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NUMERICAL APPROACH OF NONLINEAR FRACTIONAL INITIAL VALUE PROBLEMS BY COMBINATION OF THE TWO METHODS : ADOMIAN DECOMPOSITION METHOD AND JACOBI SPECTRAL COLLOCATION

A. S. ALGHAMDI, J.F. ALZAIDY, A.K. HUSSAIN

Abstract. This paper expands a numerical method that is proposed in [1] to obtain approximate solutions of initial value problems for nonlinear fractional differential equations. This approach is based on the combination of the two methods: Adomian decomposition method and Jacobi spectral collocation method. The aim of this approach is to reduce the nonlinear differential equation to a system of algebraic equations that can be solved using a numerical method. Several examples with numerical solutions are provided to demonstrate the efficiency of this approach.

1. Introduction

In the middle of the 19th century, mathematicians focused on the theoretical approach to fractional calculus. Within years, many applications of fractional calculus had appeared in other academic areas, such as chemistry, physics, financial and social sciences and even engineering[2, 3, 4].

Fractional differential equations have been solved using several methods, though the most commonly used methods are the Adomian decomposition method (ADM[5, 6],the collocation method[7, 9], the orthogonal polynomial method[10, 11], the series solution method[12], and the Laplace transform method[10, 13].

George Adomian developed the Adomian decomposition method (ADM) in [2, 14] and has been repeatedly referenced. This method is applicable to a wide range of linear and nonlinear ordinary differential equations, partial differential equations, and integral equations. The Adomian decomposition method relies on decomposing the solution into the infinite series solution[15]

$$y(x) = \sum_{k=0}^{\infty} y_k(x). \tag{1}$$

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The collocation method [7, 16, 17]is used to obtain the numerical solution of the fractional derivative as it is a global operator. However, spectral collocation methods are preferable techniques to obtain numerical solutions of nonlinear differential equations and have a number of outstanding features, such as efficiency, high accuracy, and exponential convergence. The main objective of the spectral collocation method is to approximate the unknown solution $y(x)$ using a linear combination of trial functions, such as orthogonal polynomials.The orthogonal polynomials can be chosen according to their advantages, making it particularly suitable for the problem under study.

To solve nonlinear fractional differential equation, this paper combines two methods, namely the Adomian decomposition method (ADM) and the Jacobi spectral collocation method (JSCM), in order to overcome the difficulties of convergence. The proposed approach reduces nonlinear fractional differential equations to systems of linear algebraic equations. Following this process, the resulting systems are solved using numerical methods for example Newton iterative method.

The remainder of this paper is arranged as follows. Section 2 introduces necessary definitions and the mathematical tools needed for fractional calculus, also discusses the properties of Jacobi polynomial.Section 3 shows the Adomian decomposition method that is used to solve the nonlinear term of fractional differential equations(FDEs). Section 3 ends with an explanation of how the collocation method is used to solve the linear terms of FDEs. Section 4 discusses how the algorithm is built. Section 5 presents numerical results to clarify the methods used. Section 6 is a summary of the paper.

2. Preliminaries and Useful Formulas

This section provides basic definitions and properties of fractional differential and integral operator as in [10, 18, 19].Then we mention to some properties of shifted Jacobi polynomials..

2.1. **Fractional calculus definitions.** The left-sided Riemann-Liouville fractional integral operator ${}^{R}I_a^{\alpha}$ of order $\alpha \in R^+$ on the usual Lebesgue space $L_1[a,b]$ is defined as:

$$ {}^{R}I_a^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\xi)^{\alpha-1} f(\xi) d\xi. \quad ax, \tag{2}$$

where $\Gamma(.)$ represents the Euler gamma function.

The operator ${}^{R}I_a^{\alpha}$ has the following properties:$(\beta, \gamma \in R^+)$

(i) ${}^{R}I_a^{\alpha} I_a^{\beta} = I_a^{\alpha+\beta}$,

$(ii) {}^{R}I_a^{\alpha} I_a^{\beta} = I_a^{\beta} I_a^{\alpha}$,

$(iii) {}^{R}I_a^{\alpha} x^{\gamma} = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\alpha)} x^{\alpha+\gamma}$.

The Riemann-Liouville fractional derivative operator ${}^{R}D_a^{\alpha}$ of order $\alpha \in R^+$ is defined as:

$$ {}^{R}D_a^{\alpha} f(x) = D^k I_a^{k-\alpha} f(x) = \frac{d^k}{dx^k} \frac{1}{\Gamma(k-\alpha)} \int_a^x (x-\xi)^{k-\alpha-1} f(\xi) d\xi. \tag{3}$$

where α satisfies the relation $k-1 < \alpha k$, $k \in N$, and $f \in L_1[a,b]$.

