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A STUDY OF EXTENDED BETA AND ASSOCIATED FUNCTIONS CONNECTED TO FOX-WRIGHT FUNCTION

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Abstract. In this talk, we study extended beta and associated functions (Gauss hypergeometric and confluent hypergeometric functions) connected to Fox-Wright function. For the newly extended beta, Gauss hypergeometric and confluent hypergeometric functions, we investigated some properties consisting of integral representations, differential formulas, difference formulas, Mellin transforms, transformation formulas and summation formulas. We also give statistical application of the extended beta function.

1. Introduction

Throughout this talk, $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Z}^-$ and $\mathbb{Z}_0^-$ represents sets of complex numbers, real numbers, integers, positive integers, negative integers and non-positive integers, respectively.
The classical gamma and beta functions are defined by (see [1] and [2]):

$$\Gamma(x) = \int_0^\infty t^{x-1} exp(-t)dt, \; (Re(x) > 0).$$

And

$$B(x,y) = \begin{cases} \int_0^1 t^{x-1}(1-t)^{y-1}dt, & (Re(x) > 0, Re(y) > 0), \\ \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, & (x, y \notin \mathbb{Z}_0^-). \end{cases} \tag{1}$$

The following equation holds:

$$B(x, y-x) = \frac{y}{x} B(x+1, y-x), (Re(y) > Re(x) > 0). \tag{2}$$

The classical Gauss hypergeometric function is defined by (see [3]):

$$\begin{aligned} {}_2F_1(a,b;c;z) &= \sum_{r=0}^\infty \frac{(a)_r(b)_r}{(c)_r} \frac{z^r}{r!} \\ &= \sum_{r=0}^\infty (a)_r \frac{B(b+r, c-b)}{B(b, c-b)} \frac{z^r}{r!}, \end{aligned}$$

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$$(Re(c) > Re(b) > 0, |z| < 1).$$

With the integral representation as follows:

$$ {}_2F_1(a, b; c; z) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt,$$

$$(Re(c) > Re(b) > 0, |arg(1-z)| < 1).$$

And

$$\Phi(b; c; z) = \sum_{r=0}^{\infty} \frac{(b)_r}{(c)_r} \frac{z^r}{r!} = \sum_{r=0}^{\infty} \frac{B(b+r, c-b)}{B(b, c-b)} \frac{z^r}{r!},$$

$$(Re(b) > 0, Re(c) > 0, |z| < 1).$$

With the integral representation as follows:

$$\Phi(b; c; z) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} exp(tz) dt,$$

$$(Re(c) > Re(b) > 0).$$

Where $(a)_r$ is the classical pochammer symbol defined by (see for example [4], [5] and [6]):

$$(a)_r = \begin{cases} (a)(a+1)(a+2)\cdots(a+r-1), & (r \geq 0, a \neq 0), \\ 1, & (r = 0). \end{cases}$$

With the following important relations:

$$(a)_r = \frac{\Gamma(a+r)}{\Gamma(a)}.$$

And

$$\sum_{r=0}^{\infty} (a)_r \frac{(tz)^r}{r!} = {}_1F_0(a; -; tz) = (1-tz)^{-a}. \tag{3}$$

In 1994, Chaudhry and Zubair [7] used exponential kernel to proposed the following extension of gamma function:

$$\Gamma_{\varrho}(x) = \int_0^{\infty} t^{x-1} exp\left(-t - \frac{\varrho}{t}\right) dt,$$

$$(Re(x) > 0, Re(\varrho) > 0).$$

Chaudhry and Zubair [8] used exponential kernel to introduced the following extension of beta function:

$$B_{\varrho}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} exp\left(-\frac{\varrho}{(1-t)t}\right) dt, \tag{4}$$

$$(Re(x) > 0, Re(y) > 0, Re(\varrho) > 0).$$

Chaudhry et al., [9] established the extension of Gauss and confluent hypergeometric functions by considering the extended beta function in (4) as follows:

$$F_{\varrho}(a, b; c; z) = \sum_{r=0}^{\infty} (a)_r \frac{B_{\varrho}(b+r, c-b)}{B(b, c-b)} \frac{z^r}{r!},$$

$$(\varrho \geq 0, Re(c) > Re(b) > 0, |z| < 1).$$

With the integral representation as follows:

$$F_{\varrho}(a,b;c;z)=\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}exp\left(-\frac{\varrho}{(1-t)t}\right)dt,$$
$$(\varrho>0;\varrho=0,Re(c)>Re(b)>0,|arg(1-z)|<\pi).$$

And

$$\Phi_{\varrho}(b;c;z)=\sum_{r=0}^{\infty}\frac{B_{\varrho}(b+r,c-b)}{B(b,c-b)}\frac{z^r}{r!},$$
$$(\varrho>0;\varrho=0,Re(c)>Re(b)>0).$$

With the integral representation as follows:

$$\Phi_{\varrho}(b;c;z)=\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}exp\left(tz-\frac{\varrho}{(1-t)t}\right)dt,$$
$$(\varrho>0;\varrho=0,Re(c)>Re(b)>0).$$

Two and three variables hypergeometric functions can be see in ([10]).
Lee et al., [11] presented and investigated the following extension of beta function:

$$B_{\varrho}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}exp\left(-\frac{\varrho}{(1-t)^m t^m}\right)dt,$$
$$(Re(x)>0,Re(y)>0,Re(\varrho)>0,Re(m)>0).$$

They [11] also introduced the following Gauss and confluent hypergeometric function:

$$F_{\varrho}^{m}(a,b;c;z)=\sum_{r=0}^{\infty}(a)_r\frac{B_{\varrho}^{m}(b+r,c-b)}{B(b,c-b)}\frac{z^r}{r!},$$
$$(\varrho\geq 0,Re(a)>0,Re(b)>0,Re(c)>0,Re(m)>0).$$

With the integral representation as follows:

$$F_{\varrho}^{m}(a,b;c;z)=\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}exp\left(-\frac{\varrho}{(1-t)^m t^m}\right)dt,$$
$$(\varrho>0;\varrho=0,Re(c)>Re(b)>0,Re(m)>0).$$

And

$$\Phi_{\varrho}^{m}(b;c;z)=\sum_{r=0}^{\infty}\frac{B_{\varrho}^{m}(b+r,c-b)}{B(b,c-b)}\frac{z^r}{r!},$$
$$(\varrho\geq 0,Re(b)>0,Re(c)>0,Re(m)>0).$$

With the integral representation as follows:

$$\Phi_{\varrho}^{m}(b;c;z)=\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}exp\left(tz-\frac{\varrho}{(1-t)^m t^m}\right)dt,$$
$$(\varrho>0;\varrho=0,Re(m)>0,Re(c)>Re(b)>0).$$

Ozegin and Ozarslan [12] presented the following extension of gamma and beta functions:

$$\Gamma_{\varrho}^{(\rho_1,\rho_2)}(x)=\int_0^{\infty} t^{x-1}\Phi\left(\rho_1,\rho_2,-t-\frac{\varrho}{t}\right)dt,$$
$$(Re(x)>0,Re(\varrho)>0,Re(\rho_1)>0,Re(\rho_2)>0).$$

And

$$B_{\varrho}^{(\rho_1,\rho_2)}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\Phi\left(\rho_1,\rho_2,-\frac{\varrho}{(1-t)t}\right)dt, \tag{5}$$

$$(Re(x) > 0, Re(y) > 0, Re(\varrho) > 0, Re(\rho_1) > 0, Re(\rho_2) > 0).$$

They [12] also introduced the following Gauss and confluent hypergeometric functions:

$$F_{\varrho}^{(\rho_1,\rho_2)}(a, b; c; z) = \sum_{r=0}^{\infty} (a)_r \frac{B_{\varrho}^{(\rho_1,\rho_2)}(b + r, c - b)}{B(b, c - b)} \frac{z^r}{r!},$$

$$(\varrho \geq 0, Re(c) > 0, Re(b) > 0, Re(\rho_1) > 0, Re(\rho_2) > 0, |z| < 1).$$

