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BILINEAR, BILATERAL AND TRILATERAL GENERATING RELATIONS

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ABSTRACT. The object of this paper is to present certain systematic applications of a class of bilateral generating functions for new polynomials $G_n^\alpha(x, r, \beta, k)$ due to Singhal and Srivastava (1969). Some new Bilateral and Trilateral generating relations for these polynomials. We obtain a bilinear generation relation by using an operational technique.

1. INTRODUCTION

The bilinear and bilateral generating functions are defined as, If a Function $G(x, y, t)$ can be expanded in the form

$$G(x, y, t) = \sum_{n=0}^{\infty} k_n f_n(x).f_n(y)t^n, \tag{1.1}$$

where k_n is independent of x, y and t, then $G(x, y, t)$ is called a bilinear generating function.
Again if a function $H(x, y, t)$ be expanded in powers of $'t'$ in the form

$$H(x, y, t) = \sum_{n=0}^{\infty} h_n f_n(x).g_n(y)t^n, \tag{1.2}$$

where h_n is independent of x and y. $f_n(x)$ and $g_n(x)$ are different functions of x, then by Rainville [13], $H(x, y, t)$ be bilateral generating function. Various bilinear and bilateral generating relations for the classical polynomials viz Hermite, Laguerre and Legendre are studied in ([6], [10]-[15]).
In course of discussion of group theoretic origin of certain generating functions for the hypergeometric function, ${}_2F_1(-n; \beta : y : z)$, Weisner obtained the following

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bilateral generating relation for Laguerre polynomials-

$$\sum_{n=0}^{\infty} {}_2F_{\mathrm{n}}(-n,-\nu;1+\alpha:\omega)L_n^{(\alpha)}(x)y^n = (1-y)^{-1-a\nu}(1-y+\omega y)exp\left(\frac{-xy}{1-y}\right) \times {}_1F_1\left[-\nu;\frac{xy\omega}{1+\alpha(1-y)(1-y+\omega y)}\right].$$

The above bilateral generating relation has also been established by Brafman [3] and Rainville [13] by different methods.
In 1969, Chatterjea [8] proved, by means of operational methods, the following bilateral generating relation for the ultra spherical polynomials.

$$\rho^{-2\lambda}F\left(\frac{x-t}{\rho},\frac{yt}{\rho}\right)=\sum_{r=0}^{\infty}t^r b_r(y)p_r^{\lambda}(x), \tag{1.3}$$

where

$$F(x,t)=\sum_{r=0}^{\infty}a_m t^{,} p_m^{\lambda}(x), \tag{1.4}$$

$$b_r(y)=\sum_{r=0}^{r}\left(\begin{array}{c} r\\ m\end{array}\right)a_m y^m, \tag{1.5}$$

and

$$\rho=(1-2xt+t^2)^{1/2}. \tag{1.6}$$

Mc Bride[10] presented a systematic study of obtaining generating functions $\{S_n(x), n = 0,1,...\}$ as the coefficient set in a bilinear (or bilateral) generating relations that belongs to a class of functions generated by

$$\sum_{n=0}^{\infty}A_{m,n}S_{m+n}(x)t^n=f(x,t)\{g(x,t)\}^{-m}S_m(h(x,t)), \tag{1.7}$$

where $m \geq 0$ is an integer, $A_{m,n}$ are arbitrary constants and f,g,h are arbitrary functions of x and t.
An effective method of obtaining bilateral generating functions for $S_n(x)$ defined by (1.7) was given and illustrated by Srivastava [14] as,

Theorem 1. *Let*

$$F(x,t)=\sum_{n=0}^{\infty}a_n s_n(x)t^n, \tag{1.8}$$

where $a_n \neq 0$ are arbitrary and the sequence of functions $\{S_n(x),\ n=0,1,2,....\}$ is generated by (1.7)*.*
Then

$$f(x,t)F\left[h(x,t),\frac{yt}{g(x,t)}\right]=\sum_{n=0}^{\infty}S_n(x)\sigma_n(y)t^n, \tag{1.9}$$

where

$$\sigma_n(y)=\sum_{k=0}^{n}a_k A_{k,n-k}y^k. \tag{1.10}$$

Chatterjea pointed out that the scope of the above theorem remains limited, for example, if we take

$$S_n(x) = P_n^{(\alpha,\beta)}.$$

Then no formula corresponding (1.7) is yet known and that is the reason why Singhal and Srivastava failed to apply their theorem in the case of proper Jacobi polynomials. Chatterjea [8] gave the improved version of the above ***Theorem* (1.1)** in the form of following proposition.