The fractional differential operator in Caputo sense is defined as:

$$ {}^{C}D_a^{\alpha} f(x) = \frac{1}{\Gamma(k-\alpha)} \int_a^x (x-\xi)^{k-\alpha-1} f(\xi) d\xi. \quad a < xb \tag{4} $$

where $k-1 < \alpha k$, $k \in N$. For the Caputo fractional derivative operator, we have the following some basic properties:

$$ I_a^{\alpha} {}^{C}D_a^{\alpha} f(x) = f(x) - \sum_{j=0}^{m-1} f^{(j)}(0^+) \frac{x^j}{j!}, $$

$$ {}^{C}D_a^{\alpha} x^{\gamma} = \begin{cases} 0, & \gamma \in N_0; \lceil \alpha \rceil > \gamma, \\ \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} x^{\gamma-\alpha}, & \gamma \in N_0; \lceil \alpha \rceil \leq \gamma, \end{cases} $$

where$\lceil \alpha \rceil$ is the ceiling functions , while $N_0 = \{0, 1, ...\}$.
In this paper, only Caputo fractional derivatives are considered .

2.2. **Properties Of Shifted Jacobi Polynomials.** The Jacobi polynomials $P_k^{(\eta,\delta)}(x)$ $(k = 0, 1, ...)$ of degree k are defined on the interval $[-1, 1]$, and can be generated using Rodrigue's formula[20, 21, 22]:

$$ P_k^{(\eta,\delta)}(x) = \frac{(-1)^k}{2^k k!} (1-x)^{-\eta} (1+x)^{-\delta} \frac{d^k}{dx^k} [(1-x)^{\eta+k} (1+x)^{\delta+k}], \tag{5} $$

where $k = 0, 1, ...$ and $\eta, \delta > -1$.
The qth derivative of $P_k^{(\eta,\delta)}(x)$ is provided by

$$ \frac{d^q}{dx^q} P_k^{(\eta,\delta)}(x) = \frac{\Gamma(\eta+\delta+k+1)}{2^q \Gamma(\eta+\delta+k+1)} P_{k-q}^{(\eta+q,\delta+q)}(x) \tag{6} $$

Also,the Jacobi polynomial satisfy the orthogonality relation

$$ (P_k^{(\eta,\delta)}(x), P_l^{(\eta,\delta)}(x))_{\omega(x)} = \int_{-1}^{1} P_k^{(\eta,\delta)}(x) P_l^{(\eta,\delta)}(x) \omega(x) dx = h_k \delta_{lk}, \tag{7} $$

where $\delta_{lk}$is the Kronecker delta function and
$\omega^{(\eta,\delta)}(x) = (1-x)^{\eta}(1+x)^{\delta}$, $h_k = \frac{2^{\eta+\delta+1}\Gamma(k+\eta+1)\Gamma(k+\delta+1)}{(2k+\eta+\delta+1)k!\Gamma(k+\eta+\delta+1)}$.

To use these polynomials on the interval $[0, L]$, this paper defines the shifted Jacobi polynomials by inserting the change in variable $x = \dfrac{2x}{L} - 1$. Let the shifted Jacobi polynomials $P_i^{(\eta,\delta)}(\frac{2x}{L} - 1)$ be denoted by $P_{L,i}^{(\eta,\delta)}(x)$ those can be obtained by the following recurrence formula:

$$ \begin{aligned} &P_{L,k+1}^{(\eta,\delta)}(x) = (a_k^{(\eta,\delta)}(\frac{2x}{L} - 1) - b_k^{(\eta,\delta)}(x)) P_{L,k}^{(\eta,\delta)}(x) - c_k^{(\eta,\delta)}(t) P_{L,k-1}^{(\eta,\delta)}(x), k \geq 1, \\ &P_{L,0}^{(\eta,\delta)}(x) = 1, P_{L,1}^{(\eta,\delta)}(x) = \frac{1}{L}(\eta+\delta+2)x - (\delta+1), \end{aligned} \tag{8} $$

where

$$ a_k^{(\eta,\delta)} = \frac{(2k+\eta+\delta+1)(2k+\eta+\delta+2)}{2(k+1)(k+\eta+\delta+1)}, \tag{9} $$

$$ b_k^{(\eta,\delta)} = \frac{(\delta^2-\eta^2)(2k+\eta+\delta+1)}{2(k+1)(k+\eta+\delta+1)(2k+\eta+\delta)}, \tag{10} $$

$$c_k^{(\eta,\delta)} = \frac{(k+\eta)(k+\delta)(2k+\eta+\delta+2)}{(k+1)(k+\eta+\delta+1)(2k+\eta+\delta)}, \tag{11}$$