With the integral representation as follows:

$$F_{\varrho}^{(\rho_1,\rho_2)}(a, b; c; z) = \frac{1}{B(b, c - b)} \int_0^1 t^{b-1}(1 - t)^{c-b-1}(1 - tz)^{-a} \times \Phi\left(\rho_1, \rho_2, -\frac{\varrho}{(1 - t)t}\right) dt,$$

$$(\varrho > 0; \varrho = 0, Re(c) > Re(b) > 0, |arg(1 - z)| < \pi).$$

And

$$\Phi_{\varrho}^{(\rho_1,\rho_2)}(b; c; z) = \sum_{r=0}^{\infty} \frac{B_{\varrho}^{(\rho_1,\rho_2)}(b + r, c - b)}{B(b, c - b)} \frac{z^r}{r!},$$

$$(\varrho \geq 0, Re(c) > Re(b) > 0, Re(\rho_1) > 0, Re(\rho_2) > 0).$$

With the integral representation as follows:

$$\Phi_{\varrho}^{(\rho_1,\rho_2)}(b; c; z) = \frac{1}{B(b, c - b)} \int_0^1 t^{c-1}(1 - t)^{c-b-1} exp(tz) \times \Phi\left(\rho_2, \rho_3, -\frac{\varrho}{(1 - t)t}\right) dt,$$

$$(\varrho > 0; \varrho = 0, Re(c) > Re(b) > 0).$$

Extended Appell's and Lauricella's hypergeometric functions can also be see in [13] Parmar [14] provided the following extension of gamma and beta functions:

$$\Gamma_{\varrho}^{(\rho_1,\rho_2;m)}(x) = \int_0^{\infty} t^{x-1} \Phi\left(\rho_1, \rho_2, -t - \frac{\varrho}{t^m}\right) dt,$$

$$(Re(x) > 0, Re(\varrho) > 0, Re(\rho_1) > 0, Re(\rho_2) > 0, Re(m) > 0).$$

And

$$B_{\varrho}^{(\rho_1,\rho_2;m)}(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1} \Phi\left(\rho_1, \rho_2, -\frac{\varrho}{(1 - t)^m t^m}\right) dt, \tag{6}$$

$$(Re(m) > 0, Re(y) > 0, Re(\varrho) > 0, Re(\rho_1) > 0, Re(\rho_2) > 0, Re(m) > 0).$$

He [14] also introduced the following Gauss and confluent hypergeometric functions:

$$F_{\varrho}^{(\rho_1,\rho_2;m)}(a, b; c; z) = \sum_{r=0}^{\infty} (a)_r \frac{B_{\varrho}^{(\rho_1,\rho_2;m)}(b + r, c - b)}{B(b, c - b)} \frac{z^r}{r!},$$

$$(\varrho \geq 0, Re(c) > Re(b) > 0, Re(\rho_1) > 0, Re(\rho_2) > 0, Re(m) > 0, |z| < 1).$$

With the integral representation as follows:

$$F_{\varrho}^{(\rho_1,\rho_2;m)}(a, b; c; z) = \frac{1}{B(b, c - b)} \int_0^1 t^{b-1}(1 - t)^{c-b-1}(1 - tz)^{-a} \times \Phi\left(\rho_1, \rho_2, -\frac{\varrho}{(1 - t)^m t^m}\right) dt,$$

$$(\varrho > 0; \varrho = 0, Re(c) > Re(b) > 0, Re(m) > 0, |arg(1-z)| < \pi).$$

And

$$\Phi_{\varrho}^{(\rho_1,\rho_2;m)}(b;c;z) = \sum_{r=0}^{\infty} \frac{B_{\varrho}^{(\rho_1,\rho_2;m)}(b+r, c-b)}{B(b,c-b)} \frac{z^r}{r!},$$

$$(\varrho \geq 0, Re(c) > Re(b) > 0, Re(\rho_1) > 0, Re(\rho_2) > 0, Re(m) > 0, |z| < 1).$$

With the integral representation as follows:

$$\Phi_{\varrho}^{(\rho_1,\rho_2)}(b;c;z) = \int_0^1 t^{b-1}(1-t)^{c-b-1} exp(tz)\Phi\left(\rho_1, \rho_2, -\frac{\varrho}{(1-t)^m t^m}\right) dt,$$

$$(\varrho > 0; \varrho = 0, Re(c) > Re(b) > 0, Re(m) > 0).$$

Agarwal et al., [15] provided the extended Appell's and Lauricella's hypergeometric functions for extended beta function in (6).

Ata [16] proposed the following extension of gamma and beta functions:

$$^{\psi}\Gamma_{\varrho}^{(\rho_1,\rho_2)}(x) = \int_0^{\infty} t^{x-1} {}_1\psi_1\left(\rho_1, \rho_2, -t - \frac{\varrho}{t}\right) dt,$$

$$(Re(x) > 0, Re(\varrho) > 0, Re(\rho_1) > 0, Re(\rho_2) > 1).$$

And

$$^{\psi}B_{\varrho}^{(\rho_1,\rho_2)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} {}_1\psi_1\left(\rho_1, \rho_2, -\frac{\varrho}{(1-t)t}\right) dt,$$

$$(Re(x) > 0, Re(y) > 0, Re(\varrho) > 0, Re(\rho_1) > 0, Re(\rho_2) > 1).$$

They [16] also introduced the following Gauss and confluent hypergeometric functions:

$$^{\psi}F_{\varrho}^{(\rho_1,\rho_2)}(a,b;c;z) = \sum_{r=0}^{\infty} (a)_r \frac{^{\psi}B_{\varrho}^{(\rho_1,\rho_2)}(b+r, c-b)}{B(b,c-b)} \frac{z^r}{r!},$$

$$(\varrho \geq 0, Re(a) > 0, Re(b) > 0, Re(c) > 0, Re(\rho_1) > 0, Re(\rho_2) > 1, |z| < 1).$$

With the integral representation as follows:

$$^{\psi}F_{\varrho}^{(\rho_1,\rho_2)}(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} \times {}_1\psi_1\left(\rho_1, \rho_2, -\frac{\varrho}{(1-t)t}\right) dt,$$

$$(\varrho > 0; \varrho = 0, Re(c) > Re(b) > 0, |arg(1-z)| < \pi).$$

And

$$^{\psi}\Phi_{\varrho}^{(\rho_1,\rho_2)}(b;c;z) = \sum_{r=0}^{\infty} \frac{^{\psi}B_{\varrho}^{(\rho_1,\rho_2)}(b+r, c-b)}{B(b,c-b)} \frac{z^r}{r!},$$

$$(\varrho \geq 0, Re(c) > Re(b) > 0, Re(\rho_1) > 0, Re(\rho_2) > 1, |z| < 1).$$

With the integral representation as follows:

$$^{\psi}\Phi_{\varrho}^{(\rho_1,\rho_2)}(b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} exp(tz) \times {}_1\psi_1\left(\rho_1, \rho_2, -\frac{\varrho}{(1-t)t}\right) dt,$$

$$(\varrho > 0; \varrho = 0, Re(c) > Re(b) > 0).$$

For detail discussion of Appell-Lauricela's functions see [17].
Ata and Kiymaz [18] established the following extension of gamma and beta functions:

$$\begin{aligned} {}^{\Psi}\Gamma_{\varrho}(x) &= {}^{\Psi}\Gamma_{\varrho}\left[\begin{array}{c} (\xi_i,\zeta_i)_{1,\gamma} \\ \\ (\ell_j,\varepsilon_j)_{1,\lambda} \end{array} \mid x \right] \\ &= \int_0^{\infty} t^{x-1}{}_{\tau}\Psi_{\kappa}\left(-t-\frac{\varrho}{t}\right)dt, \qquad (7) \\ &(Re(x)>0, Re(\varrho)>0). \end{aligned}$$

And

$$\begin{aligned} {}^{\Psi}B_{\varrho}^{(\rho_1,\rho_2)}(x,y) &= {}^{\Psi}B_{\varrho}\left[\begin{array}{c} (\xi_i,\zeta_i)_{1,\gamma} \\ \\ (\ell_j,\varepsilon_j)_{1,\lambda} \end{array} \mid x,y \right] \\ &= \int_0^1 t^{x-1}(1-t)^{y-1}{}_{\tau}\Psi_{\kappa}\left(-\frac{\varrho}{(1-t)t}\right)dt, \\ &(Re(x)>0, Re(y)>0, Re(\varrho)>0). \end{aligned}$$

They [18] also introduced the following Gauss and confluent hypergeometric functions:

$$\begin{aligned} {}^{\Psi}F_{\varrho}(a,b;c;z) &= {}^{\Psi}F_{\varrho}\left[\begin{array}{c} (\xi_i,\zeta_i)_{1,\gamma} \\ \\ (\ell_j,\varepsilon_j)_{1,\lambda} \end{array} \mid a,b;c;z \right] \\ &= \sum_{r=0}^{\infty}(a)_r\frac{{}^{\Psi}B_{\varrho}(b+r,c-b)}{B(b,c-b)}\frac{z^r}{r!}, \\ &(\varrho\geq 0, Re(c)>Re(b)>0). \end{aligned}$$

With the integral representation as follows:

$${}^{\Psi}F_{\varrho}(a,b;c;z) = \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}{}_{\tau}\Psi_{\kappa}\left(-\frac{\varrho}{(1-t)t}\right)dt,$$
$$(\varrho>0; \varrho=0, Re(c)>Re(b)>0, |arg(1-z)|<\pi).$$

And

$$\begin{aligned} {}^{\Psi}\Phi_{\varrho}(b;c;z) &= {}^{\Psi}\Phi_{\varrho}\left[\begin{array}{c} (\xi_i,\zeta_i)_{1,\gamma} \\ \\ (\ell_j,\varepsilon_j)_{1,\lambda} \end{array} \mid b;c;z \right] \\ &= \sum_{r=0}^{\infty}\frac{{}^{\Psi}B_{\varrho}(b+r,c-b)}{B(b,c-b)}\frac{z^r}{r!}, \\ &(\varrho\geq 0, Re(c)>Re(b)>0, |z|<1). \end{aligned}$$