***Proposition* 1.1.** *For a set of functions $S_\alpha(x)$ generated by*

$$\frac{f(x,t)}{[g(x,t)]^\alpha} S_\alpha(h(x,t)) = \sum_{n=0}^{\infty} A_n S_{\alpha+n}(x) t^n, \tag{1.11}$$

and for

$$F(x,t) = \sum_{n=0}^{\infty} a_n s_{n+m}(x) t^n, \tag{1.12}$$

where $F(x,t)$ is of arbitrary nature, the following bilateral generating relation holds.

$$\frac{f(x,t)}{[g(x,t)]^m} F\left[h(x,t), \frac{yt}{g(x,t)}\right] = \sum_{n=0}^{\infty} s_{n+m}(x) \sigma_n(y) t^n, \tag{1.13}$$

where

$$\sigma_n(y) = \sum_{k=0}^{n} a_k A_{n-k} y^k. \tag{1.14}$$

A mild generalization of (1.7) include special functions such as the Bessel function $J_\mu(x)$ which possesses a generating relation of the type:

$$\sum J_{\mu+n}(z) \frac{t^n}{n!} = \left(1 - \frac{2t}{z}\right)^{\frac{-\mu}{2}} J_\mu((z^2 - 2zt)^{1/2}), \tag{1.15}$$

where μ is an arbitrary complex number.
Mittal[12] has given a general method for deriving bilinear and bilateral generating relations for the set of polynomials $\{f_n^{(a)}(x)\}$ defined by

$$T_{a+1}^n \{f(x)\} = n!\, x^n g(x) f_n^{(a)}(x), \tag{1.16}$$

where $f(x)$ admits a formal lower series expansion in x, $g(x)$ being a function of x alone, $T_a = x(a + xQ'_x)$, $Q' = \frac{d}{dx}$ and $'a'$ is a constant. As a consequence, he obtained several generating relations for Boas and Buck type polynomials [4]
Shrivastava and Singh [15] presented a novel extension of several bilateral generating relations derived earlier by Al-Salam [2], Srivastava [17], Chatterjea [8] and others, in the form of Mixed Trilateral generating relations and applied their theorem to the Hermite, generalized Hermite, Laguerre, Bessel, Srivastava - Singhal polynomials and to the Bessel function of 1st kind.
In another publications, Srivastava and Singh [15] established the following bilateral generating relation:

$$\begin{aligned} \sum_{m=0}^{\infty} V_{\nu+m}^{(\alpha)}(x;a,k,s) R_{m,y}^q(y) t^n &= (1 - ax^\alpha t)^{\frac{-(a+s)}{a}} exp.\left[p_k(x) - p_k\left\{x(1-ax^\alpha t)^{\frac{-1}{a}}\right\}\right] \\ &\quad \times \phi_{q,\nu}\left[x(1-ax^\alpha t)^{\frac{-1}{a}}, yt^q\right], \end{aligned} \tag{1.17}$$

where $\nu = 0, 1, 2, ...$ and

$$\phi_{q,\nu}(x,t) = \sum_{m=0}^{\infty} \delta_{\nu,m} V_{\nu+m}^{(\alpha)}(x; a, k, s) t^m, \tag{1.18}$$

and

$$(1 - ax^{\alpha}t)^{\frac{(a+s)}{a}} exp. \left[p_k(x) - p_k \left\{x(1 - ax^{\alpha}t)^{\frac{-1}{a}}\right\}\right] = \sum_{n=0}^{\infty} V_n^{(\alpha)}(x; a, k, s) t^n. \tag{1.19}$$

In 1980, Agrawal and Manocha [1] deduced a bilateral generating relation.

