s^{\alpha}\{f(x)\}(\kappa)\} = f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_s^{\alpha}\left\{f\left(\frac{x^{\alpha-n}}{\alpha-n}\right)\right\}(\kappa)\ \sin\left(\kappa\frac{x^{\alpha-n}}{\alpha-n}\right)d\kappa \tag{6}$$

Theorem1.5 [13]
Let f be a function defined for $x \geq 0$, $f^{(n)}(x) \to 0$ as $x \to \infty$ and $\alpha \in (n, n+1]$, $n \in N$. Then

$$i)\ \ F_c^{\alpha}\left(D_x^{\alpha}f\left(x\right)\right) = \kappa F_s^{\alpha}\left(f^{(n)}\left(x\right)\right) - \sqrt{\frac{2}{\pi}}f^{(n)}\left(0\right). \tag{7}$$

$$ii)\ \ F_s^{\alpha}\left(D_x^{\alpha}f\left(x\right)\right) = -\kappa F_c^{\alpha}\left(f^{(n)}\left(x\right)\right). \tag{8}$$

Theorem 1.6 [13]
Let f be a function defined for $x \geq 0$, $f(x) \to 0$ as $x \to \infty$ and $\alpha \in \left(\frac{1}{2}, 1\right]$. Then,

$$F_s^{\alpha}\left(D_x^{2\alpha}f\left(x\right)\right) = -\kappa^2 F_s^{\alpha}\left(f\left(x\right)\right) + \kappa\sqrt{\frac{2}{\pi}}f\left(0\right). \tag{9}$$

Definition 1.7 [14] **(Katugampola Laplace Transform)**
Let $\alpha \in (n, n+1]$,for some $n \in N$ and $f : [0,\infty) \to R$ be a real valued function. Then, the Katugampola Laplace transform of $f(t)$ of order α is defined as

$$L_{\alpha}\{f(t)\}(p) = \widetilde{f}(p) = \int_0^{\infty} e^{-p\frac{t^{\alpha-n}}{\alpha-n}} f(t)\ t^{\alpha-n-1}\ dt. \tag{10}$$

Theorem 1.8 [14]
Let $\alpha \in (n, n+1]$, for some $n \in N$ and $f : [0,\infty) \to R$ be differentiable real valued function. Then

$$L_{\alpha}\{D_t^{\alpha}f(t)\}(p) = p\,L_{\alpha}\left\{f^{(n)}(t)\right\}(p) - f^{(n)}(0). \tag{11}$$

Lemma 1.9 [14]
Let $\alpha \in \left(\frac{k-1}{k}, 1\right]$, $k \in \mathrm{N}$ and $f\left(t\right)$ be $k\alpha-$differentiable real valued function. Then,

$$L_{\alpha}\left\{D_t^{2\alpha}f\left(t\right)\right\}(p) = p^k\,L_{\alpha}\left\{f\left(t\right)\right\}(p) - \sum_{m=0}^{k-1} p^{k-m-1}\ D_t^{m\alpha}f\left(0\right). \tag{12}$$

The following Lemma is the relation between the Katugampola Laplace transform and usual Laplace transform.

Lemma 1.10 [14]
Let $f : (0,\infty) \to R$ be a function such that $\alpha \in (n, n+1]$, $n \in N$ and $L_{\alpha}\{f(t),p\} = \widetilde{f}(p)$. Then,

$$L_{\alpha}\{f(t)\}(p) = L\{f(\,((\alpha-n)\,t)^{\frac{1}{\alpha-n}}\,)\}(p) \tag{13}$$

where $L\{f(x)\}(p) = \int_0^\infty f(t)e^{-px}dx$.

Theorem 1.11 [14] **(Convolution Theorem)**
Let $g(t)$ *and* $h\,(t)$ *be arbitrary functions. Then,*

$$L_\alpha\left\{g * h\right\} = L_\alpha\left\{g\,(t)\right\} L_\alpha\left\{h\,(t)\right\} = L_\alpha\left\{g\right\} L_\alpha\left\{h\right\}, \tag{14}$$

where $g * h$ *is the Convolutions of function* $g(t)$ *and* $h\,(t)$ *defined as*

$$g * h = \int_0^t g\,(x)\,h(t-x)dx.$$

Remark: Let $g(x)$ and $h\,(x)$ be arbitrary functions, and let ${L_\alpha}^{-1}\left\{\widetilde{g}(p)\right\} = g\,(x)$ and ${L_\alpha}^{-1}\left\{\widetilde{h}(p)\right\} = h\,(x)$. Then

$$(g * h)\,(t) = {L_\alpha}^{-1}\left\{L_\alpha\left\{(g * h)\,(t)\right\}\right\} = {L_\alpha}^{-1}\left\{L_\alpha\left\{g\,(t)\right\} {L_\alpha}^{-1}\left\{h\,(t)\right\}\right\}. \tag{15}$$