With the integral representation as follows:

$${}^{\Psi}\Phi_{\varrho}(b;c;z) = \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}exp(tz){}_{\tau}\Psi_{\kappa}\left(-\frac{\varrho}{(1-t)t}\right)dt,$$
$$(\varrho>0; \varrho=0, Re(c)>Re(b)>0).$$

Abubakar [20] presented the following extended beta functions:

$$
\begin{aligned}
{}^{\Psi}B_{\varrho}^{m}(x,y) &= {}^{\Psi}B_{\varrho}^{m}\left[\begin{array}{cc} (\xi_i,\zeta_i)_{1,\gamma} & \\ & \mid x,y \\ (\ell_j,\varepsilon_j)_{1,\lambda} & \end{array}\right] \\
&= \int_0^1 t^{x-1}(1-t)^{y-1}\,{}_{\tau}\Psi_{\kappa}\left(-\frac{\varrho}{(1-t)^m t^m}\right)dt, \qquad (8)
\end{aligned}
$$

$$(Re(x)>0, Re(y)>0, Re(\varrho)>0, Re(m)>0).$$

He [20] also provided the following Gauss hypergeometric and Kumar confluent hypergeometric functions:

$$
\begin{aligned}
{}^{\Psi}F_{\varrho}^{m}(a,b;c;z) &= {}^{\Psi}F_{\varrho}^{m}\left[\begin{array}{cc} (\xi_i,\zeta_i)_{1,\gamma} & \\ & \mid a,b;c;z \\ (\ell_j,\varepsilon_j)_{1,\lambda} & \end{array}\right] \\
&= \sum_{r=0}^{\infty}(a)_r\frac{{}^{\Psi}B_{\varrho}^{m}(b+r_1,c-b)}{B(b,c-b)}\frac{z^r}{r!}, \qquad (9)
\end{aligned}
$$

$$(\varrho \geq 0, Re(c)>Re(b)>0, Re(m)>0).$$

And

$$
\begin{aligned}
{}^{\Psi}\Phi_{\varrho}^{m}(b;c;z) &= {}^{\Psi}\Phi_{\varrho}^{m}\left[\begin{array}{cc} (\xi_i,\zeta_i)_{1,\gamma} & \\ & \mid b;c;z \\ (\ell_j,\varepsilon_j)_{1,\lambda} & \end{array}\right] \\
&= \sum_{r=0}^{\infty}\frac{{}^{\Psi}B_{\varrho}^{m}(b+r,c-b)}{B(b,c-b)}\frac{z^r}{r!}, \qquad (10)
\end{aligned}
$$

$$(\varrho \geq 0, Re(c)>Re(b)>0, |z|<1, Re(m)>0).$$

Other forms and generalizations gamma, beta, hypergeometric and confluent hypergeometric functions, reader can refer to [19]-[50].
Throughout the rest of this research paper, we consider x, y, z, a, b, c, ϱ, m, n ξ_i, ℓ_j, $\in \mathbb{C}$, ζ_i, $\varepsilon_j \in \mathbb{R}$, k, $r \in \mathbb{N}$, $Re(x)>0$, $Re(y)>0$, $Re(\varrho)>0$, $Re(m)>0$, $Re(n)>0$, $Re(a)>0$, $Re(c)>Re(b)>0$.
In the present talk, we generalized beta functions in introduced in Abubakar [20] as follows:

$$
\begin{aligned}
{}^{\Psi}B_{\varrho}^{m,n}(x,y) &= {}^{\Psi}B_{\varrho}^{m,n}\left[\begin{array}{cc} (\xi_i,\zeta_i)_{1,\gamma} & \\ & \mid x,y \\ (\ell_j,\varepsilon_j)_{1,\lambda} & \end{array}\right] \\
&= \int_0^1 t^{x-1}(1-t)^{y-1}\,{}_{\tau}\Psi_{\kappa}\left(-\frac{\varrho}{(1-t)^m t^n}\right)dt. \qquad (11)
\end{aligned}
$$

2. Integral representations of the new generalized beta function

In this section, integral representations of the new generalized beta function are investigated.
Theorem 1 The following integral representation hold true.

$$ {}^{\Psi}B_{\varrho}^{m,n}(x,y) = 2\int_0^{\frac{\pi}{2}} \sin^{2x-1}\varphi\cos^{2y-1}\varphi\;{}_{\tau}\Psi_{\kappa}\left(-\varrho\sec^{2m}\varphi\csc^{2n}\varphi\right)d\varphi. $$

Proof. Setting $t = \sin^2\varphi$, we have

$$
\begin{aligned}
{}^{\Psi}B_{\varrho}^{m,n}(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}{}_n\Psi_\kappa\left(-\frac{\varrho}{(1-t)^m t^n}\right)dt \\
&= \int_0^{\frac{\pi}{2}} (\sin^2\varphi)^{x-1}(\cos^2\varphi)^{y-1}\ {}_\tau\Psi_\kappa\left(-\frac{\varrho}{(\cos^2\varphi)^m(\sin^2\varphi)^n}\right) \\
&\qquad \times (2\sin\varphi\cos\varphi d\varphi) \\
&= 2\int_0^{\frac{\pi}{2}} \sin^{2x-1}\varphi\cos^{2y-1}\varphi\ {}_\tau\Psi_\kappa\left(-\varrho\sec^{2m}\varphi\csc^{2n}\varphi\right)d\varphi
\end{aligned}
$$

Theorem 2 The following formula hold true.

$$
{}^{\Psi}B_{\varrho}^{m,n}(x,y) = \int_0^\infty \frac{t^{x-1}}{(1-t)^{y-1}}\ {}_\tau\Psi_\kappa\left(-\frac{\varrho(1+t)^{m+n}}{t^n}\right)dt.
$$

Proof. Putting $t = \nu(1+\nu)^{-1}$, we have

$$
\begin{aligned}
{}^{\Psi}B_{\varrho}^{m,n}(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}{}_n\Psi_\kappa\left(-\frac{\varrho}{(1-t)^m t^n}\right)dt \\
&= \int_0^\infty \left(\frac{\nu}{1+\nu}\right)^{x-1}\left(\frac{1}{1+\nu}\right)^{y-1}{}_\tau\Psi_\kappa\left(-\frac{\varrho(1+\nu)^{m+n}}{\nu^m}\right)\frac{d\nu}{(1+\nu)^2} \\
&= \int_0^\infty \frac{\nu^{x-1}}{(1-\nu)^{y-1}}{}_\tau\Psi_\kappa\left(-\frac{\varrho(1+\nu)^{m+n}}{\nu^n}\right)d\nu
\end{aligned}
$$

Theorem 3 The following equation hold.

$$
\begin{aligned}
{}^{\Psi}B_{\varrho}^{m,n}(x,y) = (q-p)^{1-x-y}\int_p^q (t-p)^{x-1}(q-t)^{y-1} \\
\times\ {}_\tau\Psi_\kappa\left(-\frac{\varrho(q-p)^{m+n}}{(q-t)^m(t-p)^n}\right)dt.
\end{aligned}
$$

Proof. Substituting $t = (\nu-p)(q-p)^{-1}$, we have

$$
\begin{aligned}
{}^{\Psi}B_{\varrho}^{m,n}(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}{}_n\Psi_\kappa\left(-\frac{\varrho}{(1-t)^m t^n}\right)dt \\
&= \int_p^q \left(\frac{\nu-p}{q-p}\right)^{x-1}\left(\frac{q-\nu}{q-p}\right)^{y-1} \\
&\qquad \times\ {}_\tau\Psi_\kappa\left(-\frac{\varrho(q-p)^{m+n}}{(q-\nu)^m(\nu-p)^n}\right)\frac{d\nu}{q-p} \\
&= (q-p)^{1-x-y}\int_p^q (\nu-p)^{x-1}(q-\nu)^{y-1} \\
&\qquad \times\ {}_\tau\Psi_\kappa\left(-\frac{\varrho(q-p)^{m+n}}{(q-\nu)^m(\nu-p)^n}\right)d\nu.
\end{aligned}
$$

3. Differential formulas of the new generalized beta function

This section discussed differential formulas of the new generalized beta function
Theorem 4 The following differential formula holds.

$$\frac{d^k}{d\varrho^k}\ {}^{\Psi}B_{\varrho}^{m,n}(x,y) = (-1)^k\ {}^{\Psi}B_{\varrho}^{m,n}\left[\begin{array}{cc} (\xi_i n+\zeta_i,\xi_i)_{1,\gamma} & \\ & \mid x-kn, y-km \\ (\ell_j n+\varepsilon_j,\ell_j)_{1,\lambda} & \end{array}\right], \quad (12)$$

$$(Re(x) > kn, Re(y) > km).$$

Proof. Using mathematical induction, we have

$$\frac{d}{d\varrho}\ {}^{\Psi}B_{\varrho}^{m,n}(x,y) = \frac{d}{d\varrho}\left\{\int_0^1 t^{x-1}(1-t)^{y-1}{}_{\tau}\Psi_{\kappa}\left(-\frac{\varrho}{(1-t)^m t^n}\right)dt\right\}.$$