$$\begin{aligned}\sum_{n=0}^{\infty} L_n^{(\alpha)}(x) P_n^m(y,z) t^n \quad &= \quad \frac{\Gamma(m+\beta+1)(1-t)^{m+\beta-1}(1-t-zt)^{-(m+\beta+1)}}{\Gamma(\alpha+1)\Gamma(\beta+1)} \\ &\times exp. \left[\frac{y-(x+y)t}{1-t}\right] \times \psi_2 \left[\begin{array}{c} m+\beta+1; \alpha+1, \beta+1; \\ \frac{-xzt}{(1-t)(1-t-zt)}, \frac{-y(1-t)t}{(1-t-zt)} \end{array}\right],\end{aligned} \tag{1.20}$$

with

$$P_n^m(y,z) = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) \frac{(m+k)!}{(\alpha+k+1)!} L_{k+m}^{(\alpha)} L(y) z^k. \tag{1.21}$$

Srivastava [16], presented a systematic introduction to and several interesting applications of a general method of obtaining bilinear, bilateral or mixed multilateral generating functions for a fairly wide variety of special functions in one, two and more variables.

2. Characterizations

By making use of the formula

$$L_n^{(\alpha)}(x) = (n!)^{-1} \overset{[}{(} J = 1)](n)\pi(xD' = x + \alpha + J). \tag{2.1}$$

Al-Salam [2] proved the following theorem for characterization of the Laguerre polynomials.

***Representation* 2.1.** *Let $b_n (n = 1, 2, ...)$ be the sequence of numbers let,*

$$P_n(x) = \overset{[}{(} J = 1)](n)\pi(xQ' + x + b_j), \ (n = 1, 2, 3, ...), \tag{2.2}$$
$$P_0(x) = 1.$$

If the set $\{P_n(x)\}$; defined by means of (2.2)*, is a set of orthogonal polynomials then $P_n(x)$ is the n^{th} Laguerre polynomials.*

In the same paper, Al-Salam [2] has also given a similar result for Hermite polynomials.
Carlitz has shown that the formula

$$\int_0^1 L_m^{(\alpha)}(xt) L_n^{(\beta)}((1-x)t) x^{\alpha}(1-x)^{\beta} dx = \left(\begin{array}{c} m+n \\ m \end{array}\right) \frac{\Gamma(\alpha+m+1)\Gamma(\beta+n+1)}{\Gamma(\alpha+\beta+m+n+2)} \times L_{m+n}^{(\alpha+\beta+1)}(t), \tag{2.3}$$

$\alpha, \ \beta > -1$
can be used to characterize the Laguerre polynomials. In order to characterize the Bessel polynomials $Y_n^{(\alpha)}(x)$ of Krall and Frink [9], Al-Salam [2] has established the following theorems.

***Representation* 2.2.** *Given a sequence $\{f_{(n)}^{(\alpha)}(x)\}$ of polynomial in x where degree of $f_{(n)}^{(\alpha)}(x) = n$ and α is a parameter such that*

$$\begin{aligned} Q_r f_r^{(\gamma)}(x) &= \frac{1}{2} r(r+n+1) f_{r-1}^{(\gamma+2)}(x), \\ f_r^{(\gamma)}(0) &= 1, \\ f_r^{(\gamma)}(x) &= Y_r^{(\gamma)}(x). \end{aligned}$$

***Representation* 2.3.** *Given a sequence of functions $\{f_{(n)}^{(\alpha)}(x)\}$ such that*

$$\Delta_a f_n^{(\alpha)}(x) = \frac{1}{2} nx f_{n-1}^{(\alpha+2)}(x),$$

$$f_n^{(\alpha)}(0) = 1, \ f_0^{(\alpha)}(x) = 1.$$

Then

$$f_n^{(\alpha)}(0) = Y_0^{(\alpha)}(x).$$

***Representation* 2.4.** *If the sequence $\{f_{(n)}^{(\alpha)}(x)\}$ where $f_n^{(\alpha)}(x)$ is polynomials of degree n in x and α is a parameter satisfying $\Delta_\alpha f_n^{(\alpha)}(x) = \frac{x}{n+\alpha+1} Q_x f_n^{(\alpha)}(x)$, such that $f_n^{(\alpha)}(x) = Y_n(x)$. Then*