2. STOKES FIRST PROBLEM FOR A HEATED FLAT

Consider the Stokes first problem for a heated flat plate

$$D_t^\beta u\,(x,t) = \left(a + bD_t^\beta\right) D_x^{2\alpha} u\,(x,t)\,, \qquad x,\ t > 0, \tag{16}$$

where $u(x,t)$ is the velocity, $t$is the time, x is the distance and a, b are constants with respect to x and t, $D_x^{2\alpha}, D_t^\beta$ are Katugampola fractional derivatives of order $\alpha \in \left(\frac{1}{2}, 1\right]$ and $\beta \in (0,1]$ respectively . The corresponding initial and boundary condition of Eq.(16), are

$$u(x,0) = f_0(x), \qquad x>0 \tag{17}$$

$$\frac{\partial u\,(x,0)}{\partial t} = f_1(x), \qquad x>0 \tag{18}$$

$$u(0,t) = U, \qquad t>0 \tag{19}$$

$$u(x,t),\ \frac{\partial u\,(x,t)}{\partial x} \to 0 \quad \text{as} \quad x \to \infty. \tag{20}$$

Employing the non-dimensional quantities

$$u^* = \frac{u}{U},\ x^* = \frac{xU}{a},\ t^* = \frac{tU^2}{a},\ \ \eta = \frac{bU^2}{a^2} \tag{21}$$

Eqs.(16) to (20) are reduced to non-dimensional equations as follows (for brevity the dimensionless marks "*" are omitted here)

$$D_t^\beta u\,(x,t) = \left(1 + \eta D_t^\beta\right) D_x^{2\alpha} u\,(x,t) \tag{22}$$

$$u(x,0) = f_0(x), \qquad x>0 \tag{23}$$

$$\frac{\partial u\,(x,0)}{\partial t} = f_1(x), \qquad x>0 \tag{24}$$

$$u(0,t) = 1, \qquad t>0 \tag{25}$$

$$u(x,t),\ \frac{\partial u\,(x,t)}{\partial x} \to 0 \quad as \quad x \to \infty. \tag{26}$$

Making use of Katugampola Fourier sine integral transform and boundary conditions (25) and (26) and Theorem 1.8. Then Eqs.(22) and (23) yield

$$D_t^\beta F_s^\alpha\left\{u(x,t)\right\} = -\kappa^2\left(1 + \eta D_t^\beta\right) F_s^\alpha\left\{u(x,t)\right\} + \sqrt{\frac{2}{\pi}}\,\kappa \tag{27}$$

and

$$F_s^{\alpha}\left\{u(x,0),\kappa\right\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f_0(x)\sin\left(\kappa\frac{x^{\alpha}}{\alpha}\right)\, x^{\alpha-1}\,dx \tag{28}$$

Hence the Katugampola Laplace transform of Eq.(27) is

$$pL_{\beta}\left\{F_s^{\alpha}\left\{u(x,t)\right\}\right\} - F_s^{\alpha}\left\{f_0(x)\right\} =$$

$$-\kappa^2 L_{\beta}\left\{F_s^{\alpha}\left\{u(x,t)\right\}\right\} + \frac{1}{p}\sqrt{\frac{2}{\pi}}\,\kappa - \eta\kappa^2\left[pL_{\beta}\left\{F_s^{\alpha}\left\{u(x,t)\right\}\right\} - F_s^{\alpha}\left\{f_0(x)\right\}\right]$$

and this equation leads to

$$L_{\beta}\left\{F_s^{\alpha}\left\{u(x,t)\right\}\right\} = \frac{\left(1+\eta\kappa^2\right)F_s^{\alpha}\left\{f_0(x)\right\}}{p\left(1+\eta\kappa^2\right)+\kappa^2} + \frac{\frac{1}{p}\sqrt{\frac{2}{\pi}}\kappa}{p\left(1+\eta\kappa^2\right)+\kappa^2}\,,$$

$$=\frac{F_s^{\alpha}\left\{f_0(x)\right\}}{p+\frac{\kappa^2}{1+\eta\kappa^2}} + \frac{1}{\kappa}\sqrt{\frac{2}{\pi}}\left[\frac{1}{p} - \frac{1}{p+\frac{\kappa^2}{1+\eta\kappa^2}}\right]. \tag{29}$$