On simplifying, we obtain

$$\frac{d}{d\varrho}\ {}^{\Psi}B_{\varrho}^{m,n}(x,y) = (-1)\ {}^{\Psi}B_{\varrho}^{m,n}\left[\begin{array}{cc} (\xi_i+\zeta_i,\xi_i)_{1,\gamma} & \\ & \mid x-n, y-m \\ (\ell_j+\varepsilon_j,\ell_j)_{1,\lambda} & \end{array}\right]. \quad (13)$$

Assuming r^{th} order derivative is hold, then

$$\frac{d^r}{d^r\varrho}\ {}^{\Psi}B_{\varrho}^{m,n}(x,y) = (-1)^r\ {}^{\Psi}B_{\varrho}^{m,n}\left[\begin{array}{cc} (\xi_i r+\zeta_i,\xi_i)_{1,\gamma} & \\ & \mid x-rn, y-rm \\ (\ell_j r+\varepsilon_j,\ell_j)_{1,\lambda} & \end{array}\right].$$

The $(r+1)^{th}$ order derivative is as follows

$$\frac{d^{r+1}}{d\varrho^{r+1}}\ {}^{\Psi}B_{\varrho}^{m,n}(x,y) = \frac{d}{d\varrho}\left\{\frac{d^r}{d\varrho^r}\ {}^{\Psi}B_{\varrho}^{m,n}(x,y)\right\}. \quad (14)$$

Applying (13) to (14) and simplification, yields

$$\frac{d^{r+1}}{d\varrho^{r+1}}\ {}^{\Psi}B_{\varrho}^{m,n}(x,y) = (-1)^{r+1}$$

$$\times\ {}^{\Psi}B_{\varrho}^{m,n}\left[\begin{array}{cc} (\xi_i r+\zeta_i,\xi_i)_{1,\gamma} & \\ & \mid x-(r+1)n, y-(r+1)m \\ (\ell_j r+\varepsilon_j,\ell_j)_{1,\lambda} & \end{array}\right].$$

4. The Mellin transform of the new generalized beta function

The Mellin transform and inverse Mellin transform of the new generalized beta function is given below:
Theorem 5 The following Mellin transform formula hold.

$$\mathbf{M}\left\{{}^{\Psi}B_{\varrho}^{m,n}(x,y)\right\} = B(x+ns, y+ms)^{\Psi}\Gamma(s), \quad (15)$$

$$(Re(s) > 0, Re(x+ms) > 0, Re(y+ns) > 0).$$

Proof. On using direct substituting

$$\mathbf{M}\left\{{}^{\Psi}B_{\varrho}^{m,n}(x,y)\right\} = \int_0^{\infty}\varrho^{s-1}\ {}^{\Psi}B_{\varrho}^{m,n}(x,y)d\varrho. \quad (16)$$

Putting (11) into (16), gives

$$\mathbf{M}\left\{{}^{\Psi}B_{\varrho}^{m,n}(x,y)\right\} = \int_0^{\infty}\varrho^{s-1}\left\{\int_0^1 t^{x-1}(1-t)^{y-1}{}_{\tau}\Psi_{\kappa}\left(-\frac{\varrho}{(1-t)^m t^n}\right)dt\right\}d\varrho.$$

Interchanging the order of integrations, we have

$$\mathbf{M}\left\{{}^{\Psi}B_{\varrho}^{m,n}(x,y)\right\} = \int_0^1 t^{x-1}(1-t)^{y-1}\left\{\int_0^\infty \varrho^{s-1}{}_{\tau}\Psi_{\kappa}\left(-\frac{\varrho}{(1-t)^m t^n}\right)d\varrho\right\}dt.$$

Setting $\varrho = \nu(1-t)^m t^n$, gives

$$\mathbf{M}\left\{{}^{\Psi}B_{\varrho}^{m,n}(x,y)\right\} = \int_0^1 t^{x+ns-1}(1-t)^{y+ms-1}\left\{\int_0^\infty \nu^{s-1}{}_{\tau}\Psi_{\kappa}\left(-\nu\right)d\varrho\right\}dt. \quad (17)$$

Applying (1) and (7) to (17), we have

$$\mathbf{M}\left\{{}^{\Psi}B_{\varrho}^{m,n}(x,y)\right\} = B(x+ns,y+ms)\,{}^{\Psi}\Gamma(s).$$

Corollary 6 The following inverse Mellin transform hold true.

$${}^{\Psi}B_{\varrho}^{m,n}(x,y) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} B(x+ns,y+ms)\,{}^{\Psi}\Gamma(s)\varrho^{-s}ds,$$

$$(\sigma > 0, (Re(s) > 0, Re(x+ns) > 0, Re(y+ms) > 0).$$

5. Summation formulas of the new generalized beta function

Summations and other related formulas are introduced in the following theorems:

Theorem 7 The following formula holds.

$${}^{\Psi}B_{\varrho}^{m,n}(x,y) = {}^{\Psi}B_{\varrho}^{m,n}(x+1,y) + {}^{\Psi}B_{\varrho}^{m,n}(x,y+1). \quad (18)$$

Proof. By direct calculation

$${}^{\Psi}B_{\varrho}^{m,n}(x,y) = \int_0^1 t^{x}(1-t)^{y}\left\{t^{-1}+(1-t)^{-1}\right\}{}_{\tau}\Psi_{\kappa}\left(-\frac{\varrho}{(1-t)^m t^n}\right)dt. \quad (19)$$

On simplification of (19), we have

$${}^{\Psi}B_{\varrho}^{m,n}(x,y) = \int_0^1 t^{x}(1-t)^{y-1}{}_{\tau}\Psi_{\kappa}\left(-\frac{\varrho}{(1-t)^m t^n}\right)dt + \int_0^1 t^{x-1}(1-t)^{y}{}_{\tau}\Psi_{\kappa}\left(-\frac{\varrho}{(1-t)^m t^n}\right)dt. \quad (20)$$

Applying (11) to (20), we obtain

$${}^{\Psi}B_{\varrho}^{m,n}(x,y) = {}^{\Psi}B_{\varrho}^{m,n}(x+1,y) + {}^{\Psi}B_{\varrho}^{m,n}(x,yb+1).$$

Theorem 8 The following summation formula holds.

$${}^{\Psi}B_{\varrho}^{m,n}(x,1-y) = \sum_{r=0}^{\infty}\frac{(y)_r}{r!}\;{}^{\Psi}B_{\varrho}^{m,n}(x+r,1),\ (Re(y) < 1). \quad (21)$$

Proof. By direct calculation

$${}^{\Psi}B_{\varrho}^{m,n}(x,1-y) = \int_0^1 t^{x-1}(1-t)^{-y}{}_{\tau}\Psi_{\kappa}\left(-\frac{\varrho}{(1-t)^m t^n}\right)dt. \quad (22)$$

Applying (3) to (22), we get

$${}^{\Psi}B_{\varrho}^{m,n}(x,1-y) = \int_0^1 t^{x-1}\sum_{r=0}^{\infty}\frac{(y)_r}{r!}t^r{}_{\tau}\Psi_{\kappa}\left(-\frac{\varrho}{(1-t)^m t^n}\right)dt. \quad (23)$$

Interchanging the order of summation and integration in (23), yield

$$ {}^{\Psi}B_{\varrho}^{m,n}(x,1-y)=\sum_{r=0}^{\infty}\frac{(y)_r}{r!}\int_0^1 t^{x+r-1}{}_{\tau}\Psi_{\kappa}\left(-\frac{\varrho}{(1-t)^m t^n}\right)dt. \tag{24} $$

Applying (11) to (24), we get

$$ {}^{\Psi}B_{\varrho}^{m,n}(x,1-y)=\sum_{r=0}^{\infty}\frac{(y)_r}{r!}\ {}^{\Psi}B_{\varrho}^{m,n}(x+r,1),\ (Re(y)<1). $$

Theorem 9 The following summation formula holds.

$$ {}^{\Psi}B_{\varrho}^{m,n}(x,y)=\sum_{r=0}^{\infty}\ {}^{\Psi}B_{\varrho}^{m,n}(x+r,y+1). \tag{25} $$

Proof. By direct calculation

$$ {}^{\Psi}B_{\varrho}^{m,n}(x,y)=\int_0^1 t^{x-1}(1-t)^{y}(1-t)^{-1}{}_{\tau}\Psi_{\kappa}\left(-\frac{\rho_1}{(1-t)^m t^n}\right)dt. \tag{26} $$

Applying (3) to (26), we get

$$ {}^{\Psi}B_{\varrho}^{m,n}(x,y)=\int_0^1 t^{x-1}(1-t)^{y}\sum_{r=0}^{\infty}t^r{}_{\tau}\Psi_{\kappa}\left(-\frac{\varrho}{(1-t)^m t^n}\right)dt. \tag{27} $$