$$f_n^{(\alpha)}(x) = Y_n^{(\alpha)}(x),$$

$$Y_n^{(0)}(x) = Y_n(x).$$

***Representation* 2.5.** *Given a sequence of functions $\{f_{(n)}^{(\alpha)}(x)\}$ such that*

$$\Delta_n f_n^{(\alpha)}(x) = \frac{n}{2}(2n+\alpha+2) f_n^{(\alpha+1)}(x),$$

where

$$f_0^{(\alpha)}(x) = 1, \ \text{for every } x \text{ and } \alpha.$$

Then

$$f_n^{(\alpha)}(0) = Y_n^{(\alpha)}(x).$$

Recently, Verma and Prasad[23] considered a class of polynomial sets $\{P_n(x), n = 0, 1, 2, ...\}$ defined by

$$(1-t)^{-c} \phi \left\{ \frac{2t(x-1)}{(1-t)^2} \right\} = \sum_{n=0}^{\infty} \frac{(e)_n}{(c-\beta)_n} P_n(x) t^n, \tag{2.4}$$

where $\phi(u)$ has formal power series expansion, $\phi(0) \neq 0$. For this class of polynomials, they proved the following

***Representation* 2.6.** *The simple set of polynomials $\{P_n(x)\}$, where degree of $P_n(x) = n$, which is orthogonal and satisfies* (2.4)*, is either the set of Jacobi or Bessel polynomials.*

Further, they assumed that polynomial set $\{P_n(x)\}$, of (2.4) satisfies.

$$P_n(x) = \frac{(c)_{2n}}{n! \ (c_n)} \frac{(x-1)^n}{2} \times {}_{\mathrm{p}}F_{\mathrm{q}} \left[\begin{matrix} -n, 1-\alpha-c-n^{(\alpha_{p-2})}; \\ 1-c-2n, (\beta_{q-1}) \end{matrix} \ \frac{2}{1-x} \right], \tag{2.5}$$

where the parameters c_1 $\alpha_1, \alpha_2, ...\alpha_{p-2}, \beta_1, \beta_2, ...\beta_{q-1}$ are arbitrary complex number with $\beta_k \neq -m$ (a negative integer) and they proved.

***Representation* 2.7.** *The only hypergeometric polynomial of the type* (2.5) *which has a generating functions* (2.4) *is the set of Jacobi polynomials.*

3. Bilinear Generating relation

Singhal and Srivastava [14] considered an unified polynomial set defined by the generalized Rodrigues formula.

$$G_n^{(\alpha)}(x,r,\beta,k)=\frac{1}{n!}x^{-\alpha-kn}exp(\beta x^r)\theta^n(x^\alpha e^{-\beta x^r}),\ \theta=(x^{k+1}Q_x). \tag{3.1}$$

That provides us with an elegant generalization of the various extensions of the classical Hermite and Laguerre polynomials given, for instance by S.K. Chatterjea [8], Gould and Hopper, and Singh & Srivastava [15]. Chandel ([5],[6],[7]) introduced and studied slightly modified polynomials $T_n^{(\alpha,k)}(x,r,p)$ defined by

$$T_n^{(\alpha,k)}(x,r,p)=x^{-\alpha}e^{px^r}\Omega_x^n\{x^\alpha e^{-px^r}\},\ \Omega_x\equiv x^k\frac{d}{dx} \tag{3.2}$$

with connection

$$G_n^{(\alpha)}(x,r,p,k)=\frac{1}{n!x^{kn}}T_n^{(\alpha,k)}(x,r,p). \tag{3.3}$$

Also

$$G_n^{(\alpha)}(x,1,1,k)=\frac{1}{n!}G_{n,k}^{(\alpha)}(x), \tag{3.4}$$

$$G_n^{(\alpha+n)}(x,r,1,1)=G_n^{(\alpha+1)}(x,r,1,1)=P_{n,r}^{(\alpha)}(x), \tag{3.5}$$