The inverse Katugampola Laplace transform of Eq.(29) implies

$$F_s^{\alpha}\left\{u(x,t)\right\} = F_s^{\alpha}\left\{f_0(x)\right\} e^{-\left(\frac{\kappa^2}{1+\eta\kappa^2}\right)\frac{t^{\beta}}{\beta}} + \frac{1}{\kappa}\sqrt{\frac{2}{\pi}}\left[1 - e^{-\left(\frac{\kappa^2}{1+\eta\kappa^2}\right)\frac{t^{\beta}}{\beta}}\right], \tag{30}$$

Now considering the inverse Katugampola Fourier sine integral transform of Eq.(30). We get

$$u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\left[\begin{array}{l} F_s^{\alpha}\left\{f_0(x)\right\} e^{-\left(\frac{\kappa^2}{1+\eta\kappa^2}\right)\frac{t^{\beta}}{\beta}} \\ +\frac{1}{\kappa}\sqrt{\frac{2}{\pi}}\left(1 - e^{-\left(\frac{\kappa^2}{1+\eta\kappa^2}\right)\frac{t^{\beta}}{\beta}}\right)\end{array}\right]\sin\left(\kappa\frac{x^{\alpha}}{\alpha}\right)d\kappa \tag{31}$$

Special cases:

1. When $F_s^{\alpha}\left\{f_0(x),\kappa\right\} = 0$, then Eq.(31) yields

$$u(x,t) = \frac{2}{\pi}\int_0^{\infty}\frac{1}{\kappa}\left(1 - e^{-\left(\frac{\kappa^2}{1+\eta\kappa^2}\right)\frac{t^{\beta}}{\beta}}\right)\sin\left(\kappa\frac{x^{\alpha}}{\alpha}\right)d\kappa \tag{32}$$

2. When $F_s^{\alpha}\left\{f(x),\kappa\right\} = 0$, and $\alpha = \beta = 1$ then Eq.(31) becomes

$$u(x,t) = \frac{2}{\pi}\int_0^{\infty}\frac{1}{\kappa}\left(1 - e^{-\left(\frac{\kappa^2}{1+\eta\kappa^2}\right)t}\right)\sin\left(\kappa x\right)d\kappa \tag{33}$$

which is the result obtained by Fetacau and Corina [2].

3. THE ENERGY FRACTIONAL EQUATIONT

The time-fractional energy equation, when the Fourier's law of heat conduction is considered may be written in the form

$$\frac{d}{c\rho}D_x^{2\delta}\theta\left(x,t\right) + \frac{\nu}{c}\left(D_x^{\delta}u\left(x,t\right)\right)^2 + \frac{r\left(x,t\right)}{c\rho} = D_t^{\gamma}\theta\left(x,t\right) \tag{34}$$

where $r\left(x,t\right)$ is the radiant heating, which is neglected in this paper c is the specific heat, ρ is the density of fluid and d is the conductivity which is assumed to be constant and $D_x^{2\delta}, D_t^{\gamma}$ are Katugampola fractional derivatives of order $\delta \in \left(\frac{1}{2},1\right]$, $\gamma \in (0,1]$. The corresponding initial and boundary condition of Eq.(34), are

$$\theta(x,0) = h_0(x), \qquad x>0 \tag{35}$$

$$\theta(0,t) = T_0, \qquad t \geq 0 \tag{36}$$

$$\theta(x,t),\ \frac{\partial\theta\left(x,t\right)}{\partial x}\rightarrow 0\quad as\quad x\rightarrow\infty. \tag{37}$$

Applying the non-dimensional quantities

$$\theta^{*}=\frac{\theta}{T_0},\ \nu^{*}=\frac{u}{U},\ x^{*}=\frac{xU}{\nu},\ t^{*}=\frac{tU^2}{\nu},\ \lambda=\frac{tU^2}{cT_0},\ \mathrm{Pr}=\frac{c\rho\nu}{d} \tag{38}$$

Eqs.(34), (35) and (36) can be reduced to non-dimensional equations as follows (for brevity the dimensionless marks "*" are omitted here)

$$\frac{1}{\mathrm{Pr}}D_x^{2\delta}\theta\left(x,t\right)+\lambda\left(D_x^{\delta}u\left(x,t\right)\right)^2=D_t^{\gamma}\theta\left(x,t\right) \tag{39}$$

$$\theta(x,0)=h_0(x),\qquad x>0 \tag{40}$$

$$\theta(0,t)=1,\qquad t\geq 0 \tag{41}$$

$$\theta(x,t),\ \frac{\partial\theta\left(x,t\right)}{\partial x}\rightarrow 0\quad as\quad x\rightarrow\infty. \tag{42}$$