The explicit analytic form of the shifted Jacobi polynomials $P_{L,j}^{(\eta,\delta)}(t)$ of degree j is given by

$$P_{L,j}^{(\eta,\delta)}(x) = \sum_{k=0}^{j}(-1)^{j+k}\frac{\Gamma(j+\delta+1)\Gamma(j+k+\eta+\delta+1)}{\Gamma(k+\delta+1)\Gamma(j+\eta+\delta+1)(j-k)!k!L^k}\ x^k \tag{12}$$

and the orthogonality condition is

$$\int_0^L P_{L,j}^{(\eta,\delta)}(x)P_{L,k}^{(\eta,\delta)}(x)\omega_L^{(\eta,\delta)}(x)dt = \hbar_{L,k}^{(\eta,\delta)}\delta_{jk}, \tag{13}$$

where $\omega_L^{(\eta,\delta)}(x) = x^\delta(L-x)^\eta$ and $\hbar_{L,k}^{(\eta,\delta)} = \dfrac{L^{\eta+\delta+1}\Gamma(k+\eta+1)\Gamma(k+\delta+1)}{(2k+\eta+\delta+1)k!\Gamma(k+\eta+\delta+1)}$.

Suppose that the function $y(t)$ is square-integral in $[0,L]$, then it can be expressed in terms of shifted Jacobi polynomials as in the following form:

$$y(t) = \sum_{k=0}^{\infty} c_k P_{L,k}^{(\eta,\delta)}(t), \tag{14}$$

where the coefficients c_k are given by

$$c_k = \frac{1}{\hbar_{L,k}^{(\eta,\delta)}}\int_0^L y(t)P_{L,k}^{(\eta,\delta)}(t)\omega_L^{(\eta,\delta)}(t)dt, \qquad k = 0,1,.... \tag{15}$$

To approximate $y(t)$ by the first $(M+1)-$terms, we can write

$$y_M(t) \simeq \sum_{k=0}^{M} c_k P_{L,k}^{(\eta,\delta)}(t). \tag{16}$$

3. Adomain decomposition and Jacobi spectral collocation method

3.1. **The Adomian decomposition method(ADM).** Consider the following nonlinear differential equation

$$\mathcal{L}[y] + \mathcal{N}[y] = g(x), \tag{17}$$

where $\mathcal{L}$ is a linear operator, $\mathcal{N}$ is a nonlinear operator and $g(x)$ is a given function. To decompose the solution $y(x)$ use the infinite series solution

$$y(x) = \sum_{k=0}^{\infty} y_k(x) \tag{18}$$

and the decomposetion of the nonlinear operator$N$ in (17) is:

$$\mathcal{N}(y) = \sum_{i=0}^{\infty}\mathcal{A}_i(y_0, y_1, ..., y_i) \tag{19}$$

where$\mathcal{A}_i$ are the Adomian polynomials and are obtained by

$$\mathcal{A}_i = \frac{1}{i!}\frac{d^i}{d\lambda^i}[\mathcal{N}(\sum_{k=0}^{\infty}\lambda^k y_k)] \tag{20}$$

Substituting (18) and (19) into (17) provides the following recursive scheme:

$$\begin{cases} \mathcal{L}[y_0] = g(x). \\ \mathcal{L}[y_{i+1}] = -\mathcal{A}_i, \quad i = 0, 1, 2, ..., \end{cases} \tag{21}$$

The M- term approximation solution is defined as

$$\phi_M(x) = \sum_{i=0}^{M-1} y_i(x) \tag{22}$$

where we assume that

$$\lim_{M\to\infty} \phi_M(x) = y(x). \tag{23}$$

3.2. **Jacobi spectral Collocation method.** To fix the idea, we consider the following linear fractional differential equation:

$$\sum_{\ell=0}^{n} D^{\beta_\ell} y(x) = f(x), \quad \ell < \beta_\ell \leq \ell + 1 \tag{24}$$

with the initial conditions

$$y^{(s)}(0) = \varrho_s, \quad s = 0, 1, 2, ..., n \tag{25}$$

The solution $y(x)$in (24) can be approximated on the partition interval$[0, L]$ by a truncated series of shifted Jacobi polynomials:

$$y_r(x) = \sum_{j=0}^{r} c_j P_{L,j}^{(\eta,\delta)}(x), \tag{26}$$

where c_j are unknown coefficients.
Substituting (26) into (24) and (25), we have

$$\sum_{j=0}^{r} c_j \sum_{\ell=0}^{n} P_j^{(\eta,\delta,\beta_\ell)}(x) = f(x), \tag{27}$$

$$\sum_{j=0}^{r} c_j P_j^{(\eta,\delta,s)}(0) = \varrho_s, \quad s = 0, 1, ..., n, \tag{28}$$

where

$$P_j^{(\eta,\delta,\beta_\ell)}(x) = D^{\beta_\ell} P_j^{(\eta,\delta)}(x),$$
$$P_j^{(\eta,\delta,s)}(0) = \frac{d^s}{dx^s} P_j^{(\eta,\delta)}(0).$$

Relation (28) forms a system with $n+1$ equations and $m+1$ unknowns. To construct the remaining $m-n$ equations, we substitute Jacobi-Gauss points $x_i (i = 1, ..., m-n)$,to obtain $m-n$ equations. Therefore,the method reduces the solution of (17) to the solution of system $\Omega C = \Xi$ where Ω, C, and Ξ are

$$\Omega = \begin{pmatrix} \Omega^1 \\ \Omega^2 \end{pmatrix}, \tag{29}$$

where

$$\Omega^1_{(m-n)\times(m+1)} = \sum_{\ell=0}^{n} P_j^{(\eta,\delta,\beta_\ell)}(x_i), \qquad i = 1, ..., m-n, \quad j = 0, 1, ..., m,$$
$$\Omega^2_{(n+1)\times(m+1)} = P_j^{(\eta,\delta,s)}(0), \qquad s = 1, ..., n, \quad j = 0, 1, ..., m,$$

$$C = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix} \tag{30}$$

and vector

$$\Xi = \begin{pmatrix} \Xi^1 \\ \Xi^2 \end{pmatrix}, \tag{31}$$

where $\Xi^1_{(n+1)\times 1} = \varrho_s \quad (s = 0, 1, ..., n)$ and $\Xi^2_{(m-n)\times 1} = f(x_i) \quad (i = 1, ..., m-n)$.

For more detalis see [1, 7, 16].

4. Methodology

Suppose the following differential equation:

$$\begin{aligned} \mathcal{L}[y] + \mathcal{N}[y] &= f(x) \\ y^{(s)}(0) &= \varrho_s, \qquad s = 0, 1, ..., n. \end{aligned} \tag{32}$$

Now,by applying the ADM mentioned in the subsection 3.1 to the above problem, we get

$$y_{i+1} = -L^{-1}[\mathcal{A}_i], \quad i \geq 0. \tag{33}$$

Sometimes, L^{-1} is difficult or impossible to find. So, we can solve it by our proposed method. That is, we set

$$y_k(x) = \sum_{i=0}^{m} c_i^{(k)} P_{L,i}^{(\eta,\delta)}(x) \tag{34}$$

Using subsection 3.2, we can write

$$C^{(0)} = \Omega^{-1}\Xi^{(0)} \tag{35}$$

where

$$\begin{aligned} C^{(0)} &= [c_0^{(0)}, c_1^{(0)}, ..., c_m^{(0)}]^T \\ \Xi^{(0)} &= [f(x_1), f(x_2), ..., f(x_{m-n}), \varrho_0, \varrho_1, ..., \varrho_n]^T, \end{aligned} \tag{36}$$

and $x_1, x_2, ..., x_{m-n}$ are zeros of polynomial $P_{m-n}^{(\eta,\delta)}(t)$,while,

$$C^{(i)} = \Omega^{-1}\Xi^{(i)} \tag{37}$$

where

$$\begin{aligned} C^{(i)} &= [c_0^{(i)}, c_1^{(i)}, ..., c_m^{(i)}]^T \\ \Xi^{(i)} &= [-\mathcal{A}_{i-1}(x_1), -\mathcal{A}_{i-1}(x_2), ..., -\mathcal{A}_{i-1}(x_{m-n}), 0, 0, ..., 0]^T, \quad i = 1, ... \end{aligned} \tag{38}$$

5. Numerical Examples and Comparisons

This section illustrates some numerical examples solved in [1] to show the accuracy and applicability of the proposed method in solving the initial value problems of nonlinear FDEs. This section also compares the given results from our scheme using shifted Jacobi polynomials at various points of (η, δ) and those reported in [1], which was achieved using the shifted Legendre polynomial. These comparisons show the effectiveness and high accuracy of our method. Maple 2017 has been used to compute the numerical results with 32-digit precision in the following examples.