$$G_n^{(\alpha)}(x,r,\beta,-1)=G_n^{(\alpha-n+1)}(x,r,\beta,1)=\frac{(-x)^n}{n!}H_n^r(r,x,\beta), \tag{3.6}$$

$$G_n^{(\alpha+n)}(x,r,\beta,-1)=G_n^{(\alpha+1)}(x,r,\beta,1)=L_n^{(\alpha)}(x,r,\beta), \tag{3.7}$$

$$G_n^{(\alpha)}(x,2,1,-1)=G_n^{(-n+1)}(x,2,1,1)=\frac{(-x)^n}{n!}H_n(x), \tag{3.8}$$

$$G_n^{(\alpha+n)}(x,1,1,k)=G_n^{(\alpha+1)}(x,1,1,1)=L_n^{(x)}(x), \tag{3.9}$$

and

$$G_n^{(\alpha+n)}(x,1,1,k)=k^n y_n^{(\alpha)}(x;k),\ \alpha>-1,\ k=1,2, \tag{3.10}$$

where $H_n^{(\alpha)}(x)$, $L_n^{(\alpha)}(x)$ and $Y_n^{(\alpha)}(x;k)$ are the Hermite, Laguerre and Konhauser polynomials respectively.
It may be of interest to note that $Y_n^{(\alpha)}(x;1)=L_n^{(\alpha)}(x)$.

Singhal and Srivastava [14] discussed several interesting properties such as linear, bilinear and bilateral generating functions, pure as well as mixed recurrence relations and the differential equations associated with the class of polynomials.

$$G_n^{(\alpha)}(x,r,\beta,k),\ n=0,1,2,....$$

The object of the present investigation is to present some applications of a class of bilateral generating functions for the polynomials $G_n^{(\alpha)}(x,r,\beta,k)$ due to Srivastava [16] in deriving some new bilateral and trilateral generating relations for these polynomials. We obtain a bilinear generating relation using an operational technique. Srivastava [14] derived following Class of bilateral generating functions for the polynomials. $G_n^{(\alpha)}(x,r,\beta,k)$ as a special case of his general theorem for obtaining bilinear, bilateral or mixed multilateral generating functions for a certain class of special functions.

$$\Delta_{m,q}[x;y_1,...y_s,t]=\sum_{n=0}^{\infty}a_nG_{m+qn}^{(\alpha)}(x,r,\beta,k)\Omega_{\mu+p_n}(y_1....y_s)t^n. \tag{3.11}$$

If $a_n\neq 0$ and $\Omega(y_1,....y_s)$ is a non-vanishing function.
Then for every non-negative integer m,

$$\begin{aligned}&\sum_{n=0}^{\infty}G_{m+n}^{(\alpha)}(x,r,\beta,k)N_{n,m,q}^{p,\mu}(y_1,....,y_s;z)t^n \\ &=(1-kt)^{-m-\alpha/k}exp[\beta x^r\{1-(1-(1-kt)^{-r/k})\}] \\ &\times\Delta_{m,q}\left[x(1-xt)^{-1/k};y_1,....,y_s\frac{zt^q}{(1-kt)^q}\right],\end{aligned} \tag{3.12}$$

where

$$N_{n,m,q}^{p,\mu}(y_1,.....,y_s;z)=\sum_{r=0}^{[n/q]}\begin{pmatrix}m+n\\ n-qr\end{pmatrix}a_r\Omega_{\mu+pr}(y_1,....,y_s;z)z^r \tag{3.13}$$

is an arbitrary complex number, p and q are positive integers.

4. Applications

We derive some interesting applications of (3.1) when $m=0$ and $\Omega_\mu=1$.