Letting $g\left(x,t\right)=\lambda\left(D_x^{\delta}u\left(x,t\right)\right)^2$, then Eq.(39) can be rewritten as

$$\frac{1}{\mathrm{Pr}}D_x^{2\delta}\theta\left(x,t\right)+g(x,t)=D_t^{\gamma}\theta\left(x,t\right),\delta\in\left(\tfrac{1}{2},1\right],\ \gamma\in\left(0,1\right]. \tag{43}$$

Applying Katugampola Fourier sine integral transform with respect to "x" for Eqs. (40) and (43), we get

$$\frac{1}{\mathrm{Pr}}\left(-\kappa^2F_s^{\delta}\left\{\theta\left(x,t\right)\right\}+\sqrt{\frac{2}{\pi}}\ \kappa\right)+F_s^{\delta}\left\{g(x,t)\right\}=D_t^{\gamma}F_s^{\delta}\left\{\theta(x,t)\right\} \tag{44}$$

$$F_s^{\delta}\left\{\theta(x,0),\kappa\right\}=\sqrt{\frac{2}{\pi}}\int_0^{\infty}h_0(x)\sin\left(\kappa\frac{x^{\delta}}{\delta}\right)\ x^{\delta-1}\,dx \tag{45}$$

Hence the Katugampola Laplace transform with respect to "t" of Eq.(44) is

$$\begin{aligned}\tfrac{1}{\mathrm{Pr}}\left(-\kappa^2L_{\gamma}\left\{F_s^{\delta}\left\{\theta\left(x,t\right)\right\}\right\}+\tfrac{1}{p}\sqrt{\tfrac{2}{\pi}}\ \kappa\right)+L_{\gamma}\left\{F_s^{\delta}\left\{g(x,t)\right\}\right\}\\ =pL_{\gamma}\left\{F_s^{\delta}\left\{\theta\left(x,t\right)\right\}\right\}\ -F_s^{\delta}\left\{h_0(x)\right\}\end{aligned}$$

and this equation leads to

$$\begin{aligned}L_{\gamma}\left\{F_s^{\delta}\left\{\theta\left(x,t\right)\right\}\right\}\ &=\frac{L_{\gamma}\left\{F_s^{\delta}\left\{g(x,t)\right\}\right\}}{p+\frac{\kappa^2}{\mathrm{Pr}}}+\frac{F_s^{\delta}\left\{h_0(x)\right\}}{p+\frac{\kappa^2}{\mathrm{Pr}}}+\frac{\frac{1}{\mathrm{Pr}}\sqrt{\frac{2}{\pi}}\kappa}{p\left(p+\frac{\kappa^2}{\mathrm{Pr}}\right)}\ ,\\ &=\frac{L_{\gamma}\left\{F_s^{\delta}\left\{g(x,t)\right\}\right\}}{p+\frac{\kappa^2}{\mathrm{Pr}}}+\frac{F_s^{\delta}\left\{h_0(x)\right\}}{p+\frac{\kappa^2}{\mathrm{Pr}}}+\frac{1}{\kappa}\sqrt{\frac{2}{\pi}}\left[\frac{1}{p}-\frac{1}{p+\frac{\kappa^2}{\mathrm{Pr}}}\right]\end{aligned} \tag{46}$$

Taking the inverse Katugampola Laplace transform of Eq.(46)

$$\begin{aligned}F_s^{\delta}\left\{\theta\left(x,t\right)\right\}=\textstyle\int_0^t G(\kappa,t-\tau)F_s^{\delta}\left\{g(x,\tau)\right\}d\tau+F_s^{\delta}\left\{h_0(x)\right\}e^{-\left(\frac{\kappa^2}{\mathrm{Pr}}\right)\frac{t^{\gamma}}{\gamma}}\\ +\tfrac{1}{\kappa}\sqrt{\tfrac{2}{\pi}}\left[1-e^{-\left(\frac{\kappa^2}{\mathrm{Pr}}\right)\frac{t^{\gamma}}{\gamma}}\right]\end{aligned}\ , \tag{47}$$

where $G(\kappa,t)=L_{\gamma}{}^{-1}\left\{\frac{1}{p+\frac{\kappa^2}{\mathrm{Pr}}}\right\}=e^{-\left(\frac{\kappa^2}{\mathrm{Pr}}\right)\frac{t^{\gamma}}{\gamma}}$ is the Green's function of Eq.(44)