Interchanging the order of summation and integration in (27), yield

$$ {}^{\Psi}B_{\varrho}^{m,n}(x,y)=\sum_{r=0}^{\infty}\int_0^1 t^{x+r-1}(1-t)^{y}\ {}_{\tau}\Psi_{\kappa}\left(-\frac{\varrho}{(1-t)^m t^n}\right)dt. \tag{28} $$

Applying (11) to (28), we get

$$ {}^{\Psi}B_{\varrho}^{m,n}(x,y)=\sum_{r=0}^{\infty}\ {}^{\Psi}B_{\varrho}^{m,n}(x+r,y+1). $$

Theorem 10 The following formula holds.

$$ \begin{aligned} x\ {}^{\Psi}B_{\varrho}^{m,n}(x,y+1)=&\ \varrho(m+n){}^{\Psi}B_{\varrho}^{m,n}\left[\begin{array}{cc}(\xi_i+\zeta_i,\xi_i)_{1,\gamma} & \\ & \mid x+1-n,y-m \\ (\ell_j+\varepsilon_j,\ell_j)_{1,\lambda} & \end{array}\right]\\ &+y\ {}^{\Psi}B_{\varrho}^{m,n}(x+1,y)-\varrho n{}^{\Psi}B_{\varrho}^{m,n}\left[\begin{array}{cc}(\xi_i+\zeta_i,\xi_i)_{1,\gamma} & \\ & \mid x-n,y-m \\ (\ell_j+\varepsilon_j,\ell_j)_{1,\lambda} & \end{array}\right], \end{aligned} \tag{29} $$

$$ (Re(x)>n,Re(y)>m). $$

Proof. On using the following equation:

$$ {}^{\Psi}B_{\varrho}^{m,n}(x,y)=\mathbf{M}\left\{f^{m,n}(t:y;\varrho):x\right\}. $$

Where

$$ f^{m,n}(t:y;\varrho)=(1-t)^{y-1}H(1-t){}_{\tau}\Psi_{\kappa}\left(-\frac{\varrho}{(1-t)^m t^n}\right). \tag{30} $$

And $H(1-t)$ is Heaviside delta (see for details [49] and [51]). Differentiating (30) with respect to t, we have

$$\begin{aligned}\frac{d}{dt}f^{m,n}(t:y;\varrho) &= -(y-1)(1-t)^{y-2}H(1-t)\ {}_\tau\Psi_\kappa\left(-\frac{\varrho}{(1-t)^m t^n}\right)\\ &- (1-t)^{y-1}\delta(1-t)\ {}_\tau\Psi_\kappa\left(-\frac{\varrho}{(1-t)^m t^n}\right) - \varrho(m+n)t^{-n}\\ &\times (1-t)^{y-m-2}H(1-t)\ {}_\tau\Psi_\kappa\left[\begin{array}{c}(\xi_i+\zeta_i,\xi_i)_{1,\gamma}\\ \\ (\ell_j+\varepsilon_j,\ell_j)_{1,\lambda}\end{array}\mid -\tfrac{\varrho}{(1-t)^m t^n}\right] + \varrho n t^{-n-1}\\ &\times (1-t)^{y-m-2}H(1-t)\ {}_\tau\Psi_\kappa\left[\begin{array}{c}(\xi_i+\zeta_i,\xi_i)_{1,\gamma}\\ \\ (\ell_j+\varepsilon_j,\ell_j)_{1,\lambda}\end{array}\mid -\tfrac{\varrho}{(1-t)^m t^n}\right],\end{aligned} \tag{31}$$

where $\delta(1-t)$ is Dirac delta (see [18]).On simplification of (31), we obtain

$$\begin{aligned}-(x-1)\ {}^{\Psi}B_{\varrho}^{m,n}(x-1,y) &= {}^{\Psi}B_{\varrho}^{m,n}\left[\begin{array}{c}(\xi_i+\zeta_i,\xi_i)_{1,\gamma}\\ \\ (\ell_j+\varepsilon_j,\ell_j)_{1,\lambda}\end{array}\mid x-n-1,y-m-1\right]\\ &\times \varrho n - \varrho(m+n)\ {}^{\Psi}B_{\varrho}^{m,n}\left[\begin{array}{c}(\xi_i+\zeta_i,\xi_i)_{1,\gamma}\\ \\ (\ell_j+\varepsilon_j,\ell_j)_{1,\lambda}\end{array}\mid x-n,y-m-1\right]\\ &- (y-1)\ {}^{\Psi}B_{\varrho}^{m,n}(x,y-1).\end{aligned} \tag{32}$$

On setting $x \to x+1$ and $y \to y+1$ in (32), give the desired result in (29).

6. Beta distribution of the new generalized beta function

Considering the new generalized beta function, we defined the following beta distribution:

$$f(t) = \begin{cases}\frac{1}{{}^{\Psi}B_{\varrho}^{m,n}(h,g)}t^{h-1}(1-t)^{g-1}{}_\tau\Psi_\kappa\left(-\frac{\varrho}{(1-t)^m t^n}\right)dt, & (0<t<1),\\ 0, & \text{otherwise},\end{cases}$$

$$(h,g\in\mathbb{R}).$$

The moment of X is as given below:

$$E(X^\upsilon) = \frac{{}^{\Psi}B_{\varrho}^{m,n}(h+\upsilon,g)}{{}^{\Psi}B_{\varrho}^{m,n}(h,g)},\ (h,g\in\mathbb{R}). \tag{33}$$

On setting $\upsilon = 1$ in (33), we obtain the mean as follow:

$$\mu = E(X) = \frac{{}^{\Psi}B_{\varrho}^{m,n}(h+1,g)}{{}^{\Psi}B_{\varrho}^{m,n}(h,g)}.$$

The variance is as follow:

$$\begin{aligned}\delta &= E(X^2) - \{E(X)\}^2\\ &= \frac{{}^{\Psi}B_{\varrho}^{m,n}(h,g)\ {}^{\Psi}B_{\varrho}^{m,n}(h+2,g) - \{{}^{\Psi}B_{\varrho}^{m,n}(h,g)\}^2}{\{{}^{\Psi}B_{\varrho}^{m,n}(h,g)\}^2}\end{aligned}$$

The coefficient of variation of this distribution is given by

$$C.V = \sqrt{\frac{{}^{\Psi}B_{\varrho}^{m,n}(h+2,g)\,{}^{\Psi}B_{\varrho}^{m,n}(h,g)}{{}^{\Psi}B_{\varrho}^{m,n}(h+1,g)} - 1}$$

The moment generating function of the distribution is as follows:

$$M(t) = \sum_{r=0}^{\infty} E(X^r)\frac{t^r}{r!} = \frac{1}{{}^{\Psi}B_{\varrho}^{m,n}(h,g)} \sum_{r=0}^{\infty} {}^{\Psi}B_{\varrho}^{m,n}(h+r,g)\frac{t^r}{r!}.$$

The characteristic function of the extended distribution is given below:

$$\begin{aligned} M(exp(itx)) &= \sum_{r=0}^{\infty} E(X^r)\frac{i^r t^r}{r!} \\ &= \frac{1}{{}^{\Psi}B_{\varrho}^{m,n}(h,g)} \sum_{r=0}^{\infty} {}^{\Psi}B_{\varrho}^{m,n}(h+r,g)\frac{i^r t^r}{r!}. \end{aligned}$$

The commulative distribution (probability distribution) function is as follows:

$$F(x) = \frac{{}^{\Psi}B_{\varrho}^{m,n,z}(h+1,g)}{{}^{\Psi}B_{\varrho}^{m,n}(h,g)},$$

where

$${}^{\Psi}B_{\varrho}^{m,n;z}(h,g) = \int_0^z t^{h-1}(1-t)^{g-1}\,{}_{\tau}\Psi_{\kappa}\left(-\frac{\varrho}{(1-t)^m t^n}\right) dt,$$

is the new extended lower incomplete beta function.
The reliability function of the distribution can be express as

$$R(x) = 1 - F(x) = \frac{{}^{\Psi}B_{\varrho,z}^{m,n}(h,g)}{{}^{\Psi}B_{\varrho}^{m,n}(h,g)},$$

where

$${}^{\Psi}B_{\varrho,z}^{m,n}(h,g) = \int_z^{\infty} t^{h-1}(1-t)^{g-1}\,{}_{\tau}\Psi_{\kappa}\left(-\frac{\varrho}{(1-t)^m t^n}\right) dt,$$

is the new extended upper incomplete beta function.