Case 4.1 *(q = 1)* Firstly, on taking $a_n=\frac{\prod_{j=1}^{p}(b_j)_n}{\prod_{j=1}^{s}(c_j)_n}$ and replacing y by $-y$, the polynomials

$$\sigma_n^1(y)=N_{n,0,1}(y)\sum_{r=0}^{n}\begin{pmatrix}n\\ r\end{pmatrix}a_ry^r \tag{4.1}$$

becomes identical with the extended Laguerre polynomials.

$$\alpha_n(y:b_1,...,b_p;c_1,...,c_s)={}_{\mathrm{p+1}}F_{\mathrm{s}}\left[\begin{matrix}-n,b_1,...,b_p;\\ c_1,c_2,...,c_s\end{matrix}\ y\right]. \tag{4.2}$$

Thus from (3.1), we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}G_n^{(\alpha)}(x,r,\beta,k)\alpha_n(y:b_1,...,b_p;c_1,...,c_s)t^n \\ &=(1-kt)^{-\alpha/k}exp[\beta x^r\{1-(1-kt)^{-r/k}\}] \\ &\times F^{(2)}\left[x(1-kt)^{-\alpha/k},\frac{-yt}{(1-kt)}\right],\end{aligned} \tag{4.3}$$

where

$$F^{(2)}(x,t)=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(b_j)_n}{\prod_{j=1}^{s}(c_j)_n}G_n^{(\alpha)}(x,r,\beta,k). \tag{4.4}$$

We record, from (4.3), the following trilateral generating relation:

$$\sum_{n=0}^{\infty}\left\{\frac{n!}{(1+\alpha)_n}\right\}^{n}\overset{[}{n}](\alpha)C(x,r,\beta,-1)\times L_n^{(\alpha)}(2\sqrt{y})L_n^{(\alpha)}(-2\sqrt{y})\cdot t^n \tag{4.5}$$
$$= (1-xt)^{-\alpha/k}exp[\beta x^r\{1-(1-kt)^{-r/k}\}]$$
$$\times F^{(3)}\left[x(1-kt)^{-1/k},\frac{-yt}{(1-kt)}\right],$$

where $L_n^{(\alpha)}(z)$ are classical Laguerre polynomials and

$$F^{(3)}(x,t)=\sum_{n=0}^{\infty}\frac{(\alpha+n+1)}{(\alpha+1)_n\left(\frac{(\alpha+1)}{2}\right)_n\left(\frac{(\alpha+2)}{2}\right)}G_n^{(\alpha)}(x,r,\beta,k). \tag{4.6}$$

Next, we note the polynomials $f_n(y)$ from an Appell set provided

$$\frac{d}{dy}(f_n(y))=nf_{n-1}(y)\ (n=0,1,2,...).$$

It follows that

$$f_n(y)=\sum_{r=0}^{n}\begin{pmatrix} n \\ r \end{pmatrix}c_r y^{n-r}. \tag{4.7}$$

For some sequence (c_r) contained in (4.7), we obtain the following bilateral generating relation from (3.1)

$$\sum_{n=0}^{\infty}G_n^{(\alpha)}(x,r,\beta,k)f_n(y)t^n = (1-kyt)^{-\alpha/k}exp[\beta x^r\{1-(1-kyt)^{-r/k}\}] \tag{4.8}$$
$$\times F^{(4)}\left[x(1-kyt)^{-1/k},\frac{t}{(1-kyt)}\right],$$

where

$$F^{(4)}(x,t)=\sum Q_nC_n^{(\alpha)}(x,r,\beta,k)t^n. \tag{4.9}$$

Case 4.2 $(q\geq 1)$ Wright's generalized hypergeometric function is defined by

$${}_p\psi_s\left[\begin{array}{c}(a_1,\alpha_1),...,(a_p,\alpha_p):\\(b_1,\beta_1),...,(b_s,\beta_s)\end{array}\ z\right]=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(a_j)(a_j)_{n\alpha_j}z^n}{\prod_{j=1}^{s}\Gamma(b_j)(b_j)_{n\beta_j}n!}, \tag{4.10}$$

where the variable z and the various parameters are such that the series converges.