Inverting Eq.(47) by Katugampola Fourier sine integral transform ,we get

$$\theta(x,t)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\left[\begin{array}{l}\int_0^t G(\kappa,t-\tau)F_s^{\delta}\left\{g(x,\tau)\right\}d\tau\\ +F_s^{\delta}\left\{h_0(x)\right\}e^{-\left(\frac{\kappa^2}{\mathrm{Pr}}\right)\frac{t^{\gamma}}{\gamma}}+\frac{1}{\kappa}\sqrt{\frac{2}{\pi}}\left[1-e^{-\left(\frac{\kappa^2}{\mathrm{Pr}}\right)\frac{t^{\gamma}}{\gamma}}\right]\end{array}\right]\sin\left(\kappa\frac{x^{\delta}}{\delta}\right)d\kappa$$

$$= \sqrt{\tfrac{2}{\pi}} \int_0^\infty \sin\left(\kappa \tfrac{x^\delta}{\delta}\right) \int_0^t G(\kappa, t-\tau) F_s^\delta \{g(x,\tau)\}\, d\tau d\kappa$$
$$+ \sqrt{\tfrac{2}{\pi}} \int_0^\infty \sin\left(\kappa \tfrac{x^\alpha}{\alpha}\right) F_s^\delta \{h_0(x)\}\, e^{-\left(\frac{\kappa^2}{\mathrm{Pr}}\right)\frac{t^\gamma}{\gamma}} d\kappa$$
$$+ \frac{2}{\pi} \int_0^\infty \frac{\sin\left(\kappa \frac{x^\delta}{\delta}\right)}{\kappa} \left[1 - e^{-\left(\frac{\kappa^2}{\mathrm{Pr}}\right)\frac{t^\gamma}{\gamma}}\right] d\kappa \tag{48}$$

where $G(\kappa, t) = {L_\gamma}^{-1}\left\{\frac{1}{p + \frac{\kappa^2}{\mathrm{Pr}}}\right\} = e^{-\left(\frac{\kappa^2}{\mathrm{Pr}}\right)\frac{t^\gamma}{\gamma}}$ is the Green's function of Eq.(44)

Special cases:

1. When $F_s^\alpha \{h_0(x), \kappa\} = 0,\ \lambda = 1$, then Eq.(48) yields

$$\theta(x,t) = \sqrt{\tfrac{2}{\pi}} \int_0^\infty \sin\left(\kappa \tfrac{x^\delta}{\delta}\right) \int_0^t G(\kappa, t-\tau) F_s^\delta \{g(x,\tau)\}\, d\tau d\kappa$$
$$+ \tfrac{2}{\pi} \int_0^\infty \frac{\sin\left(\kappa \frac{x^\delta}{\delta}\right)}{\kappa} \left[1 - e^{-\left(\frac{\kappa^2}{\mathrm{Pr}}\right)t}\right] d\kappa \tag{49}$$

where $G(\kappa, t) = e^{-\left(\frac{\kappa^2}{\mathrm{Pr}}\right)\frac{t^\gamma}{\gamma}}$ is the Green's function of Eq.(44)

2. When $F_s^\delta \{h_0(x), \kappa\} = 0$, $F_s^\delta \{g(x,t), \kappa\} = 0$, and $\delta = \lambda = 1$ then Eq.(48) becomes

$$\theta(x,t) = \tfrac{2}{\pi} \int_0^\infty \frac{\sin(\kappa x)}{\kappa} \left[1 - e^{-\left(\frac{\kappa^2}{\mathrm{Pr}}\right)t}\right] d\kappa$$
$$= \int_0^\infty \tfrac{2}{\pi\kappa} \sin(\kappa x)\, d\kappa - \int_0^\infty \tfrac{2}{\pi} \sin(\kappa x)\ e^{-\left(\frac{\kappa^2}{\mathrm{Pr}}\right)t} d\kappa$$
$$= 1 - \frac{2}{\pi} \int_0^\infty \frac{\sin(\kappa x)}{\kappa}\ e^{-\left(\frac{\kappa^2}{\mathrm{Pr}}\right)t} d\kappa = 1 - erf\left(\frac{x}{2\sqrt{\frac{t}{\mathrm{Pr}}}}\right) \tag{50}$$

which is the result obtained also by Fetacau and Corina [2].