7. Gauss and confluent hypergeometric functions

The newly introduced generalized Gauss and confluent hypergeometric functions are given as follow:

$$\begin{aligned} {}^{\Psi}F_{\varrho}^{m,n}(a,b;c;z) &= {}^{\Psi}F_{\varrho}^{m,n}\left[\begin{array}{cc} (\xi_i,\zeta_i)_{1,\gamma} & \\ & \mid a,b;c;z \\ (\ell_j,\varepsilon_j)_{1,\lambda} & \end{array}\right] \\ &= \sum_{r=0}^{\infty} (a)_r \frac{{}^{\Psi}B_{\varrho}^{m,n}(b+r,c-b)}{B(b,c-b)}\frac{z^r}{r!}. \end{aligned} \tag{34}$$

And

$$\begin{aligned} {}^{\Psi}\Phi_{\varrho}^{m,n}(b;c;z) &= {}^{\Psi}\Phi_{\varrho}^{m,n}\left[\begin{array}{cc} (\xi_i,\zeta_i)_{1,\gamma} & \\ & \mid b;c;z \\ (\ell_j,\varepsilon_j)_{1,\lambda} & \end{array}\right] \\ &= \sum_{r=0}^{\infty} \frac{{}^{\Psi}B_{\varrho}^{m,n}(b+r,c-b)}{B(b,c-b)}\frac{z^r}{r!}. \end{aligned} \tag{35}$$

Theorem 11 The following integral formula holds.

$$ {}^{\Psi}F_{\varrho}^{m,n}(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \times (1-tz)^{-a} \; {}_{\tau}\Psi_{\kappa}\left(-\frac{\varrho}{(1-t)^m t^n}\right) dt. \quad (36) $$

Proof. By direct calculation, we have

$$ {}^{\Psi}F_{\varrho}^{m,n}(a,b;c;z) = \sum_{r=0}^{\infty} \left\{ \int_0^1 t^{b+r-1}(1-t)^{c-b-1} \right\} \frac{1}{B(b,c-b)} (a)_r \frac{z^r}{r!}. \quad (37) $$

Interchanging the order of summation and integration, yields

$$ {}^{\Psi}F_{\varrho}^{m,n}(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \times \sum_{r=0}^{\infty} (a)_r \frac{z^r}{r!} \; {}_{\tau}\Psi_{\kappa}\left(-\frac{\varrho}{(1-t)^m t^n}\right) dt \quad (38) $$

On simplifying, gives

$$ {}^{\Psi}F_{\varrho}^{m,n}(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \times (1-tz)^{-a} \; {}_{\tau}\Psi_{\kappa}\left(-\frac{\varrho}{(1-t)^m t^n}\right) dt. $$

Theorem 12 The following integral formulas hold.

$$ {}^{\Psi}F_{\varrho}^{m,n}(a,b;c;z) = \frac{2}{B(b,c-b)} \int_0^1 \frac{\sin^{2b-1}\varphi \cos^{2(c+b)-1}\varphi}{(1-x\sin^2\varphi)^a} \times \; {}_{\tau}\Psi_{\kappa}\left(-\varrho \sec^{2m}\varphi \csc^{2n}\varphi\right) dt, \quad (39) $$

and

$$ {}^{\Psi}F_{\varrho}^{m,n}(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^\infty \frac{t^{b-1}}{(1+t)^{c-a}} \times \{1+t(1-z)\}^{-a} \; {}_{\tau}\Psi_{\kappa}\left(-\frac{\varrho(1+t)^{m+n}}{t^n}\right) dt. \quad (40) $$

Proof. Setting $t = sin^2\varphi$ and $t = \nu(1+\nu)^{-1}$ in (36) and change of variables, we obtain the desired result in (39) and (40), respectively.

Corollary 13 The following integral formulas hold.

$$ {}^{\Psi}\Phi_{\varrho}^{m,n}(b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} exp(tz) \times \; {}_{\tau}\Psi_{\kappa}\left(-\frac{\varrho}{(1-t)^m t^n}\right) dt, \quad (41) $$

and

$$ {}^{\Psi}\Phi_{\varrho}^{m,n}(b;c;z) = \frac{exp(z)}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} exp(tz) \times {}_{\tau}\Psi_{\kappa}\left(-\frac{\varrho}{(1-t)^m t^n}\right) dt. \quad (42) $$

Theorem 14 The following integral formulas hold.

$$ {}^{\Psi}F_{\varrho}^{m,n}(a,b;c;z) = \frac{2}{B(b,c-b)} \int_0^1 \frac{\sin^{2b-1}\phi \cos^{2(c+b)-1}\phi}{(1-z\sin^2\varphi)^a} \times {}_{\tau}\Psi_{\kappa}\left(-\varrho \sec^{2m}\varphi \csc^{2n}\varphi\right) dt, \quad (43) $$

$$ {}^{\Psi}F_{\varrho}^{m,n}(a,b;c;x_1) = \frac{1}{B(b,c-b)} \int_0^{\infty} \frac{t^{b-1}}{(1+t)^{c-a}} \times \{1+t(1-z)\}^{-a} {}_{\tau}\Psi_{\kappa}\left(-\frac{\varrho(1+t)^{m+n}}{t^n}\right) dt. \quad (44) $$

Proof. Setting $t = sin^2\varphi$, and $t = \nu(1+\nu)^{-1}$ in and change of variables, we obtain the desired result in and , respectively.

Corollory 15 The following integral formula holds.

$$ {}^{\Psi}\Phi_{\varrho}^{m,n}(b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} exp(tz) \times {}_{\tau}\Psi_{\kappa}\left(-\frac{\varrho}{(1-t)^m t^n}\right) dt, $$

$$ {}^{\Psi}\Phi_{\varrho}^{m,n}(b;c;z) = \frac{exp(z)}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} exp(tz) \times {}_{\tau}\Psi_{\kappa}\left(-\frac{\varrho}{(1-t)^m t^n}\right) dt. $$

8. Differential formulas for the new generalized Gauss and confluent hypergeometric functions

This section considered differential formulas for the generalized Gauss and confluent hypergeometric functions.

Theorem 16 The following differential formulas hold.

$$ \frac{d}{dz} {}^{\Psi}F_{\varrho}^{m,n}(a,b;b;z) = \frac{ab}{c} {}^{\Psi}F_{\varrho}^{m,n}(a+1,b+1;c+1;z), \quad (45) $$

and

$$ \frac{d^k}{dz^k} {}^{\Psi}F_{\varrho}^{m,n}(a,b;c;z) = \frac{(a)_k(b)_k}{(b)_k} {}^{\Psi}F_{\varrho}^{m,n}(a+k,b+k;c+k;z). \quad (46) $$

Proof. Using , we have

$$ \frac{d}{dz} {}^{\Psi}F_{\varrho}^{m,n}(a,b;c;z) = \sum_{r=1}^{\infty} (a)_r \frac{{}^{\Psi}B_{\varrho}^{m,n}(b+r,c-b)}{B(b,c-b)} \frac{z^{r-1}}{(r-1)!}. \quad (47) $$

On setting $r \to r+1$, in (47) and applying (2), we get the desired result in (45). On successive differentiation of (45), we obtained (46).

Corollory 17 The following differential formulas hold.

$$\frac{d}{d}{}^{\Psi}\Phi_{\varrho}^{m,n}(b;c;z) = \frac{b}{c}\ {}^{\Psi}\Phi_{\varrho}^{m,n}(b+1;c+1;z), \tag{48}$$

and

$$\frac{d^k}{dz^k}{}^{\Psi}\Phi_{\varrho}^{m,n}(b;c;z) = \frac{(b)_k}{(c)_k}\ {}^{\Psi}\Phi_{\varrho}^{m,n}(b+k;c+k;z).$$

Theorem 18 The following equation hold true.

$$\begin{aligned}(b-1)B(b-1,c-b+1)\ {}^{\Psi}F_{\varrho}^{m,n}(a,b-1;c;z) &= (c-b-1)B(b,c-b-1)\\ \times\ {}^{\Psi}F_{\varrho}^{m,n}(a,b-1;c;z) - azB(b,c-b)\ {}^{\Psi}F_{\varrho}^{m,n}&(a+1,b;c;z) + \varrho(m+n)\times\\ B(b-n,c-b-m-1)\ {}^{\Psi}F_{\varrho}^{m,n}&\left[\begin{array}{cc}(\xi_i+\zeta_i,\xi_i)_{1,\gamma} & \\ & \mid a,b-n;c-m-n-1;z\\ (\ell_j+\varepsilon_j,\ell_j)_{1,\lambda} & \end{array}\right]\\ -\ {}^{\Psi}F_{\varrho}^{m,n}&\left[\begin{array}{cc}(\xi_i+\zeta_i,\xi_i)_{1,\gamma} & \\ & \mid a,b-n-1;c-m-n-2;z\\ (\ell_j+\varepsilon_j,\ell_j)_{1,\lambda} & \end{array}\right]\\ &\times \varrho nB(b-n-1,c-b-m-1),\end{aligned} \tag{49}$$

$$(Re(b) > Re(n+1), Re(c) > Re(b+m+1)).$$

Proof. Using the following formula

$$B(b,c-b)^{\Psi}F_{\varrho}^{m,n}(a,b;c;z) = \mathbf{M}\left\{f_{a,b,c}^{m,n}(t:z;\varrho):b\right\}.$$