Example 1: Consider the following initial value problem of the nonlinear fractional differential equation

$$D^{3/2}y + D^{1/2}y + Dy + y + e^y = \frac{4\sqrt{x}}{\sqrt{\pi}} + \frac{8x^{3/2}}{3\sqrt{\pi}} + 2x + x^2 + e^{x^2} \qquad x \in [0,1] \quad (39)$$

$$y(0) = y'(0) = 0. \quad (40)$$

with exact solution $y(x) = x^2$.

In Table 1, present method gives the numerical results based on the maximum absolute errors(MEAs) for N=7 ,$\eta = 24.001$ and $\delta = 0.00001$. We observe numerical results for present method are more accurate than the numerical results of the method in [1].Fig. 1 shows the matching between the numerical solution and exact solution.

Table 1. Comparison of MAEs between SADM by Legendre polynomials[1] and Present method for example 5

K	**SADM by Legendre polynomials**[1]	**Present method**
10	1.0×10^{-5}	2.1×10^{-8}
20	1.6×10^{-9}	1.7×10^{-17}
30	3.0×10^{-13}	3.9×10^{-25}

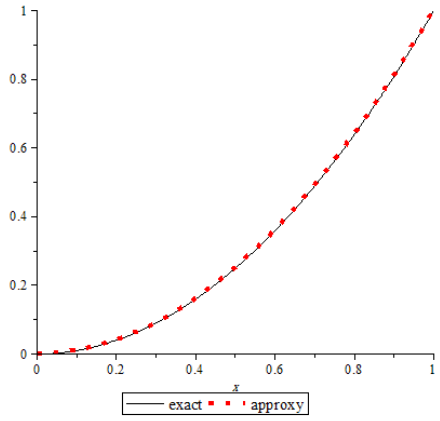

Figure 1. comparison of numerical solution and exact solution for example 5 at **K=30.**

Example 2: Consider the initial value problem of the nonlinear fractional differential equation

$$D^3y + D^{5/2}y + y^2 = x^4, \qquad x \in [0,1] \tag{41}$$

$$y(0) = y^{'}(0) = 0, \quad y^{''}(0) = 2, \tag{42}$$

with exact solution $y(x) = x^2$.

In Table 2, MEAs for N=7 and at choose values,$\eta = 30, \delta = -0.5$ are given . It is to be noted that presented method converges before SADM by shifted Legendre polynomial in [1],and the error becomes stable after 10 iteration . A comparison between numerical solution and exact solution is given in Fig. 2. The numerical solutions match well with exact solution.

TABLE 2. Comparison of MAEs between SADM by Legendre polynomials[1] and Present method for example 5

K	**SADM by Legendre polynomials**[1]	**Present method**
5	1.2×10^{-17}	3.2×10^{-28}
10	1.2×10^{-31}	8.0×10^{-32}
15	8.0×10^{-32}	8.0×10^{-32}

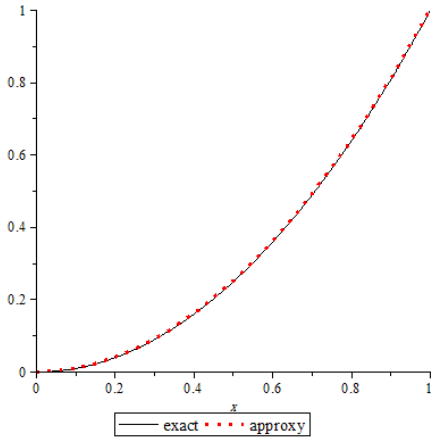

FIGURE 2. Comparison of numerical solution and exact solution for example 5 at **k=15.**

6. CONCLUSION

In this work, the proposed method was a generalization of SADM in [1]. In summary, the main advantage of this method is that is uses shifted Jacobi polynomials to find a highly accurate numerical solutions for nonlinear initial value problems of FDEs. Through comparison, the validity and accuracy of our scheme is asserted.

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A. S. Alghamdi
Faculty of Science, AlBaha University, Albaha, Saudi Arabia
E-mail address: ahlam8505@gmail.com

J.F. Alzaidy
Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia
E-mail address: j-f-h-z@hotmail.com

A.K. Hussain
Department of Material Engineering, Faculty of Engineering, Al-Mustansiriyah University,Baghdad, Iraq
E-mail address: Amina.Kass@ uomustansiriyah.edu.iq