4. THE RAYLEIGH-STOKES PROBLEM FOR A HEATED FLAT PLATE

The Rayleigh-Stokes involving a time-fractional derivatives is written as

$$D_t{}^\beta u(x,z,t) = \left(a + b D_t{}^\beta\right)\left(D_x^{2\alpha} + D_z^{2\alpha}\right) u(x,z,t)\ , \quad x,\ z,\ t > 0, \tag{51}$$

where $u(x,y,z)$ is the velocity in $xz-$plane , t is the time, and a, b are constants with respect to x and t, $D_x^{2\alpha}, D_t^\beta$ are Katugampola fractional derivatives of order $\alpha \in \left(\frac{1}{2}, 1\right]$, $\beta \in (0,1]$ respectively . The corresponding initial and boundary condition of Eq.(51), are

$$u(x,z,0) = f_0(x,z), \qquad x>0, z>0 \tag{52}$$

$$\frac{\partial u(x,z,0)}{\partial t} = f_1(x,z), \qquad x>0, z>0 \tag{53}$$

$$u(0,z,t) = u(x,0,t) = U, \qquad t>0 \tag{54}$$

$$u(x,z,t),\ \frac{\partial u(x,z,t)}{\partial x},\ \frac{\partial u(x,z,t)}{\partial z} \to 0 \quad as \quad x^2 + z^2 \to \infty. \tag{55}$$

Employing the non-dimensional quantities given by Eq.(21) and $z^* = \frac{zU}{a}$, Eqs. (51), (52), (53) and (54) reduce to non-dimensional equations as follows (for brevity the dimensionless marks "*" are omitted here)

$$D_t{}^\beta u(x,z,t) = \left(1 + \eta D_t{}^\beta\right)\left(D_x^{2\alpha} + D_z^{2\alpha}\right) u(x,z,t)\ , \quad x,\ z,\ t > 0, \tag{56}$$

$$u(x,z,0) = 0, \qquad x>0, z>0 \tag{57}$$

$$u(0,z,t)=u(x,0,t)=1, \qquad t>0 \tag{58}$$

$$u(x,z,t),\ \frac{\partial u\left(x,z,t\right)}{\partial x},\ \frac{\partial u\left(x,z,t\right)}{\partial z}\to 0 \quad as \quad x^2+z^2\to\infty. \tag{59}$$

Applying the Katugampola Fourier sine and Katugampola Laplace transforms with respect to $x-z$ and t respectively to above equations, we get

$$L_\beta\left\{F_s^\alpha\left\{u(x,z,t)\right\}\right\}=\frac{2\left(\kappa^2+\xi^2\right)}{\pi\kappa\xi p\left[p+\eta\left(\kappa^2+\xi^2\right)p+\kappa^2+\xi^2\right]}+\frac{\left(\kappa^2+\xi^2\right)\eta F_s^\alpha\left\{f_0(x,z)\right\}}{p+\eta\left(\kappa^2+\xi^2\right)p+\kappa^2+\xi^2},$$

$$=\frac{2}{\pi\kappa\xi}\left[\frac{1}{p}-\frac{1}{p+\frac{\kappa^2+\xi^2}{1+(\kappa^2+\xi^2)\eta}}\right]+\frac{\left(\kappa^2+\xi^2\right)\eta F_s^\alpha\left\{f_0(x,z)\right\}}{p+\eta\left(\kappa^2+\xi^2\right)p+\kappa^2+\xi^2} \tag{60}$$

where $F_s^\alpha\left\{f_0(x,z),\ \kappa,\xi\right\}=\frac{2}{\pi}\int_0^\infty\int_0^\infty\sin\left(\kappa\frac{x^\alpha}{\alpha}\right)\sin\left(\xi\frac{z^\alpha}{\alpha}\right)\ u(x,z,0)x^{\alpha-1}z^{\alpha-1}dxdz$
Taking the inverse Katugampola Laplace transform for Eq.(60), we get

$$\begin{aligned}F_s^\alpha\left\{u(x,z,t)\right\}=&\left(\kappa^2+\xi^2\right)\eta F_s^\alpha\left\{f_0(x,z)\right\}e^{-\left(\frac{\kappa^2+\xi^2}{1+\eta(\kappa^2+\xi^2)}\right)\frac{t^\beta}{\beta}}\\&+\tfrac{2}{\kappa\xi\pi}\left[1-e^{-\left(\frac{\kappa^2+\xi^2}{1+\eta(\kappa^2+\xi^2)}\right)\frac{t^\beta}{\beta}}\right]\end{aligned} \tag{61}$$