Where

$$\begin{aligned}f_{a,b,c}^{m,n}(t:z;\varrho) &= (1-t)^{c-b-1}(1-tz)^{-a}H(1-t)\\ &\times\ {}_{\tau}\Psi_{\kappa}\left(-\frac{\varrho}{(1-t)^m t^n}\right).\end{aligned} \tag{50}$$

Differentiating (50) with respect to t, we obtain

$$\begin{aligned}\frac{d}{dt}f_{a,b,c}^{m,n}(t:z;\varrho) &= -(c-b-1)(1-t)^{c-b-2}(1-tz)^{-a}H(1-t)\\ &\times\ {}_{\tau}\Psi_{\kappa}\left(-\frac{\varrho}{(1-t)^m t^n}\right) + az(1-t)^{c-b-1}(1-tz)^{-a-1}H(1-t)\\ &\times\ {}_{\tau}\Psi_{\kappa}\left(-\frac{\varrho}{(1-t)^m t^n}\right) - (1-t)^{c-b-1}(1-tz)^{-a}\delta(1-t)\\ &\times\left(-\frac{\varrho}{(1-t)^m t^n}\right) - \varrho(m+n)t^{-n}(1-t)^{c-b-m-2}(1-tz)^{-a}H(1-t)\\ &\times\ {}_{\tau}\Psi_{\kappa}\left[\begin{array}{cc}(\xi_i+\zeta_i,\xi_i)_{1,\gamma} & \\ & \mid -\frac{\varrho}{(1-t)^m t^n}\\ (\ell_j+\varepsilon_j,\ell_j)_{1,\lambda} & \end{array}\right] + \varrho n t^{-n-1}(1-t)^{c-b-m-2}\\ &\times(1-tz)^{-p_1}H(1-t)\left[\begin{array}{cc}(\xi_i+\zeta_i,\xi_i)_{1,\gamma} & \\ & \mid -\frac{\varrho}{(1-t)^m t^n}\\ (\ell_j+\varepsilon_j,\ell_j)_{1,\lambda} & \end{array}\right].\end{aligned} \tag{51}$$

On simplification of (51), we get

$$\begin{aligned}&-(b-1)B(b-1,c-b+1)\ {}^{\Psi}F_{\varrho}^{m,n}(a,b-1;c;z)=-(c-b-1)B(b,c-b-1)\\&\times\ {}^{\Psi}F_{\varrho}^{m,n}(a,b-1;c;z)+azB(b,c-b)\ {}^{\Psi}F_{\varrho}^{m,n}(a+1,b;c;z)-\varrho(m+n)\times\\&B(b-n,c-b-m-1)\ {}^{\Psi}F_{\varrho}^{m,n}\left[\begin{array}{c}(\xi_i+\zeta_i,\xi_i\varrho)_{1,\gamma}\\ \\ (\ell_j+\varepsilon_j,\ell_j)_{1,\lambda}\end{array}\mid a,b-n;c-m-n-1;z\right]\\&+B(b-n-1,c-b-m-1)\,{}^{\Psi}F_{\varrho}^{m,n}\left[\begin{array}{c}(\xi_i+\zeta_i,\xi_i\varrho)_{1,\gamma}\\ \\ (\ell_j+\varepsilon_j,\ell_j)_{1,\lambda}\end{array}\mid a,b-n-1;c-m-n-2;z\right]\\&\times\varrho(m+n).\end{aligned}\tag{52}$$

On simplifying (52), we get the desired result in (49).

Corollory 19 The following equation hold true.

$$\begin{aligned}&(b-1)B(b-1,c-b+1)\ {}^{\Psi}\Phi_{\varrho}^{m,n}(b-1;c;z)=(c-b-1)B(b,c-b-1)\\&\times\ {}^{\Psi}\Phi_{\varrho}^{m,n}(b-1;c;z)-zB(b,c-b)\ {}^{\Psi}\Phi_{\varrho}^{m,n}(b;c;z)-\varrho nB(b-n-1,c-b-m-1)\\&{}^{\Psi}\Phi_{\varrho}^{m,n}\left[\begin{array}{c}(\xi_i+\zeta_i,\xi_i)_{1,\gamma}\\ \\ (\ell_j+\varepsilon_j,\ell_j)_{1,\lambda}\end{array}\mid b-n-1;c-m-n-2;z\right]+B(b-n,c-b-m-1)\\&\times\varrho(m+n)\,{}^{\Psi}\Phi_{\varrho}^{m,n}\left[\begin{array}{c}(\xi_i+\zeta_i,\xi_i)_{1,\gamma}\\ \\ (\ell_j+\varepsilon_j,\ell_j)_{1,\lambda}\end{array}\mid b-n;c-m-n-1;z\right],\end{aligned}\tag{53}$$

$$(Re(b)>Re(n+1),Re(c)>Re(b+m+1)).$$

9. The Mellin transform for the extended Gauss and confluent hypergeometric functions

This section investigated the Mellin transform and inverse Mellin transform for the generalized Gauss and confluent hypergeometric functions.

Theorem 20 The following Mellin transform formula hold.

$$\mathbf{M}\left\{{}^{\Psi}F_{\varrho}^{m,n}(a,b;c;z)\right\}=\frac{B(b+ns,c-b+ms)^{\Psi}\Gamma(s)}{B(b,c-b)}\times{}_2F_1(a,b+ns;c+2(n+m)s;z),\tag{54}$$

$$(Re(s)>0,Re(b+ns)>0,Re(c-b+ms)>0).$$

Proof. By direct calculation

$$\mathbf{M}\left\{{}^{\Psi}F_{\varrho}^{m,n}(a,b;c;z)\right\}=\int_0^{\infty}\varrho^{s-1}\ {}^{\Psi}F_{\varrho}^{m,n}(a,b;c;z)d\varrho.\tag{55}$$

Putting (34) into (55), gives

$$\mathbf{M}\left\{{}^{\Psi}F_{\varrho}^{m,n}(a,b;c;z)\right\}=\frac{1}{B(b,c-b)}\int_0^{\infty}\varrho^{s-1}\sum_{r=0}^{\infty}(a)_r\frac{z^r}{r!}\times\left\{\int_0^1 t^{b+r-1}(1-t)^{c-b-1}{}_{\tau}\Psi_{\kappa}\left(-\frac{\varrho}{(1-t)^m t^n}\right)dt\right\}d\varrho.\tag{56}$$

Interchanging the order of integrations in (58) and simplification, we have

$$\mathbf{M}\left\{{}^{\Psi}F_{\varrho}^{m,n}(a,b;c;z)\right\} = \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} \times \left\{\int_0^\infty \varrho^{s-1}{}_{\tau}\Psi_{\kappa}\left(-\frac{\varrho}{(1-t)^m t^n}\right)d\varrho\right\}dt. \quad (57)$$

Setting $\varrho = \nu(1-t)^m t^n$ in (57), gives

$$\mathbf{M}\left\{{}^{\Psi}F_{\varrho}^{m,n}(a,b;c;z)\right\} = \frac{1}{B(b,c-b)}\int_0^1 t^{b+ns-1}(1-t)^{c-b+ms-1} \times (1-tz)^{-a}\left\{\int_0^\infty \nu^{s-1}\;{}_{\tau}\Psi_{\kappa}(-\nu)\,d\varrho\right\}dt. \quad (58)$$

Applying (1) and (7) to (58), we have

$$\mathbf{M}\left\{{}^{\Psi}F_{\varrho}^{m,n}(a,b;c;z)\right\} = \frac{B(b+ns,c-b+ms)^{\Psi}\Gamma(s)}{B(b,c-b)} \times {}_2F_1(a,b+ns;c+2(n+m)s;z).$$

Corollory 21 The following inverse Mellin transform formula hold true.

$${}^{\Psi}F_{\varrho}^{m,n}(a,b;c;z) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{B(b+ns,c-b+ms)^{\Psi}\Gamma(s)}{B(b,c-b)} \times {}_2F_1(a,b+ms;c+2ns;z)\varrho^{-s}ds, \quad (59)$$

$$(\sigma>0, Re(s)>0, Re(b+ns)>0, Re(c-b+ms)>0).$$

Corollary 22 The following Mellin transform formula hold.

$$\mathbf{M}\left\{{}^{\Psi}\Phi^{m,n}(b;c;z)\right\} = \frac{B(b+ns,c-b+ms)^{\Psi}\Gamma(s)}{B(b,c-b)}\Phi(b+ns;c+2ms;z), \quad (60)$$

$$(Re(s)>0, Re(b+ns)>0, Re(c-b+ms)>0).$$

Corollory 23 The following inverse Mellin transform hold true.

$${}^{\Psi}\Phi_{\varrho}^{m,n}(b;c;z) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{B(b+ns,c-b+ms)^{\Psi}\Gamma(s)}{B(b,c-b)} \times \Phi(b+ms;c+2ns;z)\varrho^{-s}ds, \quad (61)$$

$$(\sigma>0, Re(s)>0, Re(b+ns)>0, Re(c-b+ms)>0).$$

10. Transformation formulas for the generalized Gauss and confluent hypergeometric functions

In this section difference formulas for the generalized Gauss and confluent hypergeometric functions are formulated.