On setting

$$a_n=\frac{\prod_{j=1}^{m}\Gamma(a_j)(a_j)_{n\mu_j}\prod_{j=1}^{s}\Gamma(e_j)(e_j)_{n\xi_j}(q_n)!}{\prod_{j=1}^{p}\Gamma(e_j)(e_j)_{n_j}\prod_{j=1}^{l}\Gamma(d_j)(d_j)_{m\eta_j}} \tag{4.11}$$

in the polynomials

$$\overset{[}{n}]q\sigma(y)=N_{n,0,q}(y)=\sum_{r=0}^{[n/q]}\begin{pmatrix} n \\ qr \end{pmatrix}a_r y^r \tag{4.12}$$

and replacing y by $(-1)^q y$; we arrive at the following bilateral generating relation.

$$\sum_{n=0}^{\infty} G_n^{(\alpha)}(x,r,\beta,k) \times {}_{\mathrm{m+s+1}}\psi_{\mathrm{p+1}} \left[\begin{array}{c} (n,q),\{b_n,\mu_n,\{e_x,q\xi s\}\}; \\ (c_p,v_p),(q_i,q_{n1}); \end{array} y \right] \quad (4.13)$$
$$= (1-kt)^{-\alpha/k} exp[\beta x^r \{1-(1-kt)^{-r/k}\}]$$
$$\times F^{(5)} \left[x(1-kt), y\left\{ \frac{-t}{(1-kt)^q} \right\} \right],$$

where (a_p,α_p) abbreviate the array of p parameli pair $(a_1,\alpha_1),...(a_p,\alpha_p)$, and

$$F^{(5)}(x,t) = \sum_{n=0}^{\infty} a_n G_{qn}^{(\alpha)}(x,r,\beta,k). \quad (4.14)$$

Some special cases of (4.13) **are noteworthy.** For example, if in (4.13) we take $q=1,\ m=s=l=0,\ p=1,\ c_1=y+1,\ v_1=\delta$ and then applying the definition

$$L_n^{(\delta,y)}(x) = \frac{(y+1)_{n\delta}}{n!}\, {}_1\psi_1[(-n,1);(y+1,\delta);x], \quad (4.15)$$

$L_n^{(\delta,y)}(x)$ being a generalization of the Konhauser polynomials $Z_n^{(\alpha)}(x,k)$, we get

$$\sum_{n=0}^{\infty} \left\{ \frac{n!}{(y+1)_{n\delta}} \right\} G_n^{(\alpha)}(x,r,\beta,k) L_n^{(\delta,y)}(y) t^n \quad (4.16)$$
$$= (1-kt)^{-\alpha/k} exp[\beta x^r\{1-(1-kt)^{-r/k}\}] \times F^{(6)} \left[x(1-kt)^{-1/k}, \frac{-yt}{(1-kt)} \right],$$

where

$$F^{(6)}(x,t) = \sum_{n=0}^{\infty} \frac{(qn)!t^n}{(y+1)_{n\delta}} G_{qn}^{(\alpha)}(x,r,\beta,k). \quad (4.17)$$

In other hand, in terms of the Brafman polynomials defined by [3]

$$B_n^q(a_1,...,a_p;b_1,...,b_s;y) = {}_{\mathrm{p+q}}F_{\mathrm{s}} \left[\begin{array}{c} (a:-n),a_1,...a_p; \\ b_1,...b_s;\ y \end{array} \right], \quad (4.18)$$

we deduce, from (4.13), the following bilateral generating relation:

$$\sum_{n=0}^{\infty} G_n^{(\alpha)}(x,r,\beta,k) B_n^q(a_1,...,a_p;b_1,...,b_s;y) t^n \quad (4.19)$$
$$= (1-kt)^{-\alpha/k} exp[\beta x^r\{1-(1-kt)^{-r/k}\}]$$
$$\times F^{(7)} \left[x(1-kt)^{-1/k}, y\left\{ \frac{-t}{q(1-kt)} \right\}^q \right],$$

where

$$F^{(7)}(x,t) = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_n (qn)! t^n}{\prod_{j=1}^{s}(b_j)_n} G_{qn}^{(\alpha)}(x,r,\beta,k). \quad (4.20)$$

By suitably specializing the arbitrary parameters involved in (4.18), the Brafman's polynomials can be reduced to a number of familiar polynomials. Thus, the relation (4.19) may be applied in deriving a number of bilateral generating relations for $G_{qn}^{(\alpha)}(x,r,\beta,k)$.

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