Now considering the inverse Katugampola Fourier sine integral transform of Eq.(61). We get

$$\begin{aligned}u(x,z,t)=&\tfrac{2}{\pi}\int_0^\infty\int_0^\infty\sin\left(\kappa\frac{x^\alpha}{\alpha}\right)\sin\left(\xi\frac{z^\alpha}{\alpha}\right)\\&\times\left[\begin{array}{l}\left(\kappa^2+\xi^2\right)\eta F_s^\alpha\left\{f_0(x,z)\right\}e^{-\left(\frac{\kappa^2+\xi^2}{1+\eta(\kappa^2+\xi^2)}\right)\frac{t^\beta}{\beta}}\\+\tfrac{2}{\kappa\xi\pi}\left[1-e^{-\left(\frac{\kappa^2+\xi^2}{1+\eta(\kappa^2+\xi^2)}\right)\frac{t^\beta}{\beta}}\right]\end{array}\right]d\kappa d\xi\end{aligned} \tag{62}$$

Special cases:
1. When $F_s^\alpha\left\{f_0(x,z),\kappa,\xi\right\}=0$, then Eq.(62) yields

$$u(x,z,t)=\frac{4}{\pi^2}\int_0^\infty\int_0^\infty\frac{\sin\left(\kappa\frac{x^\alpha}{\alpha}\right)\sin\left(\xi\frac{z^\alpha}{\alpha}\right)}{\kappa\xi}\left[1-e^{-\left(\frac{\kappa^2+\xi^2}{1+\eta(\kappa^2+\xi^2)}\right)\frac{t^\beta}{\beta}}\right]d\kappa d\xi \tag{63}$$

2. When $F_s^\alpha\left\{f_0(x,z),\kappa,\xi\right\}=0$, and $\alpha=\beta=1$ then Eq.(62) becomes

$$u(x,z,t)=\frac{4}{\pi^2}\int_0^\infty\int_0^\infty\frac{\sin\left(\kappa x\right)\sin\left(\xi z\right)}{\kappa\xi}\left[1-e^{-\left(\frac{\kappa^2+\xi^2}{1+\eta(\kappa^2+\xi^2)}\right)t}\right]d\kappa d\xi$$

$$=1-\frac{4}{\pi^2}\int_0^\infty\frac{\sin\left(\kappa x\right)}{\kappa}\int_0^\infty\frac{\sin\left(\xi z\right)}{\xi}e^{-\left(\frac{\kappa^2+\xi^2}{1+\eta(\kappa^2+\xi^2)}\right)t}d\kappa d\xi \tag{64}$$

which is the result obtained by Fetacau and Corina [2].

5. THE ENERGY FRACTIONAL EQUATION IN $x-z$ PLANE

The time-fractional energy equation in $x-z$ plane is written as

$$\frac{d}{cq}\left[D_x^{2\delta}\theta\left(x,z,t\right)+D_z^{2\delta}\theta\left(x,z,t\right)\right]+\frac{\nu}{c}f\left(x,z,t\right)+\frac{r\left(x,z,t\right)}{qc}=D_t^\gamma\theta\left(x,z,t\right) \tag{65}$$

where $f\left(x,z,t\right)=D_x^\delta u\left(x,t\right)+D_z^\delta u\left(x,t\right)$ is known function as soon as the velocity field $u\left(x,z,t\right)$ is prescribed, $r\left(x,t\right)$ is the radiant heating, which is neglected in this paper, c is the specific heat and d is the conductivity which is assumed to be constant and $D_x^{2\delta}, D_t^{\gamma}$ are Katugampola fractional derivatives of order $\delta \in \left(\frac{1}{2}, 1\right], \ \gamma \in (0,1]$. The corresponding initial and boundary condition of Eq.(65), are

$$\theta(x,z,0) = h_0(x,z), \qquad x, z>0 \tag{66}$$

$$\theta(0,z,t) = \theta(x,0,t) = T_0, \qquad t \geq 0 \tag{67}$$

$$\theta(x,z,t),\ \frac{\partial\theta(x,z,t)}{\partial x}, \frac{\partial\theta(x,z,t)}{\partial z} \to 0 \quad as \quad x^2+z^2 \to \infty. \tag{68}$$

Using the non-dimensional quantities (38), and $z^* = \frac{zU}{\nu}$, Eqs.(65), (66) , (67) and (68) can be reduced to dimensionless equations as follows (for brevity the dimensionless marks "*" are omitted here)

$$\begin{gathered}\frac{1}{\Pr}\left[D_x^{2\delta}\theta(x,z,t) + D_z^{2\delta}\theta(x,z,t)\right] + \lambda\left[\left(D_x^{\delta}\theta(x,z,t)\right)^2 + \left(D_z^{\delta}\theta(x,z,t)\right)^2\right] \\ = D_t^{\gamma}\theta(x,z,t)\end{gathered} \tag{69}$$

$$\theta(x,z,0) = h_0(x,z), \qquad x, z>0 \tag{70}$$

$$\theta(0,z,t) = \theta(x,0,t) = 1, \qquad t \geq 0 \tag{71}$$

$$\theta(x,z,t),\ \frac{\partial\theta(x,z,t)}{\partial x}, \frac{\partial\theta(x,z,t)}{\partial z} \to 0 \quad as \quad x^2+z^2 \to \infty. \tag{72}$$