Theorem 24 The following transformation formulas hold.

$${}^{\Psi}F_{\varrho}^{m,n}(a,b;c;z) = (1-z)^{-a}\;{}^{\Psi}F_{\varrho}^{m,n}\left(a,b;c;\frac{z}{z-1}\right), \quad (62)$$

$${}^{\Psi}\Phi_{\varrho}^{m,n}(b;c;z) = exp(z)^{\Psi}\Phi_{\varrho}^{m,n}(c-b;c;-z). \quad (63)$$

Proof. On setting $t \to 1+t$ in (36), (41) and algebraic simplifications, we obtain the required results in (63) and (64), respectively.

11. Difference formulas for the generalized Gauss and confluent hypergeometric functions

In this section difference formulas for the generalized Gauss and confluent hypergeometric functions are formulated.

Theorem 25 The following difference formulas hold.

$$\Delta_a {}^{\Psi}F_{\varrho}^{m,n}(a,b;c;z) = \frac{bz}{c}\ {}^{\Psi}F_{\varrho}^{m,n}(a+1,b+1;c+1;z), \tag{64}$$

$$a\Delta_a {}^{\Psi}F_{\varrho}^{m,n}(a,b;c;z) = z\frac{d}{dz}\ {}^{\Psi}F_{\varrho}^{m,n}(a,b;c;z), \tag{65}$$

$$b\Delta_b\ {}^{\Psi}\Phi_{\varrho}^{m,n}(b;c+1;z) = -c\Delta_c {}^{\Psi}\Phi_{\varrho}^{m,n}(b;c;z), \tag{66}$$

$$\frac{d}{dz}\ {}^{\Psi}\Phi_{\varrho}^{m,n}(b;c;z) = \frac{b}{c}\ {}^{\Psi}\Phi_{\varrho}^{m,n}(b;c+1;z) - \Delta_c {}^{\Psi}\Phi_{\varrho}^{m,n}(b;c;z). \tag{67}$$

Proof. By direct calculation

$$\Delta_a\ {}^{\Psi}F_{\varrho}^{m,n}(a,b;c;z) = \ {}^{\Psi}F_{\varrho}^{m,n}(a+1,b;c;z) - \ {}^{\Psi}F_{\varrho}^{m,n}(a,b;c;z). \tag{68}$$

On simplification of (68), yields

$$\Delta_a\ {}^{\Psi}F_{\varrho}^{m,n}(a,b;c;z) = \frac{z}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1} \times (1-tz)^{-a}\ {}_{\tau}\Psi_{\kappa}\left(-\frac{\varrho}{(1-t)^m t^n}\right)dt. \tag{69}$$

And

$$\Delta_a\ {}^{\Psi}F_{\varrho}^{m,n}(a+1,b+1;c+1;z) = \frac{z}{B(b+1,c-b)}\int_0^1 t^{b}(1-t)^{c-b-1} \times (1-tz)^{-a-1}\ {}_{\tau}\Psi_{\kappa}\left(-\frac{\varrho}{(1-t)^m t^n}\right)dt. \tag{70}$$

Using equation (2) and (70) in (69), we obtain (64). Using differential formula in (45), we get (65). Applying differential operator to (41), we obtain (66). Using differential formula in (48) and (11), we have (67).

12. Conclusion and recommendation

In this research paper, we introduced and investigated new generalized beta function, we also gave certain of its properties such as integral representations, differential formulas, difference formulas, Mellin transform, Mellin inversion formula and summation formulas. We also gave some statistical applications by introducing beta distribution and its corresponding mean, variance and moment generating function. The following particular cases can be drawn from the new introduced generalized beta, Gauss and confluent hypergeometric functions, if the parameters are replaced appropriately:

For $\varrho = 0$ and $m = n = 1$, then

$$B(x,y) = {}^{\Psi}B_0^{1,1}\left[\begin{array}{cc} (1,0)_{1,1} & \\ & \mid x,y \\ (1,1)_{1,1} & \end{array}\right],$$

$$F(a,b;c;z) = {}^{\Psi}F_{0}^{1,1}\left[\begin{array}{cc}(1,0)_{1,1} & \\ & \mid a,b;c;z \\ (1,1)_{1,1} & \end{array}\right],$$

And

$$\Phi(b;c;z) = {}^{\Psi}\Phi_{0}^{1,1}\left[\begin{array}{cc}(1,0)_{1,1} & \\ & \mid b;c;z \\ (1,1)_{1,1} & \end{array}\right].$$

For $\varrho \neq 0$ and $m = n = 1$, then

$$B_{\varrho}(x,y) = {}^{\Psi}B_{\varrho}^{1,1}\left[\begin{array}{cc}(1,0)_{1,1} & \\ & \mid x,y \\ (1,1)_{1,1} & \end{array}\right],$$

$$F_{\varrho}(a,b;c;z) = {}^{\Psi}F_{\varrho}^{1,1}\left[\begin{array}{cc}(1,0)_{1,1} & \\ & \mid a,b;c;z \\ (1,1)_{1,1} & \end{array}\right],$$

And

$$\Phi_{\varrho}(b;c;z) = {}^{\Psi}\Phi_{\varrho}^{1,1}\left[\begin{array}{cc}(1,0)_{1,1} & \\ & \mid b;c;z \\ (1,1)_{1,1} & \end{array}\right].$$

For $\varrho \neq 0$ and $m = n = 1$, then

$$^{\Psi}B_{\varrho}(x,y) = {}^{\Psi}B_{\varrho}^{1,1}\left[\begin{array}{cc}(\xi_i,\zeta_i)_{1,\gamma} & \\ & \mid x,y \\ (\ell_j,\varepsilon_j)_{1,\lambda} & \end{array}\right],$$

$$^{\Psi}F_{\varrho}(a,b;c;z) = {}^{\Psi}F_{\varrho}^{1,1}\left[\begin{array}{cc}(\xi_i,\zeta_i)_{1,\gamma} & \\ & \mid a,b;c;z \\ (\ell_j,\varepsilon_j)_{1,\lambda} & \end{array}\right],$$

And

$$^{\Psi}\Phi_{\varrho}(b;c;z) = {}^{\Psi}\Phi_{\varrho}^{1,1}\left[\begin{array}{cc}(\xi_i,\zeta_i)_{1,\gamma} & \\ & \mid b;c;z \\ (\ell_j,\varepsilon_j)_{1,\lambda} & \end{array}\right].$$

if $\varrho \neq 0$ and $m = n$, then

$$^{\Psi}B_{\varrho}^{m}(x,y) = {}^{\Psi}B_{\varrho}^{m,m}\left[\begin{array}{cc}(\xi_i,\zeta_i)_{1,\gamma} & \\ & \mid x,y \\ (\ell_j,\varepsilon_j)_{1,\lambda} & \end{array}\right],$$

$$^{\Psi}F_{\varrho}^{m}(a,b;c;z) = {}^{\Psi}F_{\varrho}^{m,m}\left[\begin{array}{cc}(\xi_i,\zeta_i)_{1,\gamma} & \\ & \mid a,b;c;z \\ (\ell_j,\varepsilon_j)_{1,\lambda} & \end{array}\right],$$

And

$$^{\Psi}\Phi_{\varrho}^{m}(b;c;z) = {}^{\Psi}\Phi_{\varrho}^{m,m}\left[\begin{array}{cc}(\xi_i,\zeta_i)_{1,\gamma} & \\ & \mid b;c;z \\ (\ell_j,\varepsilon_j)_{1,\lambda} & \end{array}\right].$$

B, F and Φ denote classical beta, Gauss and confluent hypergeometric functions (see [1], [2] and [3]); B_{ϱ}, F_{ϱ} and Φ_{ϱ} are beta, Gauss and confluent hypergeometric functions defined in (refer to [7], [8] and [9]); B_{ϱ}^{m}, F_{ϱ}^{m} and Φ_{ϱ}^{m} are beta, Gauss and confluent hypergeometric functions introduced in [11]; $B_{\varrho}^{(\rho_1,\rho_2)}$, $F_{\varrho}^{(\rho_1,\rho_1)}$ and

$\Phi_{\varrho}^{(\rho_1,\rho_2)}$ are beta, Gauss and confluent hypergeometric functions established in [12]. Other generalized beta, Gauss and confluent hypergeometric functions can also be generated from the generalized functions ([14], [18]). The new generalized beta, Gauss and confluent hypergeometric functions as generalization of many known generalization, hence they become of paramount important from application point of view in the field of mathematical physics, statistic, engineering and other applied mathematics related areas . This generalization of beta, Gauss and confluent hypergeometric function can be use to study Appell's and Lauricella's hypergeometric functions (see for examples [10], [13], [15], [52] and [53]) and Riemann-Liouville and Capotu fractional derivative operators (see, [54], [55] and [56]).

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