Letting $g(x,z,t) = \lambda\left[\left(D_x^{\delta}\theta(x,z,t)\right)^2 + \left(D_z^{\delta}\theta(x,z,t)\right)^2\right]$, then Eq.(69) can be rewritten as

$$\frac{1}{\Pr}\left[D_x^{2\delta}\theta(x,z,t) + D_z^{2\delta}\theta(x,z,t)\right] + g(x,t,z) = D_t^{\gamma}\theta(x,z,t), \delta \in \left(\tfrac{1}{2}, 1\right], \ \gamma \in (0,1]. \tag{73}$$

By following the same steps as in section 3, we get

$$\begin{aligned}&\theta(x,z,t) = \tfrac{2}{\pi}\int_0^\infty\int_0^\infty \sin\left(\kappa\tfrac{x^\delta}{\delta}\right)\sin\left(\xi\tfrac{z^\delta}{\delta}\right) \\ &\times\left[\begin{array}{l}\int_0^t G(\kappa,\xi,t-\tau)F_s^{\delta}\{g(x,z,\tau)\}\,d\tau \\ +F_s^{\delta}\{h_0(x,z)\}\,e^{-\left(\frac{\kappa^2+\xi^2}{\Pr}\right)\frac{t^\gamma}{\gamma}} + \frac{2}{\kappa\xi\pi}\left[1-e^{-\left(\frac{\kappa^2+\xi^2}{\Pr}\right)\frac{t^\gamma}{\gamma}}\right]\end{array}\right]d\kappa d\xi\end{aligned} \tag{74}$$

where $G(\kappa,\xi,t) = L_\gamma{}^{-1}\left\{\frac{1}{p+\frac{\kappa^2+\xi^2}{\Pr}}\right\} = e^{-\left(\frac{\kappa^2+\xi^2}{\Pr}\right)\frac{t^\gamma}{\gamma}}$ is the Green's function of Eq.(73).

Special cases:

1. When $F_s^{\alpha}\{h_0(x,z),\kappa,\xi\} = 0,\ \lambda = 1$, then Eq.(74) yields

$$\begin{aligned}&\theta(x,z,t) = \tfrac{2}{\pi}\int_0^\infty\int_0^\infty \sin\left(\kappa\tfrac{x^\delta}{\delta}\right)\sin\left(\xi\tfrac{z^\delta}{\delta}\right) \\ &\times\left[\begin{array}{l}\int_0^t G(\kappa,\xi,t-\tau)F_s^{\delta}\{g(x,z,\tau)\}\,d\tau \\ +\frac{2}{\kappa\xi\pi}\left[1-e^{-\left(\frac{\kappa^2+\xi^2}{\Pr}\right)t}\right]\end{array}\right]d\kappa d\xi\end{aligned} \tag{75}$$

where $G(\kappa,\xi,t) = e^{-\left(\frac{\kappa^2+\xi^2}{\Pr}\right)t}$ is the Green's function of Eq.(73)

2. When $F_s^{\alpha}\{h_0(x,z),\kappa,\xi\} = 0$, $F_s^{\delta}\{g(x,z,t),\kappa,\xi\} = 0$, and $\delta = \lambda = 1$ then Eq.(74) becomes

$$\theta(x,z,t) = \frac{4}{\pi^2}\int_0^\infty\int_0^\infty \frac{\sin(\kappa x)\sin(\xi z)}{\kappa\xi}\left[1-e^{-\left(\frac{\kappa^2+\xi^2}{\Pr}\right)t}\right]d\kappa d\xi \tag{76}$$

6. CONCLUSION

In this paper we have presented some results about Stokes first problem, the Rayleigh-Stokes problem and energy equation . Exact solutions of these equations are obtained by using the Katugampola Fourier sine integral transform and Katugampola Laplace transform . the Katugampola fractional derivative is considered in both Stokes first problem and Rayleigh-Stokes problem, where the order of the fractional is considered as $\alpha, \delta \in \left(\frac{1}{2}, 1\right]$, $\beta,\gamma \in (0,1]$. Special cases have been considered in the cases $\alpha,\ \delta\ = 1$ and $\beta,\ \gamma = 1$.

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